THE LIMITS OF DETERMINACY IN SECOND ORDER ARITHMETIC

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Abstract. We establish the precise bounds for the amount of determinacy provable in second order arithmetic. We show that for every natural number n, second order arithmetic can prove that determinacy holds for Boolean combinations of n many $\mathbf{\Pi}^0_3$ classes, but it cannot prove that all finite Boolean combinations of $\mathbf{\Pi}^0_3$ classes are determined. More specifically, we prove that Π^1_{n+2}-$\mathsf{CA}_0 \vdash n$-$\mathbf{\Pi}^0_3$-DET, but that Δ^1_{n+2}-$\mathsf{CA} \nvdash n$-$\mathbf{\Pi}^0_3$-DET, where n-$\mathbf{\Pi}^0_3$ is the nth level in the difference hierarchy of $\mathbf{\Pi}^0_3$ classes.

We also show some conservativity results that imply that reversals for the theorems above are not possible. We prove that for every true Σ^1_4 sentence T (as for instance n-$\mathbf{\Pi}^0_3$-DET) and every $n \geq 2$, Δ^1_n-$\mathsf{CA}_0 + T + \Pi^1_\infty$-$\mathsf{TI} \nvdash \Pi^1_n$-$\mathsf{CA}_0$ and Π^1_{n-1}-$\mathsf{CA}_0 + T + \Pi^1_\infty$-$\mathsf{TI} \nvdash \Delta^1_n$-$\mathsf{CA}_0$.

1. Introduction

The general enterprise of calibrating the strength of classical mathematical theorems in terms of the axioms (typically of set existence) needed to prove them was begun by Harvey Friedman in [1971]. In that paper he worked primarily in the set theoretic settings of subsystems (and extensions) of ZFC. Actually, almost all of classical mathematics can be formalized in the language of second order arithmetic. This language consists of that of ordinary (first order) arithmetic and membership only for numbers in sets of numbers. It also has variables for, and quantification over, numbers and subsets of the natural numbers $\mathbb{N}$. (In terms of representing classical mathematics in this setting, we are restricting our attention to the countable case for algebraic and combinatorial results and, analogously, the separable situation for analytic or topological ones.) Moreover, the standard theorems of classical (countable) mathematics can be established in systems requiring only the basic axioms for arithmetic and set existence axioms just for subsets of $\mathbb{N}$. Realizing this, Friedman [1975] moved to the setting of second order arithmetic and subsystems of its full theory Z_2 which assumes some basic axioms of arithmetic and the existence of each subset of $\mathbb{N}$ defined by a formula of second order arithmetic. (See §2.3 for the precise definitions of our language, structures, axiom systems and their models.) Many researchers have since contributed to this endeavor but the major systematic developer and expositor since Friedman has been Stephen Simpson and the basic source for both background material and extensive results is his book [2009].

Five subsystems of Z_2 of strictly increasing strength emerged as the core of the subject with the vast majority of classical mathematical theorems being provable in one of them. (Formal definitions of the subsystems of second order arithmetic used in this paper are in §2.3.)

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Indeed, relative to the weakest of them (RCA_0, which corresponds simply to computable mathematics) almost all the theorems studied turned out to be equivalent to one of these five systems. Here the equivalence of a theorem T to a system S means that not only is the theorem T provable in S but that $\mathsf{RCA}_0 + T$ proves all the axioms of S as well. Thus the system S is precisely what is needed to establish T and gives a characterization of its (proof theoretic) strength. It is this approach that gives the subject the name of Reverse Mathematics. In standard mathematics one proves a theorem T from axioms S. Here one then tries to reverse the process (over a weak base theory) by proving the axioms of S from T (and RCA_0). In fact, for those theorems that are not true effectively (essentially the same as provable in RCA_0), the vast majority turn out to be equivalent in the sense of reverse mathematics to one of the two weakest of these systems (WKL_0 or ACA_0). Only a handful are equivalent to one of the two stronger systems (ATR_0 or Π^1_1-CA_0) and just a couple lie beyond them. (See Simpson [2009] for examples.) There is one now known (Mummert and Simpson [2005]) to be at the next level (Π^1_2-CA_0) and (as far as we know) none that are not known to be provable a bit beyond this level but still at one less than Π^1_3-CA_0.

In this paper we supply a natural hierarchy of theorems that require, respectively, each of the natural levels of set existence assumptions going from the previous known systems all the way up to full second order arithmetic. These theorems come from the realm in which the subject began, axioms of determinacy. The subject here is that of two person games with complete information. Our *games* (at least in this section) are played by two players I and II. They alternate *playing* natural numbers with I playing first to produce a *play of the game* which is a sequence $x \in \omega^\omega$. A *game G_A is specified* by a subset A of ω^ω. We say that I *wins a play x of the game G_A specified by A* if $x \in A$. Otherwise II wins that play. A *strategy* for I (II) is a function σ from strings p of even (odd) length into ω. We say that the game G_A *is determined* if there is a winning strategy for I or II in this game. (More details and terminology can be found in §2.1 where we switch from Baire space, ω^ω, to Cantor space, 2^ω to better match the standard setting for reverse mathematics and for other technical reasons. Basic references are Moschovakis [2009] and Kechris [1995].)

Now, in its full form, the Axiom of Determinacy says that all games G_A are determined. It has many surprising implications (e.g. all sets of reals are Lebesgue measurable) that contradict the Axiom of Choice. So instead, one considers restricted classes of games: If Γ is a class of sets A (of reals), then we say that Γ *is determined* if G_A is determined for every $A \in \Gamma$. We denote the assertion that Γ is determined by Γ *determinacy* or Γ-DET. One is then interested in determinacy for "simple" or easily definable classes Γ. One hopes that, for such Γ, determinacy will be provable (in some system of set theory including the axiom of choice) and that many of the consequences of the full axiom will then follow for sets in Γ. This has, indeed, turned out to be the case and a remarkably rich and extensive theory with applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems and other areas has been developed. (In addition to the basic texts mentioned above, one might refer, for example, to Kechris and Becker [1996], Kechris and Louveau [1987], Kechris and Miller [2004] and Foreman [2000] for sample applications.) A crowning achievement of this theory has been the calibration of the higher levels of determinacy for Γ in the projective hierarchy of sets of reals that begins with the analytic sets and progresses by complementation and projection. Here work of Martin, Steel and Woodin (Martin and Steel [1989], Woodin [1988]) has precisely characterized the large cardinal assumptions needed to prove each level of determinacy in the projective hierarchy (and beyond).

We are here concerned with lower levels of determinacy. Friedman's first foray [1971] into the area that grew into reverse mathematics dealt with these issues. He famously proved that Borel determinacy is not provable in ZFC without the power set axiom. Indeed, he showed that one needed $\aleph_1$ many iterations of the power set to prove it. Martin [1975], then showed that Borel determinacy is provable in ZFC and provided a level by level analysis of Borel hierarchy and the number of iterations of the power set needed to establish determinacy at those levels. Moving from set theory to second order arithmetic and the realm of what is now called reverse mathematics, Friedman [1971] also showed that $\mathbf{\Sigma}^0_5$ determinacy (i.e. for $G_{\delta\sigma\delta\sigma}$ sets or see §2.2) is not provable in full second order arithmetic. Martin [1974a], [n.d., Ch. I] improved this to $\mathbf{\Sigma}^0_4$ determinacy (i.e. $F_{\sigma\delta\sigma}$ sets). He also presented [1974] [n.d., Ch. 1] a proof of $\mathbf{\Delta}^0_4$ determinacy (sets that are both $F_{\sigma\delta\sigma}$ and $G_{\delta\sigma\delta}$) that he said could be carried out in Z_2. This seemed to fully determine the boundary of determinacy that is provable in second order arithmetic and left only the first few levels of the Borel hierarchy to be analyzed from the viewpoint of reverse mathematics.

The first very early result (essentially Steel [1976] see also Simpson [2009 V.8]) was that $\mathbf{\Sigma}^0_1$ (open) determinacy is equivalent (over RCA_0) to ATR_0. Tanaka [1990] then showed that Π^1_1-CA_0 is equivalent to $\mathbf{\Sigma}^0_1 \wedge \mathbf{\Pi}^0_1$ (intersections of open and closed sets) determinacy. Moving on to $\mathbf{\Sigma}^0_2$ (F_σ) determinacy, Tanaka [1991] showed that it was equivalent to an unusual system based on closure under monotonic Σ^1_1 definitions. At the level of $\mathbf{\Delta}^0_3$ (sets in both $\mathrm{F}_{\sigma\delta}$ and $G_{\delta\sigma}$) determinacy, MedSalem and Tanaka [2007] showed that each of Π^1_2-CA_0 + Π^1_3-TI_0 (an axiom system of transfinite induction) and Δ^1_3-CA_0 + Σ^1_3-IND_0 (induction for Σ^1_3 formulas) prove $\mathbf{\Delta}^0_3$-DET but Δ^1_3-CA_0 alone does not. They improve these results in [2008] by showing that $\mathbf{\Delta}^0_3$-DET is equivalent (over Π^1_3-TI_0) to another system based on transfinite combinations of Σ^1_1 inductive definitions. Finally, Welch [2009] has shown that Π^1_3-CA_0 proves not only $\mathbf{\Pi}^0_3$ ($\mathrm{F}_{\delta\sigma}$) determinacy but even that there is a β-model (Definition 2.9) of Δ^1_3-CA_0 + $\mathbf{\Pi}^0_3$-DET. In the other direction, he has also shown that, even augmented by an axiom about the convergence of arithmetical quasi-inductive definitions, Δ^1_3-CA_0 does not prove $\mathbf{\Pi}^0_3$-DET. The next level of determinacy is then $\mathbf{\Delta}^0_4$.

Upon examining Martin's proof of $\mathbf{\Delta}^0_4$-determinacy as sketched in [1974] and then later as fully written out in [n.d., Ch. 1], it seemed to us that one cannot actually carry out his proof in Z_2. Essentially, the problem is that the proof proceeds by a complicated induction argument over an ordering whose definition seems to require the full satisfaction relation for second order arithmetic This realization opened up anew the question of determining the boundary line for determinacy provable in second order arithmetic. We answer that question in this paper by analyzing the strength of determinacy for the finite levels of the difference hierarchy on $\mathbf{\Pi}^0_3$ ($\mathrm{F}_{\sigma\delta}$) sets, the m-$\mathbf{\Pi}^0_3$ sets. (As defined in §2.2, these give a natural hierarchy for the finite Boolean combinations of $\mathbf{\Pi}^0_3$ sets.)

In the positive direction (§4), we carefully present a variant of Martin's proof specialized and simplified to the finite levels of the difference hierarchy on $\mathbf{\Pi}^0_3$ along with the analysis needed to determine the amount of comprehension used in the proof for each level of the hierarchy. (It is a classical theorem of Kuratowski [1958] that extending the hierarchy into the transfinite to level $\aleph_1$ gives all the $\mathbf{\Delta}^0_4$ sets. Martin's proof proceeds by an induction encompassing all these levels.) This establishes an upper bound for the provability of the m-$\mathbf{\Pi}^0_3$ sets in Z_2.

Theorem 1.1. *For each $m \geq 1$, Π^1_{m+2}-$CA_0 \vdash m$-$\mathbf{\Pi}^0_3$-DET.*

In the other direction, we prove that this upper bound is sharp in terms of the standard subsystems of second order arithmetic.

Theorem 1.2. *For every $m \geq 1$, Δ^1_{m+2}-CA does not prove m-$\mathbf{\Pi}^0_3$-DET.*

Corollary 1.3. *Determinacy for the class of all finite Boolean combinations of $\mathbf{\Pi}^0_3$ classes of reals (ω-$\mathbf{\Pi}^0_3$-DET) cannot be proved in second order arithmetic. As these classes are all (well) inside $\mathbf{\Delta}^0_4$, Z_2 does not prove $\mathbf{\Delta}^0_4$-DET.*

Note that by Theorem 1.1, any model of second order arithmetic in which the natural numbers are the standard ones (i.e. $\mathbb{N}$) does satisfy ω-$\mathbf{\Pi}^0_3$-DET and so the counterexample for its failure to be a theorem of Z_2 must be nonstandard. Of course, it can be constructed by a compactness argument or, equivalently, as an ultraproduct of the L_{α_n} defined below. In contrast, the counterexamples from Friedman [1971] and Martin [1974a], [n.d., Ch. 1] are all even β-models, so not only with its numbers standard but all its "ordinals" (well orderings) are true ordinals (well orderings) as well.

We can reformulate this limitative result in the setting of set theory by noting the following conservation result.

Proposition 1.4. *ZFC^- (ZFC with collection but without the power set axiom) and even with a definable well ordering of the universe assumed as well, is a Π^1_4 conservative extension of Z_2.*

Proof: (Sketch) This fact should be "well known" and certainly follows from the extensive analysis of the relations between models of (subsystems of) second order arithmetic and those of the form $L(X)$ in Simpson [2009, VII]. Basically, given a model $\mathcal{M}$ of Z_2 and an X a set in $\mathcal{M}$, one defines an interpretation $L^{\mathcal{M}}(X)$ (defined over the well orderings of $\mathcal{M}$) of $L(X)$ in $\mathcal{M}$ and checks that it is a model of ZFC^-. This is detailed work but fairly straight forward for Z_2. Much of the work in Simpson [2009 VII.7] (whose ideas are described there as "probably well known but we have been unable to find bibliographic references for them") is devoted to showing that the facts required for the interpretation to satisfy the basic axioms of set theory can be derived in ATR_0 and the Shoenfield absoluteness theorem (VII.4.14) in Π^1_1-CA_0. As a guide we note that the basic translations and interpretations between second order arithmetic and a simple set theory are in VII.3. The relations with L and $L(X)$ and their basic properties are in VII.4. The material needed to verify the comprehension axioms is in VII.5 with Theorem VII.5.9(10) and the proof of its Corollary VII.5.11 being the closest to what is needed here. The material for replacement is in VII.6 with Lemma VII.6.15 and Theorem VII.6.16 being the closest to what is needed here. Finally, as $L^{\mathcal{M}}(X)$ is a model of $V = L(X)$, $L^{\mathcal{M}}(X)$ satisfies even global choice, i.e. there is a definable well ordering of the universe. Thus $L^{\mathcal{M}}(X)$ is a model of ZFC^- (and global definable choice).

Now if $T = \forall X \exists Y \forall Z \exists W \varphi$ is Π^1_4 and false in some $\mathcal{M} \vDash \mathsf{Z}_2$, we take X to be the counterexample and consider $L^{\mathcal{M}}(X)$ and any $Y \in L^{\mathcal{M}}(X)$. By assumption, $\mathcal{M} \vDash \exists Z \forall W \neg\varphi(X, Y)$ and so by Shoenfield absoluteness, $L^{\mathcal{M}}(X) \vDash \exists Z \forall W \neg\varphi(X, Y)$. Thus T fails in $L^{\mathcal{M}}(X)$ for the desired contradiction. $\square$

It is worth pointing our that Π^1_4 is as far as conservation results can go for ZFC^- over Z_2. As Simpson [2009, VII.6.3] points out, Feferman and Levy have produced a model of Z_2 in which a Σ^1_3-AC axiom fails. Now each Σ^1_3-AC axiom is clearly a Σ^1_5 sentence and a theorem of ZFC^-. Now if $T = \forall X \exists Y \forall Z \exists W \varphi$ is Π^1_4 and false in some $\mathcal{M} \models \mathsf{Z}_2$, we take X to be the counterexample and consider $L^{\mathcal{M}}(X)$ and any $Y \in L^{\mathcal{M}}(X)$. By assumption, $\mathcal{M} \models \exists Z \forall W \neg\varphi(X, Y)$ and so by Shoenfield absoluteness, $L^{\mathcal{M}}(X) \models \exists Z \forall W \neg\varphi(X, Y)$. Thus T fails in $L^{\mathcal{M}}(X)$ which is a model of ZFC^-.

Corollary 1.5. *Determinacy for the class of all finite Boolean combinations of $\mathbf{\Pi}^0_3$ classes of reals (ω-$\mathbf{\Pi}^0_3$-DET) and so, a fortiori, $\mathbf{\Delta}^0_4$-DET cannot be proved in ZFC$^-$.*

In fact, our counterexamples that establish Theorem 1.2 (the games with no strategy in L_{α_n} as described below) are all given by effective versions of the m-$\mathbf{\Pi}^0_3$ sets where the initial $\mathbf{\Pi}^0_3$ sets are effectively defined, i.e. given by recursive unions and intersections starting with closed sets that are "lightfaced", i.e. defined by Π^0_1 formulas of first order arithmetic without real (second order) parameters. (See the remarks in the last paragraph of §2.2. We use notations such as Π^0_1 and Π^0_3 to denote these "lightfaced' versions of the Borel classes.) This gives rise to a Gödel like phenomena for second order arithmetic with natural mathematical Σ^1_2 statements saying that specific games have strategies and containing no references to provability.

Theorem 1.6. *There is a Σ^1_2 formula $\varphi(x)$ with one free number variable x, such that, for each $n \in \omega$, $Z_2 \vdash \varphi(\underline{n})$ but $Z_2 \nvdash \forall n \varphi(n)$.*

Of course, on their face Theorems 1.1 and 1.2 along with Corollary 1.3 produce a sequence of Π^1_3 formulas $\psi(n)$ that have the same proof theoretic properties while eliminating the references to syntax and recursion theory present in the φ of Theorem 1.6. They simply state the purely mathematical propositions that all n-$\mathbf{\Pi}^0_3$ games are determined.

The plan for the first few sections of this paper is as follows. We provide basic definitions and notations about games and determinacy, hierarchies of sets and formulas and subsystems of second order arithmetic in §2. Basic facts about Gödel's constructible universe L not found in the standard texts are given in §3. The proof of Theorem 1.1 is in §4.

To prove Theorem 1.2, we will work in set theory or more specifically in fragments of ZFC + $\mathsf{V{=}L}$ instead of directly in second order arithmetic. We can do this as Simpson [2009, VII] shows how one can move back and forth between systems of second order arithmetic and subsystems of ZFC + $\mathsf{V{=}L(X)}$, once one has ATR_0 (or at times Π^1_1-CA_0) as a base theory. (Note also that, as mentioned above, these systems are equivalent to even weaker forms of determinacy than any we consider and so are provable even in $\mathbf{\Pi}^0_3$-DET.)

Let α_n denote the first n-admissible ordinal, and $\mathbb{R}$ the set of subsets of ω. (See §3 for the basic definitions.) By Lemma 3.2 and Simpson [2009, VII.5.3], $\mathbb{R} \cap L_{\alpha_n}$ is a β-model of Δ^1_{n+1}-CA for $n \geq 2$. Let Th_{α_n} denote the true theory of L_{α_n}. We think of Th_{α_n} as a subset of ω, or as the characteristic function (in 2^ω) for the set of indices of sentences of set theory (without parameters) true in L_{α_n}. As every element of L_{α_n} is definable in L_{α_n} without parameters (Lemma 3.6), the Gödel-Tarski undefinability of truth (a simple diagonal argument at this point) says that $Th_{\alpha_n} \notin L_{\alpha_n}$. Our plan is to show that, for $n \geq 2$, (n-1)-Π^0_3-DET does not hold in L_{α_n} by showing that, if it did, Th_{α_n} would be a member of L_{α_n}.

As the L_{α_n} are β-models, Theorem 1.2 can be immediately improved in various ways by including in the base theory any axioms true in all β-models, as for example:

Theorem 1.7. *For every $n \geq 1$, Σ^1_{n+2}-DC + Π^1_∞-TI does not prove n-Π^0_3-DET.*

Proof: Every β-model is obviously a model of Π^1_∞-TI and, as $L_{\alpha_{n+1}} \models \Delta^1_{n+1}$-CA, it also satisfies Σ^1_{n+2}-DC (which is stronger than Δ^1_{n+1}-CA) by Simpson [2009, VII.6.18]. □

The key result needed to prove Theorem 1.2 is the following.

Lemma 1.8. *For every $n \geq 2$, there is a game G that is $(n$-$1)$-Π^0_3, such that, if we interpret the play of each player as the characteristic function of a set of sentences in the language of set theory, then*

1. *If* I *plays Th_{α_n}, he wins.*
2. *If* I *does not play Th_{α_n} but* II *does, then* II *wins.*

We prove this Lemma in §5 but we now show how it implies our main result, Theorem 1.2.

Proof of Theorem 1.2: As we mentioned above, $\mathbb{R} \cap L_{\alpha_n}$ is a β-model of Δ^1_{n+1}-CA and so it suffices to show that $L_{\alpha_n} \not\models (n$-$1)$-$\Pi^0_3$-DET. Let G be as in the Lemma; and suppose it is determined in L_{α_n}. Player II cannot have a winning strategy for G in L_{α_n} because if II has a winning strategy σ in L_{α_n}, σ would also be a winning strategy in V as $\mathbb{R} \cap L_{\alpha_n}$ is a β-model (and σ being a winning stratgey for G is a Π^1_1 property). But, I has a winning strategy for G in V by clause (1) of the Lemma. So, I must have a winning strategy σ for G in L_{α_n}. Again, as $\mathbb{R} \cap L_{\alpha_n}$ is a β-model, σ is truly a winning strategy for I (in V). We claim that if II plays so as to simply copy I's moves, then σ has to play Th_{α_n}. If not, then at some first point I plays a bit that is different from Th_{α_n}. At this point II could stop copying I and just continue playing Th_{α_n} and he would win (by clause (2) of the Lemma). Thus σ would not be a truly winning strategy for I (in V). We conclude that σ computes Th_{α_n} as the sequence of I's plays against II's copying his moves and so $Th_{\alpha_n} \in L_{\alpha_n}$ for the desired contradiction. □

In the spirit of reverse mathematics one should now ask for reversals showing that m-$\mathbf{\Pi}^0_3$-DET implies Π^1_{m+2}-CA$_0$ or something along these lines. Nothing like this is, however, possible. MedSalem and Tanaka [2007] have shown that even full Borel ($\mathbf{\Delta}^1_1$) determinacy does not imply even Δ^1_2-CA$_0$. (Indeed they also show that it does not imply either induction for Σ^1_3 formulas or Π^1_2-TI$_0$.) Their proofs proceed via constructing countably coded β-models and appealing to the second Gödel incompleteness theorem.

In §6, we provide a very different approach that applies to any true (or even consistent with ZFC) Σ^1_4 sentence T and shows that no such sentence can imply Δ^1_2-CA$_0$ even for β-models. The counterexamples are all initial segments of $L(X)$ (for X a witness to the Σ^1_4 sentence). We also show that for any such T and $n \geq 2$, Δ^1_n-CA$_0$ + T + Π^1_∞-TI $\nvdash \Pi^1_n$-CA$_0$ and Π^1_n-CA$_0$ + T + Π^1_∞-TI $\nvdash \Delta^1_{n+1}$-CA$_0$ (even for β-models). These results are also suitably generalized to Σ^1_m theorems of ZFC in an optimal way. As Borel and m-$\mathbf{\Pi}^0_3$ determinacy are Π^1_3 theorems of ZFC, these general conservation results apply and we have the following results:

Borel-DET + Π^1_∞-TI $\nvdash$ Δ^1_2-CA$_0$.

For $n \geq 0$, Δ^1_{n+2}-CA$_0$ + n-$\mathbf{\Pi}^0_3$-DET + Π^1_∞-TI $\nvdash \Pi^1_{n+2}$-CA$_0$.

Thus, by Theorem 1.1, for $n \geq 0$, n-$\mathbf{\Pi}^0_3$-DET is a consequence of Π^1_{n+2}-CA$_0$ incomparable (in terms of provability) with Δ^1_{k+2}-CA$_0$ for all $k \leq n$ while Δ^1_{n+2}-CA$_0$ + n-$\mathbf{\Pi}^0_3$-DET is a system strictly between Δ^1_{n+2}-CA_0 and Π^1_{n+2}-CA_0 given by a mathematically natural proposition.

2. Definitions and Notations

2.1. **Games and Determinacy.** From now on, our basic playing field is not Baire space but Cantor space, 2^ω, the set of infinite (length ω) binary sequences (reals), x which we identify with the set $X \subseteq \mathbb{N}$ with its characteristic function x. Our *games* are played by two players I and II. They alternate *playing* 0 or 1 with I playing first to produce a *play of the game* which is a real x. A *game* G_A *is specified* by a subset A of 2^ω. We say that I *wins a play* x *of the game* G_A *specified by* A if $x \in A$. Otherwise II wins that play. A *strategy* for I (II) is a function σ from binary strings p of even (odd) length into $\{0,1\}$. The intuition here is that at any finite string of even (odd) length, a *position in the game* at which it is I's (II's) *turn to play*, the strategy σ instructs I (II) to play $\sigma(p)$. We say that a position q (play x) is *consistent with* σ if, for every $p \subset q$ (x) of even (odd) length, $\sigma\hat{}\sigma(p) \subseteq q$ (x). (We use $\hat{}$ for concatenation of strings and confuse a number i with the string $\langle i \rangle$.) The stratgey σ is a winning strategy for I (II) in the game G_A if every play consistent with σ is in A $(\bar{A})$. (We use $\bar{A}$ for $2^\omega \backslash A$, the *complement* of A in Cantor space.)

We say that the game G_A *is determined* if there is a winning strategy for I or II in this game. If Γ is a class of sets A (of reals), then we say that Γ *is determined* if G_A is determined for every $A \in \Gamma$. We denote the assertion that Γ is determined by Γ*determinacy* or Γ-DET.

Note that easy codings translate between games on Baire space to ones on Cantor space. Given an $A \subseteq \omega^\omega$ we code it by $\hat{A} \subseteq 2^\omega$ with elements precisely the ones of the form $\hat{f} = 0^{f(0)}10^{f(1)}10^{f(2)}\dots10^{f(n)}1\dots$ where we use 0^k to denote the sequence consisting of k many 0's. A stratgey for G_A can be gotten effectively from one for $\hat{A} = \{x \in 2^\omega | [\exists n \forall m > n(x(2m+1) = 0]$ or $[\forall n \exists m > n(x(2m) = 1)$ & for the f such that $x = \hat{f}$, $f \in A]\}$. In the other direction, given an $A \subseteq 2^\omega$ a strategy for G_A can be found effectively from one for $G_{\check{A}}$ where $\check{A} = \{f \in \omega^\omega | \exists n(f(2n+1) \notin \{0,1\}$ or $f \in A\}$. Thus if Γ is rich enough to be closed under these operations, it makes no difference whether we play in Baire space or Cantor space. It is clear that once Γ is at least at the level of $\mathbf{\Pi}^0_3$ $(\mathrm{F}_{\sigma\delta})$ (or even $\mathbf{\Delta}^0_3$) Γ determinacy in one space is equivalent to it in the other, and in this paper we consider only classes from $\mathbf{\Pi}^0_3$ and above. For the record, we note that there are significant differences at levels below $\mathbf{\Delta}^0_3$. In contrast to the standard low level results for Baire space mentioned in §1, Nemoto, MedSalem and Tanaka [2007] show that in Cantor space $\mathbf{\Delta}^0_1$-DET is equivalent (over RCA_0) to each of $\mathbf{\Sigma}^0_1$-DET and WKL_0. ACA_0 is equivalent to $(\mathbf{\Sigma}^0_2 \wedge \mathbf{\Pi}^0_2)$-DET. $\mathbf{\Delta}^0_2$-DET is equivalent to each of $\mathbf{\Sigma}^0_2$-DET and ATR_0 while, as indicated above, $\mathbf{\Delta}^0_3$-DET is equivalent to the same level of determinacy in Baire space.

As usual, we let $\breve{\Gamma} = \{\bar{A} | A \in \Gamma\}$. Idiosyncratically, and only for the purposes of the next Lemma, we let $A^* = \{0\hat{}x | x \in A\} \cup \{1\hat{}x | x \in A\}$ and $\Gamma^* = \{A^* | A \in \Gamma\}$.

Lemma 2.1. (RCA_0) *If* $\Gamma^* \subseteq \Gamma$ *and* Γ *is determined then so is* $\breve{\Gamma}$.

Proof: A winning stratgey σ for I (II) in G_{A^*} is easily converted into one τ for II (I) in $G_{\bar{A}}$: $\tau(p) = \sigma(\tau(p) = \sigma(\emptyset)\hat{}p)$ $(\sigma(0\hat{}p))$. □

This is as much as we need about determinacy for our proof of its failure for various Γ in specified models of Δ^1_{n+2}-CA_0 in §5. To show that determinacy holds for these Γ in Π^1_{n+2}-CA_0 in §4, we need to generalize these notions a bit.

We now allow games to be played on arbitrary binary trees T. The idea here is that we replace 2^ω by $[T]$, the *set of paths through* T, $\{x|\forall n(x\upharpoonright n \in T)\}$. (So the basic notion takes T to be $2^{<\omega}$, the tree of all finite binary sequences.) A *(binary) tree* is a set of finite binary sequences closed under initial segments and with no dead ends, i.e. $\forall p \in T(p\hat{\ }0 \in T$ or $p\hat{\ }1 \in T)$. If T is a tree and $p \in T$ then T_p is the subtree of all $q \in T$ comparable with p, $T_p = \{q \in T|q \subseteq p \text{ or } p \subseteq q\}$. (A *subtree of a tree* T is a subset T which is also a tree.) A *play (real)* x *is one in the game on* T if $x \in [T]$. A *game* $G(A,T)$ is now specified by an $A \subseteq [T]$. The other notions defined above are generalized accordingly with the positions restricted to T and the plays to $[T]$. A *quasistrategy for* I *(*II*) in* $G(A,T)$ is a subtree S of T such that if $p \in S$ is of odd (even) length then the immediate successors in S are precisely the same as the ones in T. Thus a strategy for I (II) can be viewed as a quasistrategy in which each $p \in S$ of even (odd) length has precisely one immediate successor in S. A position p or play x in $G(A,T)$ being *consistent with* S now just means that $p \in S$ or $x \in [S]$. We say that S is a *winning quasistrategy for* I *(*II*)* if every x consistent with S is in A $(\bar{A})$. If I (II) does not have a winning strategy in $G(A,T)$ then we define II's (I's) *nonlosing quasistrategy* W in this game to be the set of all $p \in T$ such that I (II) has no winning strategy in $G(A,T_p)$. If this set exists (e.g. we have enough comprehension as when A is arithmetic and we have at least Π^1_2-CA_0), it is easily seen to be a quasistrategy for II (I) by induction on the length of $p \in W$.

2.2. **Classes and Hierarchies.** We have to deal with several classes and hierarchies of both formulas and sets that have standard but at times conflicting (or overlapping) designations. We try to use as single unambiguous system through out the paper while choosing notations whose meaning in specific contexts should be evident to readers familiar with work in the appropriate areas.

We begin with formulas. Here we have formulas of both second order arithmetic and set theory. We use Σ^0_n, Π^0_n, Σ^1_n and Π^1_n in the usual way for the formulas of second order arithmetic with the understanding (typical in reverse mathematics) that the second order quantifiers range over subsets of $\mathbb{N}$ (or equivalently reals) and the formulas may include free set variables (i.e. ones over reals) at no cost to the quantifier complexity. For formulas of set theory, we use the notations Σ_n and Π_n in the usual way with the understanding that at the bottom level we allow bounded quantifiers in the $\Sigma_0 = \Pi_0$ formulas and may also include free variables.

We will be concerned with sets of reals and not sets of numbers and so define the relevant classes (of sets) only that setting. From the set theoretic point of view we have the usual topologically based Borel hierarchy beginning with the open or closed sets and then progressing by alternating applications of countable intersections and unions. Recursion theoretically, the open (closed) sets are those defined by Σ^0_1 (Π^0_1) formulas where the free set variables other than the one representing the reals in the set being defined are replaced by specific real parameters. (It is worth noting that the closed sets are precisely those of the form $[T]$ for some tree T.) One then moves up the Borel hierarchy as usual. We are only concerned with the first few levels and, in particular, the $\mathbf{\Pi}^0_3$ sets are those A of the form $\cap A_k$ where the $A_k \in \mathbf{\Sigma}^0_2$, i.e. they are of the form $A_k = \cup A_{k,j}$ where the $A_{k,j}$ are $\mathbf{\Pi}^0_1$, i.e. closed, sets. They are also the classes of sets of reals defined by formulas at the indicated level of the arithmetic hierarchy with real parameters.

The sets that we will be analyzing from the viewpoint of their determinacy are the finite Boolean combinations of $\mathbf{\Pi}^0_3$ sets. These are typically laid out in the difference hierarchy of the m-$\mathbf{\Pi}^0_3$ sets.

Definition 2.2. A set A (of reals) is m-$\mathbf{\Pi}^0_3$ if there are $\mathbf{\Pi}^0_3$ sets $A_0, A_1, \ldots, A_{m-1}, A_m = \emptyset$ such that

$$x \in A \Leftrightarrow \text{the least } i \text{ such that } x \notin A_i \text{ is odd.}$$

We say that *the sequence* $\langle A_i | i \leq m \rangle$ *represents* A (as an m-$\mathbf{\Pi}^0_3$ set).

Remark 2.3. Without loss of generality, we may assume that the A_i in Definition 2.2 are nested downward, i.e. $A_i \supseteq A_{i+1}$ by simply replacing A_i by $\cap_{j \leq i} A_j$. We assume that any given representation (as an m-$\mathbf{\Pi}^0_3$ set) is of this form.

This hierarchy is called the difference hierarchy since the sets at level m are generated by taking unions of differences of $\mathbf{\Pi}^0_3$ sets. More specifically, they are Boolean combinations of $\mathbf{\Pi}^0_3$ sets of the form $(A_0 \backslash A_1) \cup (A_2 \backslash A_3) \ldots \cup (A_{m-2} \backslash A_{m-1})$ if m is even and $(A_0 \backslash A_1) \cup (A_2 \backslash A_3) \ldots \cup (A_{m-3} \backslash A_{m-2}) \cup A_{m-1}$ if m is odd. (We write $A \backslash B$ for $A \cap \bar{B}$.) Of course, when m is even, $A_m = \emptyset$ and so $A_{m-1} = A_{m-1} \backslash A_m$ and we can view it as being of the first form as well. The converse holds as well, i.e. every set A of the form $(A_0 \backslash A_1) \cup (A_2 \backslash A_3) \ldots \cup (A_{m-2} \backslash A_{m-1})$ is m-$\mathbf{\Pi}^0_3$ for m even and each of the form $(A_0 \backslash A_1) \cup (A_2 \backslash A_3) \ldots \cup A_{m-1}$ is m-$\mathbf{\Pi}^0_3$ for m odd. One can see this as analogous to the difference hierarchy on r.e. sets.

It is easy to see that all finite Boolean combinations of $\mathbf{\Pi}^0_3$ sets are m-$\mathbf{\Pi}^0_3$. The sets that are a union of differences of $\mathbf{\Pi}^0_3$ sets are clearly closed under union and include the $\mathbf{\Pi}^0_3$ sets. On the other hand, if A is represented by $\langle A_i | i \leq m \rangle$, then $\bar{A}$ is represented by $\langle 2^\omega, A_i | i \leq m \rangle$. Thus $\cup_m m$-$\mathbf{\Pi}^0_3$ contains the $\mathbf{\Pi}^0_3$ sets and is closed under union and complementation.

An advantage of Definition 2.2 as a hierarchy on the finite Boolean combinations of $\mathbf{\Pi}^0_3$ sets is that it can immediately be extended into the transfinite simply by allowing sequences A_i for $i < \alpha$ with the understanding that limit ordinals are even. It is a classical theorem of Kuratowski [1958] that as α ranges over the countable ordinals these sets are precisely the $\mathbf{\Delta}^0_4$ ones, i.e. those that are both $\mathbf{\Sigma}^0_4$ and $\mathbf{\Pi}^0_4$. (See MedSalem and Tanaka [2007] for a proof in ACA_0.)

In §4, we prove, in Π^1_{m+2}-CA_0, that all m-$\mathbf{\Pi}^0_3$ sets are determined for each $m \in \mathbb{N}$. Our negative results in §5, however, use a different, dual representation of the finite Boolean combinations of $\mathbf{\Pi}^0_3$ sets.

Definition 2.4. A set $A \subseteq 2^\omega$ is $\mathbf{\Pi}^0_{3,n}$ if there exist $\mathbf{\Pi}^0_3$ sets $A_0, ..., A_n$ such that $A_n = 2^\omega$ and

$$x \in A \iff \text{the least } i \text{ such that } x \in A_i \text{ is even.}$$

We say that *the sequence* $\langle A_i | i \leq m \rangle$ *represents* A (as a $\mathbf{\Pi}^0_{3,n}$ set).

Remark 2.5. Note that without loss of generally we may assume that the A_i in Definition 2.4 are nested upward, i.e. $A_i \subseteq A_{i+1}$ by simply replacing A_i by $\cup_{j \geq i} A_j$. We assume that any given representation (as a $\mathbf{\Pi}^0_{3,n}$ set) is of this form.

This formulation does not extend into the transfinite to capture the $\mathbf{\Delta}^0_4$ sets as did that of the m-$\mathbf{\Pi}^0_3$ sets. Indeed, it is easy to see that at level ω one gets all the $\mathbf{\Sigma}^0_4$ sets represented by sequences $\langle A_i | i < \omega \rangle$ of $\mathbf{\Pi}^0_3$ sets. (The issue here is that the $\mathbf{\Pi}^0_3$ sets are closed under countable intersections and so $\{x | \forall i < \beta (x \in A_i)\}$ is still $\mathbf{\Pi}^0_3$ while they are not closed under countable (as opposed to finite) unions and so $\{x|\exists i<\beta(x\in A_i)\}$ is only $\mathbf{\Sigma}^0_4$. Thus Definition 2.4 is suited to a difference hierarchy built on the $\mathbf{\Sigma}^0_3$ rather than the $\mathbf{\Pi}^0_3$ sets. Nonetheless, at the finite levels of the two hierarchies are closely enough related to allow us to use both in our calculations of the limits of determinacy provable in fragments of second order arithmetic.

Lemma 2.6. (RCA_0) $\mathbf{\Pi}^0_{3,2n+1}=(2n+1)\text{-}\mathbf{\Pi}^0_3$ *and* $\check{\mathbf{\Pi}}^0_{3,2n}=(2n)\text{-}\mathbf{\Pi}^0_3$.

PROOF: Consider any sequence $A_0\subseteq A_1\subseteq\cdots\subseteq A_{m-1}$ of $\mathbf{\Pi}^0_3$ sets. Let A be the $\mathbf{\Pi}^0_{3,m}$ set represented by $\langle A_0,\ldots,A_{m-1},2^\omega\rangle$. Let B be the m-$\mathbf{\Pi}^0_3$ set represented by $\langle A_{m-1},\ldots,A_1,\emptyset\rangle$. We claim that $A=B$ when m is odd and $A=\bar{B}$ when m is even. This would prove the lemma as A or B could be any m-$\mathbf{\Pi}^0_3$ or $\mathbf{\Pi}^0_{3,m}$ set.

Consider $x\in 2^\omega$. Let i_x be the least i such that $x\in A_i$. So, $x\in A$ if and only if i_x is even. Let j_x be the least j such that $x\notin A_{m-1-j}$. So, $x\in B$ if and only if j_x is odd. Since i_x-1 is the greatest i such that $x\notin A_i$, we have that $m-1-j_x=i_x-1$. It then follows that i_x and j_x have the same parity if m is even and different parities if m is odd. Hence $x\in A\iff x\notin B$ if m is even and $x\in A\iff x\in B$ if m is odd. □

Corollary 2.7. *For each* $m\geq 1$, $\mathsf{RCA}_0\vdash m\text{-}\mathbf{\Pi}^0_3\text{-}DET\leftrightarrow\mathbf{\Pi}^0_{3,m}\text{-}DET$.

PROOF: It is immediate that $(\mathbf{\Pi}^0_1)^*\subseteq\mathbf{\Pi}^0_1$ and so the same property clearly holds for $\mathbf{\Pi}^0_3$ as it is generated from $\mathbf{\Pi}^0_1$ by unions and intersections which preserves the desired closure property. The required closure property then holds for m-$\mathbf{\Pi}^0_3$, $\mathbf{\Pi}^0_{3,m}$, m-$\check{\mathbf{\Pi}}^0_3$ and $\check{\mathbf{\Pi}}^0_{3,m}$ (simply replace each A_i in the representation by A_i^*). Thus the corollary holds by Lemmas 2.6 and 2.1. □

We also point out that the classes we have considered all have "*light faced*" versions, i.e. ones defined by the same classes of formulas but without any reals as parameters. In terms of our definitions, the light faced (or recursive) closed sets are those of the form $[T]$ for T a recursive tree. The class of these sets is denoted by Π^0_1. The light faced class Π^0_3 is defined from Π^0_1 as in the boldface version except that now the closed sets $A_{k,j}$ required to define $A\in\Pi^0_3$ must be uniformly recursive, i.e. there is a set of uniformly recursive trees $T_{k,j}$ (i. e. the relation $p\in T_{k,j}$ is a recursive one of the triple $\langle p,k,j\rangle$) such that $A_{k,j}=[T_{k,j}]$. The light faced m-Π^0_3 and $\Pi^0_{3,n}$ sets are then defined from the Π^0_3 sets exactly as in the bold faced versions in Definitions 2.2 and 2.4, respectively. Of course, each of these light faced classes are contained in the corresponding bold face class. Our negative results about the failure of determinacy in various subsystems of second order arithmetic as exhibited in the particular models of §5 all provide light face counterexamples.

2.3. **Systems of Reverse Mathematics.** We first briefly review the five standard systems of reverse mathematics. For the analysis of even the lowest levels of determinacy one begins near the top of this list and moves well beyond. Details, general background, and results, as well as many examples of reversals, can be found in Simpson [2009], the standard text on reverse mathematics. Each of the systems is given in the (two sorted) language of second order arithmetic, that is, the usual first order language of arithmetic augmented by set variables with their quantifiers and the membership relation $\in$ between numbers and sets. A structure for this language is one of the form $\mathcal{M}=\langle M,S,+,\times,<,0,1,\in\rangle$ where M is a set (the set of "numbers" of $\mathcal{M}$) over which the first order quantifiers and variables of our language range; $S \subseteq 2^M$ is the collection of subsets of "numbers" in $\mathcal{M}$ over which the second order quantifiers and variables of our language range; $+$ and $\times$ are binary functions on M; $<$ is a binary relation on M while 0 and 1 are members of M.

Each subsystem of second order arithmetic that we consider contains the standard basic axioms for $+$, $\cdot$, and $<$ (which say that $\mathbb{N}$ is an ordered semiring). In addition, they all include a form of induction that applies only to sets (that happen to exist):

(I_0) $\quad (0 \in X \wedge \forall n\,(n \in X \to n+1 \in X)) \to \forall n\,(n \in X)$.

We call the system consisting of I_0 and the basic axioms of ordered semirings P_0. All five standard systems are defined by adding various types of set existence axioms to P_0.

(RCA_0) Recursive Comprehension Axioms: This is a system just strong enough to prove the existence of the computable sets. In addition to P_0 its axioms include the schemes of Δ^0_1 comprehension and Σ^0_1 induction:

$(\Delta^0_1\text{-}\mathsf{CA}_0)$ $\quad \forall n\,(\varphi(n) \leftrightarrow \psi(n)) \to \exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$ for all Σ^0_1 formulas φ and Π^0_1 formulas ψ in which X is not free.

$(\mathrm{I}\Sigma^0_1)$ $\quad (\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1))) \to \forall n\,\varphi(n)$ for all Σ^0_1 formulas φ.

The next system says that every infinite binary tree has an infinite path.

(WKL_0) Weak König's Lemma: This system consists of RCA_0 plus the statement that every infinite subtree of $2^{<\omega}$ has an infinite path.

We next move up to arithmetic comprehension.

(ACA_0) Arithmetic Comprehension Axioms: This system consists of RCA_0 plus the axioms $\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$ for every arithmetic formula (i.e. Σ^0_n for some n) φ in which X is not free.

The next system says that arithmetic comprehension can be iterated along any countable well order. As noted in §1, it is equivalent to $\mathbf{\Sigma}^0_1$-determinacy in Baire space (Steel [1976]).

(ATR_0) Arithmetical Transfinite Recursion: This system consists of RCA_0 plus the following axiom. If X is a set coding a well order $<_X$ with domain D and Y is a code for a set of arithmetic formulas $\varphi_x(z, Z)$ (indexed by $x \in D$) each with one free set variable and one free number variable, then there is a sequence $\langle K_x \mid x \in D\rangle$ of sets such that if y is the immediate successor of x in $<_X$, then $\forall n\,(n \in K_y \leftrightarrow \varphi_x(n, K_x))$, and if x is a limit point in $<_X$, then K_x is $\bigoplus\{K_y \mid y <_X x\}$.

The systems that we actually need climb up to full second order arithmetic, Z_2, (i.e. comprehension for all formulas). They are characterized by the syntactic level of the second order formulas for which they include comprehension axioms.

$(\Pi^1_n\text{-}\mathsf{CA}_0)$ These systems consists of RCA_0 plus the Π^1_n comprehension axioms: $\exists X\,\forall k\,(k \in X \leftrightarrow \varphi(k))$ for every Π^1_n formula φ in which X is not free.

The first of these, $\Pi^1_1\text{-}\mathsf{CA}_0$, is the last of the five standard systems. As noted in §1, it is equivalent to $(\mathbf{\Sigma}^0_1 \wedge \mathbf{\Pi}^0_1)$-determinacy in Baire space (Tanaka [1990]). If one assumes $\Pi^1_n\text{-}\mathsf{CA}_0$ for every n then one has full second order arithmetic.

$(\Pi^1_\infty\text{-}\mathsf{CA}_0)$ or (Z_2) This system consists of $\Pi^1_n\text{-}\mathsf{CA}_0$ for every n.

We also need the intermediate Δ^1_n classes and corresponding axioms.

(Δ^1_n-CA_0) Δ^1_n Comprehension Axioms: $\forall k\,(\psi(k) \leftrightarrow \neg\varphi(k)) \to \exists X\,\forall k\,(k \in X \leftrightarrow \varphi(k))$ for every Σ^1_n formulas φ and ψ in which X is not free.

The other set existence axioms that play a role in our analysis are various versions of the axiom of choice in the setting of second order arithmetic.

(Σ^1_n-AC_0) This system consists of ACA_0 plus the Σ^1_n choice axioms $\forall n \exists Y \varphi(n,Y) \to \exists Z \forall n \varphi(n,(Z)_n)$ for every Σ^1_n formula φ and where $(Z)_n = \{i | \langle i,n\rangle \in Z\}$.

(Σ^1_n-DC_0) This system consists of ACA_0 plus the Σ^1_n dependent choice axioms $\forall n \forall X \exists Y \varphi(n,X,Y) \to \exists Z \varphi(n,(Z)^n,(Z)_n)$ for every Σ^1_n formula φ and where $(Z)^n = \{\langle i,m\rangle \,|\, \langle i,m\rangle \in Z\ \&\ m < n\}$

(Strong Σ^1_n-DC_0) This system consists of ACA_0 plus the strong Σ^1_n dependent choice axioms $\exists Z \forall n \forall Y (\varphi(n,(Z)^n,Y) \to \varphi(n,(Z)^n,(Z)_n)$ for every Σ^1_n formula φ.

See Simpson [2009, VII.6] for the relations among these choice principles and various comprehension axioms.

If we strengthen the basic induction axiom I_0 by replacing it with induction for all formulas φ of second order arithmetic we get full induction

(I) $(\varphi(0)\ \wedge\ \forall n\,(\varphi(n)\ \to\ \varphi(n+1))\ \to\ \forall n\,(\varphi(n))$ for every formula φ of second order arithmetic.

Each of the systems above has an analog in which I_0 is replaced by I. It is designated by the same letter sequence as above but without the subscript 0, as for example, RCA in place of RCA_0.

Definition 2.8. If $\mathcal{M}$ is a structure for second order arithmetic and its first order part M is $\mathbb{N}$ then $\mathcal{M}$ is an *ω-model*.

Obviously if an ω-model $\mathcal{M}$ is a model of one of the systems above, such as Π^1_n-CA_0, then it is also a model of the analogous system, such as Π^1_n-CA.

Restricted versions of the full induction axioms are designated as follows:

(Σ^1_n-IND) $(\varphi(0)\ \wedge\ \forall n\,(\varphi(n) \to \varphi(n+1)) \to \forall n\,(\varphi(n))$ for every Σ^1_n formula φ of second order arithmetic.

Definition 2.9. An ω-model $\mathcal{M}$ is a *β-model* if, for every Σ^1_1 sentence φ with parameters in $\mathcal{M}$, $\varphi \Leftrightarrow \mathcal{M} \vDash \varphi$.

In particular, if $\mathcal{M}$ is a β-model then any well ordering in $\mathcal{M}$, i.e. a linear ordering in S with no infinite descending chain in S, is actually a well ordering (where S is the second order part of $\mathcal{M}$). There are also some induction axioms and systems (even stronger than Σ^1_n-IND_0) for transfinite procedures that serve as proof theoretical approximations to being a β-model.

(Π^1_∞-TI_0) This system consists of ACA_0 plus all the transfinite induction axioms $\forall X(X$ is a well-ordering $\&\ \forall j(\forall i(i <_X j \to \psi(i)) \to \psi(j)) \to \forall j \psi(j)$, where by "$X$ is a well-ordering" we mean that the relation $<_X = \{\langle i.j\rangle \,|\, \langle i,j\rangle \in X\}$ given by the pairs of numbers in X is a well ordering, i.e. has no infinite descending chains, and ψ is any formula of second order arithmetic in which X is not free.

Obviously, every β-model satisfies Π^1_∞-TI_0. The systems (Π^1_n-TI_0) are defined analogously with ψ restricted to Π^1_n formulas.

3. Facts about L

We assume a familiarity with the basic facts about L and Σ_1 admissible sets of the form L_α as can be found, for example, in Devlin [1984, I-II] or Barwise [1975, I-II]. Generalizing from 1 to n we say that an L_α is Σ_n *admissible* if it satisfies Σ_n bounding: For any $\delta < \alpha$ and any Π_{n-1}/L_α formula $\varphi(x,y)$, if $L_\alpha \vDash \forall\gamma < \delta\exists y\varphi(\gamma,y)$, then there is a $\lambda < \alpha$ such that $L_\alpha \vDash \forall\gamma < \delta\exists y \in L_\lambda\varphi(\gamma,y)$. This property is equivalent (for initial segments of L) to the requirements that the set satisfy (in addition to the standard (KP) axioms of admissibility) Δ_{n-1} collection and Σ_{n-1} separation (also called comprehension). It is also equivalent to there not being a function f with domain some $\delta < \alpha$ which is Σ_n (or equivalently Π_{n-1}) over L_α whose range is unbounded in α or L_α. One more equivalent is that for any Σ_n/L_α function f, $f \upharpoonright \gamma \in L_\alpha$ for every $\gamma < \alpha$. Thus we have the usual bounded quantifier manipulation rules: for $\varphi \in \Pi_{n-1}$, $\forall x \in t\exists y\varphi$ is equivalent to a Σ_n formula. We also say that α is n*-admissible* if L_α is Σ_n admissible.

We need a few facts about Σ_n Skolem functions and projections which are deep and difficult theorems of Jensen's fine structure theory in general (as in Devlin [1984, VI]) but become quite simple via standard arguments when enough admissibility is assumed. We also need a couple of special facts about the ordinals less than β_0, the first ordinal which is Σ_n admissible for all n. All of these ought to be at least known folklore but seem to be missing from the couple of standard texts we consulted.

Proposition 3.1. *If α is Σ_k admissible then L_α has a parameterless Σ_{k+1} Skolem function.*

Proof: For a Σ_{k+1} formula $\exists x\varphi(x,\bar{y})$ with Gödel number n we let $f(n,\bar{y}) = x \Leftrightarrow \varphi(x,\bar{y}) \wedge (\forall x' <_L x)(\neg\varphi(x,\bar{y}))$. The first conjunct is Π_k while the second one is equivalent to a Σ_k formula by Σ_k admissibility. □

Lemma 3.2. *If L_α is Σ_n admissible then it satisfies Δ_n comprehension, i.e. for any $u \in L_\alpha$ and Σ_n formulas $\varphi(z)$ and $\psi(z)$ such that $L_\alpha \vDash \forall z(\varphi(z) \leftrightarrow \neg\psi(z))$, $\{z \in u | \varphi(z)\} \in L_\alpha$.*

Proof: For $n = 1$ this is a standard fact. We now proceed by induction on $n \geq 2$. Consider any Σ_{n-2} formulas $\varphi(x,y,z)$ and $\psi(x,y,z)$ such that $L_\alpha \vDash \forall z(\exists x\forall y\varphi(x,y,z) \leftrightarrow \neg\exists x\forall y\psi(x,y,z))$ and any $u \in L_\alpha$. We must show that $\{z \in u | L_\alpha \vDash \exists x\forall y\varphi(x,y,z)\} \in L_\alpha$. Define in L_α a function on every $z \in u$ by $f(z) = x \Leftrightarrow [\forall y\varphi(x,y,z) \wedge (\forall x' <_L x)\exists y'\neg\varphi(x',y',z)] \vee [\forall y\psi(x,y,z) \wedge (\forall x' <_L x)\exists y'\neg\psi(x',y',z)]$. By Σ_n admissibility, the formula defining f is equivalent to one that is Δ_n and so its range is bounded, say by L_γ. We then have that $\{z \in u | L_\alpha \vDash \exists x\forall y\varphi(x,y,z)\} = \{z \in u | L_\alpha \vDash \exists x \in L_\gamma\forall y\varphi(x,y,z)\} = \{z \in u | L_\alpha \vDash \forall x \in L_\gamma\exists y\neg\psi(x,y,z)\}$. Again by Σ_n admissibility these last two definitions are equivalent to ones that are Π_{n-1} and Σ_{n-1}, respectively and so by Δ_{n-1} comprehension which we have by induction, they belong to L_α as required. □

Lemma 3.3. *If α is $(k+1)$-admissible, then there are unboundedly many $\gamma < \alpha$ such that $L_\gamma \preccurlyeq_k L_\alpha$. (We write $M \preccurlyeq_k N$ to denote the assertion that M is a Σ_k* elementary submodel *of N, i.e. for any Σ_k formula φ with parameters in M, $M \vDash \varphi \Leftrightarrow N \vDash \varphi$.)*

Proof: Consider any $\delta < \alpha$. We want to show there is a γ between δ and α such that $L_\gamma \preccurlyeq_k L_\alpha$. Let h be a (parameterless) Σ_k-Skolem function for L_α. Let H_1 be the Σ_k-Skolem hull of L_δ in L_α. Since α is $(k+1)$-admissible, H_1 is bounded in L_α; let $\delta_1 < \alpha$ be the least ordinal such that $H_1 \subseteq L_{\delta_1}$. We now iterate this process. Let H_l be the Σ_k-Skolem hull of

$L_{\delta_{l-1}}$ in L_α, and let δ_l be the least ordinal such that $H_l \subseteq L_{\delta_l}$. As for δ_1, one can show that all the δ_l are $< \alpha$. Let $\gamma = \sup_{l\in\omega} \delta_l$. First, let us show that $L_\gamma \preccurlyeq_k L_\alpha$. Suppose that $i < k$ and that we know that $L_\gamma \preccurlyeq_i L_\alpha$; we claim that $L_\gamma \preccurlyeq_{i+1} L_\alpha$. Let φ be a Π_i formula with parameters in L_{δ_l} for some l such that $L_\alpha \vDash \exists x\varphi(x)$. Then, there exists an $x \in H_{l+1}$ such that $L_\alpha \vDash \varphi(x)$, and since $L_\gamma \preccurlyeq_i L_\alpha$ we have that $L_\gamma \vDash \varphi(x)$. This proves our claim, and hence $L_\gamma \preccurlyeq_k L_\alpha$.

Next, we need to show that $\gamma < \alpha$. The reason is that the function $l \mapsto \delta_l \colon \omega \to L_\alpha$ is Σ_{k+1}:

$\delta_l = \mu\beta(\exists\beta_0, \dots, \beta_l \le \beta(\beta_0 = \delta \wedge \beta_l = \beta \wedge \bigwedge_{i<l} \forall x \in L_{\beta_i}(h(x) \in L_{\beta_{i+1}})))$.

Since α is $(k+1)$-admissible and h is Σ_k, the formula in parenthesis is Σ_k. Finding the least such β is Δ_{k+1}. Therefore, again by $(k+1)$-admissibility, this function has to be bounded below α, and hence $\gamma = \sup_{l\in\omega} \delta_l < \alpha$. □

Lemma 3.4. *If $L_\gamma \preccurlyeq_k L_\beta$ and β is Σ_{k-1} admissible then γ is Σ_k admissible.*

Proof: We proceed by induction on $k \ge 1$. Consider k and $L_\gamma \preccurlyeq_k L_\beta$ with β Σ_{k-1} admissible. (By convention, Σ_0 admissibility is just transitivity and so every L_β is Σ_0 admissible.) If the Lemma fails, there is a Π_{k-1}/L_γ function f with domain some $\delta < \gamma$ and range unbounded in L_γ. Suppose, for $\eta < \delta$, $f(\eta) = y \Leftrightarrow \psi(\eta, y)$ where ψ is Π_{k-1}/L_γ.For each $\eta < \delta$ and $y = f(\eta)$, $L_\beta \vDash \psi(\eta, y)$ since $L_\gamma \preccurlyeq_k L_\beta$. Thus $L_\beta \vDash \exists x \forall\eta < \delta \exists y <_L x(\psi(\eta, y))$. By the usual rules for manipulating bounded quantifiers, there is a Π_{k-1} formula $\theta(x)$ which is equivalent to $\forall\eta < \delta\exists y <_L x(\psi(\eta, y))$ in any Σ_{k-1} admissible L_ζ for $\zeta \ge \gamma$. As L_β is Σ_{k-1} admissible by assumption, $L_\beta \vDash \exists x\theta(x)$. As $L_\gamma \preccurlyeq_k L_\beta$, $L_\gamma \vDash \exists x\theta(x, \eta, y)$. Finally, as L_γ is Σ_{k-1} admissible by induction, $L_\gamma \vDash \exists x\forall\eta < \delta\exists y <_L x(\psi(\eta, y))$ for the desired contradiction. □

Lemma 3.5. *For $\gamma < \beta_0$, L_γ is countable in $L_{\gamma+1}$.*

Proof: For $\gamma = \omega$, the conclusion is immediate. We now proceed by induction on γ. For a successor ordinal $\gamma + 1$, the conclusion follows simply from the countability of the formulas paired with the (inductively given) countability of the parameter space L_γ. If γ is a limit ordinal less than β_0, it is not Σ_n admissible for some n, i.e. there is a Σ_n/L_γ map f from some $\delta < \gamma$ with range unbounded in γ. We now define a counting of L_γ by combining the countings of δ and $L_{f(\zeta)}$ for each $\zeta < \delta$ (which are uniformly definable as they are each in $L_{f(\zeta)+1}$ by induction). □

Lemma 3.6. *Every member of L_{α_n} is Σ_{n+1} definable over L_{α_n} without parameters.*

Proof: By Lemma 3.1 we may take f to be a parameterless Σ_{n+1} Skolem function for L_{α_n}. Let H be the Skolem hull of the empty set under f. As L_{α_n} is not $(n+1)$-admissible (by Lemmas 3.3 and 3.4), H is unbounded in L_{α_n}. On the other hand, By Lemma 3.5, H is transitive. (Say $\delta \in H$ and so a counting of δ also belongs to H but then so does the value of this counting on every $k \in \omega$.) Thus $H = L_{\alpha_n}$ and so every element of L_{α_n} is Σ_{n+1} definable (as the value of f on some $k \in \omega$) over L_{α_n} without parameters. □

4. Proving m-$\mathbf{\Pi}^0_3$ determinacy in Π^1_{m+2}-CA$_0$

This section is devoted to proving Theorem 1.1: For each $m \ge 1$, $\Pi^1_{m+2}\text{-}\mathsf{CA}_0 \vdash m\text{-}\mathbf{\Pi}^0_3$-DET. Our proof is a variant of Martin's proof [1974], [n.d., Ch. 1] of $\mathbf{\Delta}^0_4$-DET specialized and simplified to the finite levels of the difference hierarchy on $\mathbf{\Pi}^0_3$. Our proof of the main Lemma (Lemma 4.9) is perhaps somewhat different from his. In addition, we supply the analysis needed to see precisely how much comprehension is needed at every level.

We fix $m \in \omega$, $m \geq 1$. As m-$\mathbf{\Pi}^0_3$ determinacy is a Π^1_3 sentence and strong Σ^1_{m+2}-DC_0 is Π^1_4 conservative over Π^1_{m+2}-CA_0 by Simpson [2009, VII.6.20], we assume strong Σ^1_{m+2}-DC_0 as well as Π^1_{m+2}-CA_0. In particular, we fix an m-$\mathbf{\Pi}^0_3$ set A given by the nested sequence $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_m = \emptyset$ of $\mathbf{\Pi}^0_3$ sets. We let $A_i = \cap A_{i,k}$ for $A_{i,k} \in \mathbf{\Sigma}^0_2$ and $A_{i,k} = \cup A_{i,k,j}$ for $A_{i,k,j} \in \mathbf{\Pi}^0_1$.

We note that this proof does not use any determinacy beyond open determinacy, that (as we have seen) can be proved in systems much weaker than those we are using here.

We consider strings $s \in \omega^{\leq m}$ (so of length at most m) and subgame trees S of the full binary tree. When s is specified, we let $l = m - |s|$ and frequently make definitions and proofs depend on the parity of l (or equivalently of m when $s = \emptyset$ or $|s|$ is otherwise fixed).

Definition 4.1. We define relations $P^s(S)$ by induction on $|s| \leq m$:

(1) $P^\emptyset(S) \Leftrightarrow$ there is a winning strategy for I (II) if l (or equivalently m) is even (odd) in $G(A,S)$.

For $|s| = n+1$ we also divide into cases based on the parity of l.

(1)
(2) If l is even, $P^s(S) \Leftrightarrow$ there is a quasistrategy U for I in S such that
  (a) $[U] \subseteq A \cup A_{m-n-1,s(n)}$ and
  (b) $P^{s\upharpoonright n}(U)$ fails.
(3) If l is odd, $P^s(S) \Leftrightarrow$ there is a quasistrategy U for II in S such that
  (a) $[U] \subseteq \bar{A} \cup A_{m-n-1,s(n)}$ and
  (b) $P^{s\upharpoonright n}(U)$ fails.

A quasistrategy U *witnesses* $P^s(S)$ if U is as required in the appropriate clause ((1), (2) or (3)) of the above definition.

It is not hard to show that for any s, if l is even (odd), and I (II) has a winning strategy in $G(A,S)$, then $P^s(S)$ holds. The converse is also true, but we would need to use m-$\mathbf{\Pi}^0_3$ determinacy to prove it. Our first goal is to prove that for every s of length $n+1$ and any S, either $P^s(S)$ holds or $P^{s\upharpoonright n}(S)$ holds (Lemma 4.9). Then, in the proof of Theorem 1.1, we will use this result to show that, if $P^\emptyset(T)$ does not hold, we can build a winning strategy for II (I) in the case when m is even (odd).

Definition 4.2. A quasistrategy U *locally witnesses* $P^s(S)$ if, $|s| = n+1$ and U is a quasistrategy for I (II) if l is even (odd) and there is a $D \subseteq S$ such that for every $d \in D$ there is a quasistrategy R^d for II (I) if l is even (odd) in S_d such that

(i) $\forall d \in D \cap U(U_d \cap R^d$ witnesses $P^s(R^d))$
(ii) $[U] \setminus \cup_{d\in D} [R^d] \subseteq A$ $(\bar{A})$ if l is even (odd)
(iii) $\forall p \in S \exists^{\leq 1} d \in D(d \subseteq p \;\&\; p \in R^d)$

Remark 4.3. Simple quantifier counting shows that the relations $P^s(S)$, U witnesses $P^s(S)$ and U locally witnesses $P^s(S)$ are $\Sigma^1_{|s|+2}$, $\Pi^1_{|s|+1}$ and $\Sigma^1_{|s|+2}$, respectively.

We now prove a number of Lemmas in Π^1_{m+2}-CA_0 + strong Σ^1_{m+2}-DC_0 to establish our theorem. Note that when we perform inductions on something like $|s|$ we do not have to worry about what induction axioms we need as $|s| \le m \in \omega$.

Lemma 4.4. *If U locally witnesses $P^s(S)$ then U witnesses $P^s(S)$.*

PROOF: Suppose l is even. We show that clause (2a) of Definition 4.1 is satisfied. Consider $x \in [U]$. If $x \in A$ there is nothing to prove. If not, by Definition 4.2(ii), $x \in [R^d]$ for some $d \in D$. Then by (i) $U_d \cap R^d$ witnesses $P^s(R^d)$ and so by Definition 4.1 (2a) $x \in A_{m-n-1,s(n)}$ as required. If l is odd, the same argument with $\bar{A}$ in place of A and clause (3a) in place of (2a) shows that clause (3a) is satisfied as required.

We now proceed by induction on $|s| = n+1 \le m$ to show that part (b) of clause (2) or (3) of Definition 4.1 holds. We begin with $n = 0$. Suppose that m is odd and so the associated l with s in the definitions of $P^s(S)$ and U locally witnessing $P^s(S)$ is even. If (2b) does not hold, i.e. $P^\emptyset(U)$ holds, there is a winning stratgey τ for II in $G(A, U)$ (as m is odd). We claim there is a $d \in D$ consistent with τ such that every $x \supseteq d$ in $[U]$ and consistent with τ is in $[R^d]$. Suppose not. Now note that every $e \notin \cup_{d \in D, d \subset e} R^d$ in U and consistent with τ has a $\hat{d} \supseteq e$ in D (and so a minimal one) consistent with τ, for if not then, for any $x \supseteq e$ in $[U]$ and consistent with τ, $x \notin \cup_{d \in D} R^d$. By Definition 4.2(ii) we would then have $x \in A$ contradicting our choice of τ as a winning stratgey for II in $G(A, U)$. Next, note that, by our assumption, any such $\hat{d}$ has a (minimal) extension $\hat{e}$ consistent with τ and not in $R^{\hat{d}}$. By Definition 4.2(iii), no $e' \supset \hat{d}$ with $e' \subset \hat{e}$ is in D and so $\hat{e}$ has the same properties assumed about e. We can continue this procedure to build a sequence e_j each consistent with τ such that $\cup e_j = x \notin \cup_{d \subset x, d \in D} R^d$ and so $x \in U \setminus \cup_{d \in D} R^d \subseteq A$ for the same contradiction to our choice of τ.

So we have a $d \in D$ consistent with τ such that every $x \supseteq d$ in $[U]$ and consistent with τ is in $[R^d]$. Thus the natural restriction of τ is a winning strategy for II in $G(A, U_d \cap R^d)$, i.e. $P^\emptyset(U_d \cap R^d)$ contradicting Definition 4.2(i) and so establishing (2b) of Definition 4.1 as required.

If m is even and so the associated l for s in the definition of $P^s(S)$ is odd, the same proof works with the following pairs interchanged: I and II; 2 and 3; A and $\bar{A}$ (except that $G(A, U)$ is unchanged).

Next, we consider $n > 1$ and assume for definiteness that $m - n$ $(= m - |s \restriction n|)$ is odd and, for the sake of a contradiction, that $P^{s\restriction n}(U)$ holds. Let $\hat{S}$ be II's quasistrategy in U witnessing $P^{s\restriction n}(U)$. We will build $\hat{U}$, $\hat{D}$ and $\{\hat{R}^d : d \in \hat{D}\}$ locally witnessing $P^{s\restriction n-1}(\hat{S})$. By the induction hypothesis this would imply $P^{s\restriction n-1}(\hat{S})$, contradicting that $\hat{S}$ witnesses $P^{s\restriction n}(U)$. Roughly speaking, to build I's quasistrategy $\hat{U}$, player I would try to fall out of $\cup_{d\in D}[R^d]$ and hence end up in A. For the $d \in D$ such that I has no strategy to exit R^d, we will add $d \in \hat{D}$ and define $\hat{R}^d \subseteq R^d$ such that $P^{s\restriction n-1}(\hat{R}^d)$.

Let $\hat{D} = \{d \in \hat{S} \cap D | G(\overline{[R^d]}, \hat{S}_d) \text{ is a win for II}\}$ (a Σ^1_2 set). For $d \in \hat{D}$ we let $\hat{R}^d$ be II's nonlosing quasistrategy in $G(\overline{[R^d]}, \hat{S}_d)$ (each a $\mathbf{\Pi}^1_2$ set and so the indexed collection is given by Π^1_3-CA_0) so, of course, $\hat{R}^d \subseteq R^d$ and $\hat{R}^d$ is a quasistrategy for II in U_d. As $\hat{S}$ witnesses $P^{s\restriction n}(U)$, $[\hat{S}] \subseteq \bar{A} \cup A_{m-n,s(n-1)}$ and so $[\hat{R}^d] \subseteq \bar{A} \cup A_{m-n,s(n-1)}$. By Definition 4.2(i), $U_d \cap R^d$ witnesses $P^s(R^d)$ and so, in particular, $P^{s\restriction n}(U_d \cap R^d)$ fails and $\hat{R}^d$ is not a witness for it. As condition (3a) has already been verified for $[U] \supseteq [\hat{R}^d]$, (3b) must fail with $\hat{R}^d$ and so there is a witness $\hat{U}^d$ for each $P^{s\restriction n-1}(\hat{R}^d)$. The indexed sequence $\{\hat{U}^d : d \in \hat{D}\}$ then exists by $\Sigma^1_{|s|}$-AC_0 by Remark 4.3. Finally, we use Σ^1_2-CA_0 and Σ^1_2-AC_0 to define the sequence $\sigma_{q,d}$ of winning strategies for I in $G(\overline{[R^d]}, \hat{S}_q)$ for $d \in D$ and $q \notin \hat{R}^d$ (which is taken to be $\emptyset$ for $d \in D \backslash \hat{D}$) when one exists.

We now (arithmetically in the above parameters) define a quasistrategy $\hat{U}$ for I in $\hat{S}$:

(1) If $p \in \hat{U}$ and there is no $d \in D$ such that $d \subseteq p$ and $p \in R^d$ then the immediate extensions of p in $\hat{U}$ are the same as those in $\hat{S}$.
(2) For each $q \in \hat{U}$ and $d \in D$ such that q is a minimal extension of d in $R^d \backslash \hat{R}^d$, the play in $\hat{U}$ following q agrees with that of $\sigma_{q,d}$ until we reach a $p \notin R^d$.
(3) For $d \in \hat{D} \cap \hat{U}$, let $\hat{U}_d \cap \hat{R}^d = \hat{U}^d$.

We claim that $\hat{U}$ (with $\hat{D}$ and $\hat{R}^d$) locally witnesses $P^{s\restriction n-1}(\hat{S})$. Property (i) of Definition 4.2 follows from (3) in the definition of $\hat{U}$ and our choice of $\hat{U}^d$ for $d \in \hat{D}$. For (ii), note that, by (2) in the definition of $\hat{U}$, $[\hat{U}] \cap [R^d] \subseteq [\hat{R}^d]$ while $\hat{U}$ is a quasistrategy in $\hat{S}$ which is one in U. Thus $[\hat{U}] \backslash \cup_{d\in D} [\hat{R}^d] \subseteq [U] \backslash \cup_{d \in D} [R^d] \subseteq A$ by Definition 4.2(ii) as required. Property (iii) is immediate from the definitions and the corresponding condition for U being a local witness for $P^s(S)$ as $\hat{D} = D \cap \hat{S}$ and $\hat{R}^d \subseteq R^d$. Thus by induction, $\hat{U}$ witnesses $P^{s\restriction n-1}(\hat{S})$ for the desired contradiction (with clause (3b) in the definition of $\hat{S}$ being a witness for $P^{s\restriction n(U)}$).

Finally, when $n = 1$ (still assuming that $m - n$ is odd), the proof is much the same as when $n > 1$ except that when we choose a witness $\hat{U}^d$ for $P^{s\restriction n-1}(\hat{R}^d)$ we must take a winning strategy for I in $G(A, \hat{R}^d)$ and we need to show that $[\hat{U}] \subseteq A$ to see that it witnesses $P^{\emptyset}(\hat{S})$ for the desired contradiction (with clause (3b) in the definition of $\hat{S}$ being a witness for $P^{s\restriction n(U)}$). The point here is that if we stay in some $\hat{R}^d$ then we follow $\hat{U}^d$ which is a winning strategy for I in $G(A, \hat{R}^d)$. If we leave $\hat{R}^d$ then we leave R^d by (2) in the definition of $\hat{U}$. If we leave every R^d, we follow $\hat{S}$ and so stay in U and also wind up A by clause (ii) of the definition of U being a local witness for $P^s(S)$.

For $m - n$ even, the proof is the same except that we interchange I and II, clauses (2) and (3) of Definition 4.1, A and $\bar{A}$ and $G(\overline{[R^d]}, \hat{S}_d)$ and $G([R^d], \hat{S}_d)$. □

Definition 4.5. $P^s(S)$ *fails everywhere* if $P^s(S_p)$ fails for every $p \in S$. By Remark 4.3, this relation is $\Pi^1_{|s|+2}$.

Lemma 4.6. *If $P^s(S)$ fails then there is a quasistrategy W for* I *if l is odd (for* II *if l is even) in S such that $P^s(W)$ fails everywhere.*

Proof: For $|s| = 0$, if l is odd (even), II (I) does not have a winning strategy in $G(A, S)$. Let W be I's (II's) nonlosing quasistrategy in this game. It exists by Π^1_2-CA_0 and, by definition, II (I) does not have a winning strategy in $G(A, W_p)$ for any $p \in W_p$ and so $P^s(W)$ fails everywhere.

Now suppose $|s| = n+1$ and, for definiteness, that l is even. Let D be the $\Sigma^1_{|s|+2} \wedge \Pi^1_{|s|+2}$ set of minimal $d \in S$ for which $P^s(S_d)$. (If there are none, then $P^s(S)$ already fails everywhere.) By $\Sigma^1_{|s|+2}$-AC_0 let U^d witness $P^s(S_d)$ for every $d \in D$.

Consider now the game $G(B, S)$ where $B = \{x \in [S] | \exists d \in D(d \subseteq x)\}$. We claim that I has no winning stratgey in this game. If there were one σ, then we could define a quasistrategy U for I in S by following σ until a position $d \in D$ is reached at which point we move into

U^d. With D and $R^d = S_d$ for $d \in D$, it is easy to see that U locally witnesses $P^s(S)$ which by the previous Lemma is the desired contradiction.

Thus we can let W be II's nonlosing quasistrategy in $G(B, S)$ and σ_p be a winning strategy for I for $p \in S \backslash W$. (These exist as above.) If W is not as required, there is a $q \in W$ and a witness $\hat{U}$ that $P^s(W_q)$. Define a quasistrategy U for I in S_q by $U \cap W_q = \hat{U}$. If we ever reach a position $p \notin W$, we switch to following σ_p until we reach a position $d \in D$ at which point we switch to U^d. If we now let $\hat{D} = D \cup \{q\}$, $R^q = W_q$ and $R^d = S_d$ for $d \in D$, it is easy to see that together with U we have a local witness for $P^s(S_q)$. By the previous Lemma then $P^s(S_q)$ and so by the definition of D there is a $p \subseteq q$ in D contradicting the choice of W as a nonlosing strategy for II in $G(B, S)$.

The argument for $s = n + 1$ and odd l is the same except that we interchange I and II and $G(B, S)$ and $G(\bar{B}, S)$. □

Definition 4.7. For $n + 1 = |s|$, W *strongly witnesses* $P^s(S)$ if, for all $p \in W$, W_p witnesses $P^s(S_p)$, i.e. W witnesses $P^s(S)$ and $P^{s\upharpoonright n}(W)$ fails everywhere. By Remark 4.3, this relation is $\Pi^1_{|s|+1}$.

Lemma 4.8. *If $P^s(S)$ then there is a W that strongly witnesses it.*

Proof: Assume, for definiteness, that l is even. Let U witness $P^s(S)$. By Definition 4.1(2b), $P^{s\upharpoonright n}(U)$ fails. By Lemma 4.6, let W be a quasistrategy for I in U such that $P^{s\upharpoonright n}(W)$ fails everywhere. As $[W] \subseteq [U]$, Definition 4.1(2a) holds of W and so it strongly witnesses $P^s(S)$ as required. The proof for l odd is the same except that we interchange I and II as well as clauses (2) and (3) of Definition 4.1. □

Lemma 4.9. *If $|s| = n + 1$ then at least one of $P^s(S)$ and $P^{s\upharpoonright n}(S)$ holds.*

Proof: We prove the Lemma by reverse induction on $n < m$. Suppose for definiteness that $m - n$ is odd. If $P^s(S)$ fails we define by induction on the length of positions (using strong Σ^1_{m+2}-DC_0) a quasistrategy U for II in S along with $D \subseteq S$ and R^d for $d \in D$ locally witnessing $P^{s\upharpoonright n}(S)$ if $n > 0$ and witnessing $P^{\emptyset}(S)$ if $n = 0$. Thus, by Lemma 4.4 when $n > 0$, $P^{s\upharpoonright n}(S)$ holds as required. We will decide which q are in U by recursion on the length of q, starting with $\emptyset \in U$ and going up the tree S. Along the way we will enumerate some of these q into D and define a quasistrategy R^q for I. Inside each R^q we have to make sure to define U such that $U \cap R^q$ witnesses $P^{s\upharpoonright n}(R^q)$. For $x \in [U] \setminus \bigcup_{d \in D}[R^d]$ we need to make sure $x \in \bar{A}$, and we will do this as follows. On the one hand, we will define U so that x belongs to $\bar{A} \cup A_{m-n-2,j}$ for every $j \in \omega$, and hence $x \in \bar{A} \cup A_{m-n-2}$. On the other hand, we will make sure that $x \notin A_{m-n-1,s(n),j}$ for any j, and hence $x \notin A_{m-n-1}$. Then we use that $\bar{A} \cup A_{m-n-2} \setminus A_{m-n-1} \subseteq \bar{A}$ to show that $x \in \bar{A}$. We will go along the path x satisfying the requirements $x \in \bar{A} \cup A_{m-n-2,j}$ and $x \notin A_{m-n-1,s(n),j}$ one j at the time. To keep track of which requirements need to be satisfied, along the construction we will define the notion of a position $q \in U$ marking a $j \in \omega$ (with the j's marked in order along any path). Simultaneously we also define certain auxiliary quasistrategies W^q and $\hat{R}^q$.

Of course, $\emptyset \in U$ and we say that it marks 0. If $n = m - 1$, by lemma 4.6, we let $W^{\emptyset}$ be a quasistrategy for II in S such that $P^s(W^{\emptyset})$ fails everywhere. If $n < m - 1$, we know by (reverse induction) that $P^{s\hat{}0}(S)$ holds. In this case we apply Lemma 4.8 (and strong Σ^1_{m+2}-DC_0) to choose a $W^{\emptyset}$ strongly witnessing this fact (a $\Pi^1_{|s|+2}$ and so at worst Π^1_{m+1} relation). (Note that in this case, $P^s(W^{\emptyset})$ also fails everywhere by Definition 4.7.)

Now assume by induction (on positions in U) that $q \in U$ marks j and that q belongs to a quasistrategy W^q for II in S_q such that $P^s(W^q)$ fails everywhere and if $n < m-1$ then W^q strongly witnesses $P^{s\hat{\ }j}(S_q)$.

Recall that $A_{m-n-1,s(n),j}$ is a closed set. If $G(A_{m-n-1,s(n),j}, W^q)$ is not a win for II (a Π^1_2 relation), put q in D and let $\hat{R}^q$ be I's nonlosing quasistrategy in this game (a Π^1_2 set in the parameters). Define R^q to be $\hat{R}^q$ on W^q and to agree with S elsewhere in S_q. Now $[\hat{R}^q] \subseteq A_{m-n-1,s(n),j}$ since it is a nonlosing quasistrategy for this closed set. Next note that, by definition, $A_{m-n-1,s(n),j} \subseteq A_{m-n-1,s(n)}$. Thus if $P^{s\restriction n}(\hat{R}^q)$ fails, $\hat{R}^q$ would witness $P^s(W^q)$ contrary to our assumption that $P^s(W^q)$ fails everywhere. So we may take U^q to be a witness for $P^{s\restriction n}(\hat{R}^q)$ (a $\Pi^1_{|s|}$ relation) . We now continue to define U by letting it agree with U^q on $\hat{R}^q$ and let no $p \in \hat{R}^q$ into D nor mark any j'.

If beginning in this way with q we ever first reach a $p \notin \hat{R}^q$ (which we take to be q if $G(A_{m-n-1,s(n),j}, W^q)$ is a win for II), we continue in U by following a winning strategy τ_p for II in $G(A_{m-n-1,s(n),j}, W^q)$ (a Π^1_1 relation) until we first reach a $q' \supset p$ with $[W^q_{q'}] \cap A_{m-n-1,s(n),j} = \emptyset$. We must reach such a q' since τ_p is a winning strategy for II in $G(A_{m-n-1,s(n),j}, W^q)$ and $A_{m-n-1,s(n),j}$ is a closed set. We do not mark any q^* with $p \subseteq q^* \subset q'$ nor put any such into D. We now say that q' marks $j+1$. $P^s(W^q_{q'})$ fails everywhere since $P^s(W^q)$ does. If $n = m-1$, we let $W^{q'} = W^q_{q'}$. If $n < m-1$, $P^{s\hat{\ }j+1}(W^q_{q'})$ holds by our reverse induction on n and we let $W^{q'}$ strongly witness this fact (a $\Pi^1_{|s|+2}$ and so at worst Π^1_{m+1} relation). Of course, it also strongly witnesses $P^{s\hat{\ }(j+1)}(S_{q'})$ as required for our inductive definition of U.

We now prove that, if $n > 0$, U (with D and R^d) locally witnesses $P^{s\restriction n}(S)$. First note that, by our definitions, $U_d \cap R^d = U_d \cap \hat{R}^d = U^d$ for $d \in D$. Thus condition (i) of Definition 4.2 holds as U^d witnesses $P^{s\restriction n}(\hat{R}^d)$. As we never put any $q \supset d \in D$ into D until we have left $\hat{R}^d$, condition (iii) holds as well. We now turn to condition (ii) and consider any $x \in [U]$. Let $\emptyset = q_0 \subset q_1 \ldots \subset q_i \subset \ldots$ be the strictly increasing sequence of q's contained in x such that q_j marks j. By construction, each $q_j \in D$. If the sequence terminates at some $d = q_k$, then, by definition, x never leaves $\hat{R}^d$ and so $x \in [R^d]$ and (ii) holds trivially for this x. If not, then, for every j, $x \notin A_{m-n-1,s(n),j}$ as $x \in [W^{q_j}_{q_{j+1}}]$ and $[W^{q_j}_{q_{j+1}}] \cap A_{m-n-1,s(n),j} = \emptyset$. Thus $x \notin A_{m-n-1,s(n)} = \cup_j A_{m-n-1,s(n),j}$. As $A_{m-n-1,s(n)} \supseteq A_{m-n-1}$, $x \notin A_{m-n-1}$. Now if $n+1 = m$, $x \notin A_0$ and so $x \notin A$ and condition (ii) is satisfied. If $n+1 < m$ then, as W^{q_j} witnesses $P^{s\hat{\ }j}(S_{q_{j+1}})$, $x \in \bar{A} \cup A_{m-n-2,j}$ for each j. (Note that as $m-n$ is odd so is $m - |s\hat{\ }j|$ and we are in case (3) of Definition 4.1.) Thus $x \in \bar{A} \cup A_{m-n-2}$ as $A_{m-n-2} = \cap_j A_{m-n-2,j}$. As, by our case assumptions, $m-n-1$ is even, $A_{m-n-2} \backslash A_{m-n-1} \subseteq \bar{A}$ by the definition of A in terms of the sequence A_i. Thus $x \in \bar{A}$ and condition (ii) is again satisfied.

Finally, if $n = 0$ we argue that U is a wining quasistrategy for II in $G(A, S)$. Consider any $x \in [U]$. If there is a $d \in D$ such that $x \in [\hat{R}^d]$ then $x \in [U^d]$ by construction. Now U^d is a witness for $P^{\emptyset}(\hat{R}^d)$ (as $n = 0$, $s \restriction n = \emptyset$), i.e. U^d is a winning strategy for II in $G(A, \hat{R}^d)$. Thus $x \in \bar{A}$ as required. On the other hand, if leaves every $\hat{R}^d$ then, by the argument above, $x \in \bar{A}$ as well.

For $m-n$ even we interchange I and II as well as $G(A_{m-n-1,s(n),j}, W^q)$ and $G(\bar{A}_{m-n-1,s(n),j}, W^q)$ in the definition of U. In the verification that U is as desired note that the case that $n+1 = m$ cannot occur. Otherwise, we interchange I and II, cases (2) and (3) of Definition 4.1, even and odd as well as A and $\bar{A}$. □

Proof: (of Theorem 1.1) If $G(A,T)$ is not a win for II (I) if m is odd (even), $P^{\emptyset}(T)$ fails. By Lemma 4.6, there is a quasistrategy $W^{\emptyset}$ for I (II) such that $P^{\emptyset}(W^{\emptyset})$ fails everywhere. We define a quasistrategy U for I (II) in $W^{\emptyset}$ by induction on $|p|$ for $p \in U$. For $|p| = j+1$, we simultaneously define (using strong Σ^1_3-DC_0) a quasistrategy W^p for I (II) which strongly witnesses $P^{\langle j\rangle}(W^{p\restriction j}_p)$. Of course, $\emptyset \in U$ and we already have $W^{\emptyset}$. Suppose then that $p \in U$, $|p| = j+1$, and W^p has been defined. The immediate successors q of p in U are precisely its immediate successors in W^p. By our choice of W^p, $P^{\emptyset}(W^p)$ fails everywhere and so, in particular, $P^{\emptyset}(W^p_q)$ fails for each immediate successor q of p in U. By Lemma 4.9, then $P^{\langle j\rangle}(W^p_q)$ holds and, by Lemma 4.8 (and strong Σ^1_3-DC_0) we may choose a W^q that strongly witnesses it to continue our induction.

Consider now any play $x \in [U]$. By construction $x \in [W^{x\restriction j+1}]$ for every j and $W^{x\restriction j+1}$ witnesses $P^{\langle j\rangle}(W^{x\restriction j}_{x\restriction j+1})$. By Definition 4.1 (2a) (4.1 (3a)) if m is odd (even), $x \in A \cup A_{m-1,j}$ ($\bar{A} \cup A_{m-1,j}$) for every j. As $\cap_j A_{m-1,j} = A_{m-1} \subseteq A$ ($\bar{A}$) (by the definition of A in terms of the sequence A_i), $x \in A$ ($\bar{A}$). Thus U is a winning quasistrategy for I (II) in $G(A,T)$ as desired. □

5. The Game: Failures of Determinacy in L

This section is dedicated to proving Lemma 1.8. Let T_n consist of KP + Σ_n-$\mathsf{admissibility}$ + $\mathsf{V{=}L}$ + $\forall\alpha \in Ord(L_\alpha$ is not Σ_n-admissible). So L_{α_n} is the only wellfounded model of T_n. Each player will have to play the characteristic function of a complete consistent extension of T_n, or otherwise they loose. The goal now is to be able to identify the player playing the true theory of L_{α_n} using a Boolean combination of Π^0_3 formulas.

To define G we first define $2n+4$ many Π^0_3 conditions in order:

$$(R_{\mathrm{I}}0), (R_{\mathrm{II}}0), (R_{\mathrm{II}}1), (R_{\mathrm{II}}1), ..., (R_{\mathrm{II}}(n+1)), (R_{\mathrm{II}}(n+1)).$$

Then, if the first condition satisfied by the play is of the form $(R_{\mathrm{I}}k)$, I wins, and if it is of the form $(R_{\mathrm{II}}k)$, then II wins. If no condition is ever satisfied, we will let one of the two players win; for now it does not matter which.

We now define these conditions one at a time, so we can make the motivations and the pertinent definitions along the way. We will later rearrange the conditions to reduce the level of the game in the $\Pi^0_{3,n}$ hierarchy. Every time we define a new pair of conditions $(R_{\mathrm{I}}k)$ and $(R_{\mathrm{II}}k)$ we will show that they are Π^0_3, and that if any of these conditions is the first one to be satisfied, the choice of the winner is compatible with (1) and (2) of Lemma 1.8. At the end, we will show that, if one of the players is actually playing Th_{α_n}, then some condition along the way will have to be satisfied, and hence (1) and (2) of Lemma 1.8 hold.

We begin with Conditions $(R_{::}0)$.

$(R_{\mathrm{I}}0)$: II does not play a complete consistent extension of T_n
$(R_{\mathrm{II}}0)$: I does not play a complete consistent extension of T_n

Claim 5.1. Conditions $(R_{\mathrm{I}}1)$ and $(R_{\mathrm{II}}1)$ are Σ^0_1.

Proof: Saying that the set of sentences played form a complete and consistent theory is Π^0_1. Saying that the axioms of T_n are included in these sets is also Π^0_1. □

If either of these conditions are satisfied, the game ends and a player playing Th_{α_n} cannot loose at this stage. In this case it does not matter what the rest of the conditions say. So, from now on, when we describe the rest of the conditions, we will assume neither of these two conditions apply.

The term model of a complete theory T consistent with T_n is built as follows: Consider the set of formulas with one free variable $\varphi(x)$ such that $T \vdash \exists! x\varphi(x)$. Think of this formula φ as a term t_φ representing the unique element satisfying φ. So, given two such formulas φ and ψ, we say that $t_\varphi = t_\psi$ if T proves that φ and ψ are equivalent. Also, we let $t_\varphi \in t_\psi$ if T proves that the witness of φ belongs to the witness of ψ. This defines the term model of T. Since T_n proves that every nonempty class has a $<_L$-least element, every sentence $\exists x\varphi(x)$ in T has a term witnessing it. So, this term model we just built is a model of T and, furthermore, from T we can compute its full elementary diagram. Note that the term model of Th_{α_n} is isomorphic to L_{α_n} because very element of L_{α_n} is definable without parameters by Lemma 3.6.

Let $\mathcal{M}_{\rm I}$ be the term model of the theory built by I and $\mathcal{M}_{\rm II}$ the one built by II. So, from the play, we can compute the full elementary diagrams of $\mathcal{M}_{\rm I}$ and $\mathcal{M}_{\rm II}$. These models might be wellfounded or not. But recall that the only wellfounded model of T_n is L_{α_n}. All the other models of T_n are ill-founded. We now continue with the next pair of conditions.

$(R_{\rm I}1)$: $\mathcal{M}_{\rm II}$ is not an ω-model.
$(R_{\rm II}1)$: $\mathcal{M}_{\rm I}$ is not an ω-model.

Claim 5.2. Conditions $(R_{\rm I}1)$ and $(R_{\rm II}1)$ are Σ^0_2.

Proof: To say that a term model $\mathcal{M}$ is an ω-model, we have to say that for every term t, if $\mathcal{M} \models t \in \omega$, then there exists a number n such that $\mathcal{M} \models t = 1 + 1 + ... + 1$, where 1 is added n times. This is a Π^0_2 formula about $\mathcal{M}$. □

We remark that the proof of the claim above works when both players have actually played complete consistent extensions of T_n. Formally, we think of $(R_{\rm II}1)$ and $(R_{\rm II}1)$ as the Σ^0_2 formulas given in the proof of the claim. The interpretation of these Σ^0_2 formulas will match what we first called $(R_{\rm I}1)$ and $(R_{\rm II}1)$ above the claim, when conditions $(R_{\rm I}0)$ and $(R_{\rm II}0)$ are not satisfied, but that is the only case where we care. Similar remarks will apply to the conditions $(R_{::}k)$ below. That is, we supply an intuitive description of sets in our list and then a formal Π^0_3 definition that makes the intuition true in the cases of interest. As we continue to define more conditions on the winning set, keep in mind that our goal as we go along is to guarantee for each condition that if I plays the theory of L_{α_n} then he does not loose. If he does not play Th_{α_n} and II does, then II does not loose. At the end we will prove that someone wins the game.

L_{α_n} is an ω-model, so a player playing Th_{α_n}, cannot loose at this step either. From now on assume (for the intuitive descriptions of our clauses) that both $\mathcal{M}_{\rm I}$ and $\mathcal{M}_{\rm II}$ are ω-models of T_n and that at least one of them is L_{α_n}. Let us move on to the third pair of conditions.

$(R_{\rm I}2)$: $\mathbb{R}_{\mathcal{M}_{\rm I}} \subseteq \mathbb{R}_{\mathcal{M}_{\rm II}}$.
$(R_{\rm II}2)$: $\mathbb{R}_{\mathcal{M}_{\rm II}} \subseteq \mathbb{R}_{\mathcal{M}_{\rm I}}$.

If both players are playing Th_{α_n}, then $(R_{\rm I}2)$ holds, and hence I wins as needed for (1) of Lemma 1.8. So, from now on, let as assume that one player is playing Th_{α_n} and the other one is not. Next, note that L_{α_n} has no proper sub-models satisfying T_n, and hence the player playing Th_{α_n} cannot loose by either of these conditions.

Claim 5.3. Conditions $(R_{\mathrm{I}}2)$ and $(R_{\mathrm{II}}2)$ are Π^0_3.

Proof: For $(R_{\mathrm{I}}2)$, it is enough to say that every ordinal in $\mathcal{M}_{\mathrm{I}}$, is also an ordinal in $\mathcal{M}_{\mathrm{II}}$: If a linear ordering α belongs to both $On^{\mathcal{M}_{\mathrm{I}}}$ and $On^{\mathcal{M}_{\mathrm{II}}}$, it has to actually be wellfounded as we are assuming one of $\mathcal{M}_{\mathrm{I}}$ or $\mathcal{M}_{\mathrm{II}}$ is wellfounded. So, by transfinite induction, this implies that L_α also belongs to both $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. Now, note that since every ordinal α in $\mathcal{M}_{\mathrm{I}}$ is countable in $\mathcal{M}_{\mathrm{I}}$ (Lemma 3.5) there is always a real representing a linear ordering that is isomorphic to α.

To say that two ordinals α and β (even in different models) are the same, we say that there is a real x coding α in $\mathcal{M}_{\mathrm{I}}$ and a real y coding an ordinal β in $\mathcal{M}_{\mathrm{II}}$, such that $x = y$. Even if x and y belong to different models, we can say they represent the same real by saying that for every natural number n, $\mathcal{M}_{\mathrm{I}} \models n \in x \iff \mathcal{M}_{\mathrm{II}} \models n \in y$. Therefore, the Π^0_3 sentence we need is

$$\begin{aligned} \forall\alpha &\in On^{\mathcal{M}_{\mathrm{I}}}\, \exists x \in \mathbb{R}^{\mathcal{M}_{\mathrm{I}}}, y \in \mathbb{R}^{\mathcal{M}_{\mathrm{II}}}, \beta \in On^{\mathcal{M}_{\mathrm{II}}} \big[(\mathcal{M}_{\mathrm{I}} \models x \text{ codes } \alpha) \\ \wedge (\mathcal{M}_{\mathrm{II}} &\models y \text{ codes } \beta) \wedge \forall n \in \omega \big(\mathcal{M}_{\mathrm{I}} \models n \in x \iff \mathcal{M}_{\mathrm{II}} \models n \in y\big)\big]. \end{aligned} \tag{5.1}$$

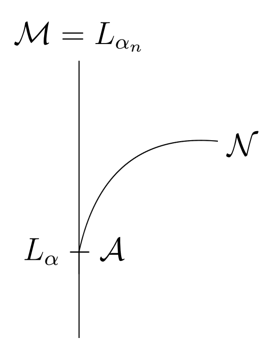
□

We remark again, that the claim should actually say: If $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are ω-models of T_n and at least one of them is wellfounded, then $(R_{\mathrm{I}}2)$ and $(R_{\mathrm{II}}2)$ are equivalent to Π^0_3 properties. In any other case, we think of $(R_{\mathrm{I}}2)$ and $(R_{\mathrm{II}}2)$ as given formally by these Π^0_3 properties, which, in cases not of interest, might not have the intended meaning.

To this point, our proof follows the ideas in Martin [1974a] and [n.d., Ch. 1] where he shows that Σ^0_4-determinacy is not true in the least β-model of second order arithmetic $\mathbb{R}\cap L_{\beta_0}$. These, in turn, are based on those of H. Friedman's [1971] proof that Σ^0_5 determinacy fails in $\mathbb{R} \cap L_{\beta_0}$. The ideas in the rest of the proof are new.

From now on, we assume that no condition listed so far holds and so, in particular, not both of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ are well founded. Again, we are interested in the plays where one is well founded and so actually L_{α_n}.

Definition 5.4. We use $\mathcal{M}$ to denote the wellfounded model of the pair $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{N}$ for the ill-founded one. The player playing $\mathcal{M} \cong L_{\alpha_n}$ should win the game but we do not know which one it is. There is a least ordinal $\alpha \in On^{\mathcal{M}}$ that is not in $\mathcal{N}$. Let $\mathcal{A}$ be the well founded part of $\mathcal{N}$. So, we have that $L_\alpha \cong \mathcal{A}$. Let us call $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$ the initial segments of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ that correspond to L_α and $\mathcal{A}$ in $\mathcal{M}$ and $\mathcal{N}$.



We abuse notation slightly by using $\mathcal{A}$ to also denote the isomorphism between $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$. That is, $\mathcal{A}$ is the set of pairs of terms $(z,w) \in \mathcal{M}_{\mathrm{I}} \times \mathcal{M}_{\mathrm{II}}$ representing elements $z \in \mathcal{A}_{\mathrm{I}}$ and $w \in \mathcal{A}_{\mathrm{II}}$ which are the same set under the isomorphism $\mathcal{A}_{\mathrm{I}} \cong \mathcal{A}_{\mathrm{II}}$.

Claim 5.5. $\mathcal{A}$ is a Σ^0_2 set. Also, $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$ are Σ^0_2 sets.

Proof: For $\mathcal{A}$ we have the following Σ^0_2 definition.

$$(z,w) \in \mathcal{A} \iff \exists x \in \mathbb{R}^{\mathcal{M}_{\mathrm{I}}}, y \in \mathbb{R}^{\mathcal{M}_{\mathrm{II}}} \big(\mathcal{M}_{\mathrm{I}} \vDash x \text{ codes } z \quad \wedge \quad \mathcal{M}_{\mathrm{II}} \vDash y \text{ codes } w \quad \wedge \\ \forall n \in \omega \big(\mathcal{M}_{\mathrm{I}} \vDash n \in x \iff \mathcal{M}_{\mathrm{II}} \vDash n \in y\big)\big).$$

For $\mathcal{A}_{\mathrm{I}}$ we have that $z \in \mathcal{A}_{\mathrm{I}} \iff \exists w\ ((z,w) \in \mathcal{A})$, and analogously for $\mathcal{A}_{\mathrm{II}}$. □

(At this point, Martin's proof ([1974a] and [n.d., Ch. 1]) that Σ^0_4-DET fails in L_{β_0} has a Σ^0_4 winning condition that says that there exists a $\alpha \in \mathcal{M}_{\mathrm{I}}$ such that $L^{\mathcal{M}_{\mathrm{I}}}_{\alpha} = \mathcal{A}_{\mathrm{I}}$.)

The next $n-1$ pairs of conditions all have basically the same form. The conditions $(R_{::}3)$ are a bit simpler than the rest and we describe them first. We will then generalize them and define the conditions $(R_{::}(2+k))$ for $k > 2$ all at once.

We start by defining the sets of ordinals in one model that have witnesses for Σ_1 formulas with parameters in $\mathcal{A}$ that have no witnesses in the other model. The expectation is that the witnesses are in the nonwellfounded model $\mathcal{N}$. We will show later that either there are many such witnesses and the set of witnesses shows that the $On^{\mathcal{N}} \setminus \mathcal{A}$ has no least element; or there are few such witnesses and there is an ordinal $\beta \in \mathcal{N}$ such that $\mathcal{A} \preccurlyeq_1 L^{\mathcal{N}}_{\beta}$. In the former case, we have identified the nonwellfounded model and we can end the game. In the latter case, we continue with the next pair of conditions, now assuming the existence of such a β.

Let

$$C_{\mathcal{M}_{\mathrm{II}},1} = \{\beta \in On^{\mathcal{M}_{\mathrm{II}}} : \exists (x_1,x_2) \in \mathcal{A},\ \varphi \in \Delta_0, \\ \big(\big(\exists z \in L^{\mathcal{M}_{\mathrm{II}}}_{\beta}\ \mathcal{M}_{\mathrm{II}} \vDash \varphi(z,x_2)\big) \wedge \big(\mathcal{M}_{\mathrm{I}} \vDash \neg\exists y \varphi(y,x_1)\big)\big)\}.$$

Note that this condition says more than that $\mathcal{A}$ is not a Σ_1 substructure of $L^{\mathcal{M}_{\mathrm{II}}}_{\beta}$ as it asserts that there are no witnesses for the formula φ even in $\mathcal{M}_{\mathrm{I}}$. Thus its negation does not imply that $\mathcal{A}$ is a Σ_1 substructure of $L^{\mathcal{M}_{\mathrm{II}}}_{\beta}$. $C_{\mathcal{M}_{\mathrm{I}},1}$ is defined analogously. These sets are Σ^0_2 because, as we noted before, $\mathcal{A}$ is Σ^0_2 and the second line of the definition is computable form the elementary diagrams of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. We are now ready to define the next pair of conditions.

$(R_{\mathrm{I}}3)$: $C_{\mathcal{M}_{\mathrm{II}},1}$ has no least element.
$(R_{\mathrm{II}}3)$: $C_{\mathcal{M}_{\mathrm{I}},1}$ has no least element.

Clearly a player playing a wellfounded model cannot loose because of these conditions. Also, these conditions are Π^0_3 as they say, for example, that $\forall\beta(\beta \notin C_{\mathcal{M}_{\mathrm{II}},k} \vee \exists\gamma < \beta(\gamma \in C_{\mathcal{M}_{\mathrm{II}},k}))$ and $C_{\mathcal{M}_{\mathrm{II}},k}$ is Σ^0_2.

Suppose now that neither of these conditions is satisfied.

Claim 5.6. There is a $\beta \in On^{\mathcal{N}} \setminus \mathcal{A}$ such that $\mathcal{A} \preccurlyeq_1 L^{\mathcal{N}}_{\beta}$.

Proof: Suppose not, that is, for every $\gamma \in On^{\mathcal{N}} \setminus \mathcal{A}$, there is a Σ_1 formula with parameters in $\mathcal{A}$, that is true in L_γ but not in $\mathcal{A}$. Since conditions $(R_{\mathrm{I}}3)$ and $(R_{\mathrm{II}}3)$ are not satisfied, $C_{\mathcal{N},1}$ has a least element δ. Notice that δ cannot be in $\mathcal{A}$ because if the witness z for a Δ_0

formula $\varphi(z, x_2)$ is in $L_{\mathcal{N}_\delta} \subseteq \mathcal{A}$, then $\exists y \varphi(y, x_1)$ also holds in $\mathcal{M}$. Let $\delta > \gamma_0 > \gamma_1 > \gamma_2 > ...$ be a descending sequence in $On^{\mathcal{N}} \setminus \mathcal{A}$, converging down to the cut $(On^A, On^N \setminus \mathcal{A})$. For each i, there is a Δ_0 formula φ_i with parameters from $\mathcal{A}$ and a $<_L$-least witness $z_i \in L_{\gamma_i}$ such that $\mathcal{N} \vDash \varphi_i(z_i)$ but $\mathcal{A} \vDash \neg\exists y \varphi_i(y)$. By thinning out the sequence if necessary, we may assume that $z_i \in L_{\gamma_i}$ but $z_i \notin L_{\gamma_{i+1}}$. So $\{z_i : i \in \omega\}$ is an $<_L^{\mathcal{N}}$-descending sequence. Since $\gamma_i \notin C_{\mathcal{N},1}$, $\mathcal{M} \vDash \exists y \varphi_i(y)$. Let y_i be the $<_L^{\mathcal{M}}$-least such witness. Since $\mathcal{M}$ is wellfounded, the sequence $\{y_i : i \in \omega\}$ cannot be a $<_L^M$-descending sequence. So, there exists $i < j$ such that $z_j <_L^{\mathcal{N}} z_i$ but $y_i <_L^{\mathcal{M}} y_j$. Therefore, L_{γ_j} is a witness in L_{γ_j+1} for x showing the truth in $\mathcal{N}$ of the Δ_0 formula

$$\psi(x) \equiv \exists z \in x \varphi_j(z) \wedge \forall z \in x \neg\varphi_i(z)$$

but there is no witness for $\psi(x)$ in $\mathcal{M}$. This shows that $\gamma_j + 1$ is in $C_{\mathcal{N},1}$, contradicting our choice of δ as the least element of $C_{\mathcal{N},1}$. □

We now turn to defining the rest of the conditions. Again, the idea is to define conditions such that they hold only if they have identified the nonwellfounded model, and if they do not hold, they give us information about the models that we can assume when we define the following conditions. We will show that if for some k, none of the conditions $(R_{::}1), ..., (R_{::}(2+k))$ is satisfied, then α is $(k+1)$-admissible. Since L_{α_n} has no n-admissible ordinals, we will be able to argue that some condition $(R_{::}(2+k))$ will have to be satisfied for some $k < n$. This will be proved, of course, by induction on k, but the induction hypothesis needs to be stronger. We show that if no condition $(R_{::}1), ..., (R_{::}(2+k))$ holds then there exists β_1 and β_2 such that

$$(\bigstar_k)(\beta_1, \beta_2): \quad \beta_1 \in On^{\mathcal{M}_{\mathrm{I}}} \setminus \mathcal{A}_{\mathrm{I}} \wedge \mathcal{M}_{\mathrm{I}} \vDash \beta_1 \text{ is } (k-1)\text{-admissible} \wedge$$
$$\beta_2 \in On^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{A}_{\mathrm{II}} \wedge \mathcal{M}_{\mathrm{II}} \vDash \beta_2 \text{ is } (k-1)\text{-admissible} \wedge$$
$$L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}.$$

Here $\equiv_{k,\mathcal{A}}$ means Σ_k-elementary equivalence with parameters from $\mathcal{A}$. In other words, $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$ means that for every Σ_k formula $\varphi(z)$, and every $x_1 \in \mathcal{A}_{\mathrm{I}}$ and $x_2 \in \mathcal{A}_{\mathrm{II}}$, if x_1 and x_2 represent the same set (i.e. $(x_1, x_2) \in \mathcal{A}$), then $\mathcal{M}_{\mathrm{I}} \vDash \varphi(x_1) \iff \mathcal{M}_{\mathrm{II}} \vDash \varphi(x_2)$.

By Claim 5.6, we already know there are ordinals satisfying $(\bigstar_1)$: Just let $\beta_1 = \alpha \in \mathcal{M}$ and let $\beta_2 \in \mathcal{N}$ be such that $\mathcal{A} \preccurlyeq_1 L_{\beta_2}^{\mathcal{N}}$.

Property $(\bigstar_k)$ is useful because, on the one hand, it is simple to define (it is Π_2^0 as we prove below) and, on the other hand, it has strong consequences as we will show in Lemma 5.10. For instance, we will show that the existence of β_1 and β_2 satisfying it implies that α is $(k+1)$-admissible.

Lemma 5.7. $(\bigstar_k)$ *is a* Π_2^0 *property.*

Proof: We already showed that $\beta_1 \notin \mathcal{A}_{\mathrm{I}} \wedge \beta_2 \notin \mathcal{A}_{\mathrm{II}}$ is a Π_2^0 relation. Whether β_i is $(k-1)$-admissible is just one sentence in the theory of the structure. To check that $L_{\beta_1}^{\mathcal{M}_{\mathrm{I}}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\mathrm{II}}}$, we need to say that, for every Σ_k formula $\varphi(x)$, and for every pair of parameters $(x_1, x_2) \in \mathcal{A}$, $\mathcal{M}_{\mathrm{I}} \vDash \varphi(x_1) \iff \mathcal{M}_{\mathrm{II}} \vDash \varphi(x_2)$. Recall that $\mathcal{A}$ is Σ_2^0 (Claim 5.5). Counting quantifiers we see that $(\bigstar_k)$ is Π_2^0. □

Definition 5.8. S_k is the set of formulas that are Boolean combinations of formulas of the form $(\forall x \in z)\ \psi(z, \bar{y})$, where ψ is Σ_k.

Lemma 5.9. *If $L_{\beta_1}^{\mathcal{M}_{\rm I}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\rm II}}$, then $L_{\beta_1}^{\mathcal{M}_{\rm I}}$ and $L_{\beta_2}^{\mathcal{M}_{\rm II}}$ satisfy the same S_k-sentences with parameters from $\mathcal{A}$ substituted for the free variables z and $\bar{y}$.*

Proof: Let φ be a sentence of the form $\forall x \in z\, \psi(x, z, \bar{y})$ with parameters from $\mathcal{A}$ where ψ is Σ_k. Since φ has no free variables, z and $\bar{y}$ are parameters from $\mathcal{A}$, and so every $x \in z$ is also in $\mathcal{A}$. Since $L_{\beta_1}^{\mathcal{M}_{\rm I}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\rm II}}$, we have that for every $x \in z$, $L_{\beta_1}^{\mathcal{M}_{\rm I}} \vDash \psi(x, z, \bar{y}) \Leftrightarrow L_{\beta_2}^{\mathcal{M}_{\rm II}} \vDash \psi(x, z, \bar{y})$. So, of course, $L_{\beta_1}^{\mathcal{M}_{\rm I}} \vDash \varphi \Leftrightarrow L_{\beta_2}^{\mathcal{M}_{\rm II}} \vDash \varphi$. Finally, if ϕ is a Boolean combination of sentences of this form, then we also have $L_{\beta_1}^{\mathcal{M}_{\rm I}} \vDash \phi \Leftrightarrow L_{\beta_2}^{\mathcal{M}_{\rm II}} \vDash \phi$. □

Now we define sets that generalizes $C_{\mathcal{M}_{\rm I},1}$ and $C_{\mathcal{M}_{\rm II},1}$. This time, the sets consist of the ordinals in one model ($\mathcal{N}$) that have witnesses for S_k formulas with parameters in $\mathcal{A}$ that have no witnesses in the other model ($\mathcal{M}$). We will show later that: either there are lots of such witnesses and the set $C_{\mathcal{N},k}^{\beta_1,\beta_2}$ shows that $On^{\mathcal{N}} \setminus \mathcal{A}$ has no least element, in which case we have identified the nonwellfounded model, or there are few such witnesses and we can find ordinals satisfying ($\bigstar_k$).

For $\beta_1 \in \mathcal{M}_{\rm I}$ and $\beta_2 \in \mathcal{M}_{\rm II}$, let

$$C_{\mathcal{M}_{\rm II},k}^{\beta_1,\beta_2} = \{\beta \in \beta_2 : \exists (x_1, x_2) \in \mathcal{A},\ \exists \varphi \in S_{k-1} \\ \big(\exists z \in L_{\beta}^{\mathcal{M}_{\rm II}}\ L_{\beta_2}^{\mathcal{M}_{\rm II}} \vDash \varphi(z, x_2)\big) \wedge \big(L_{\beta_1}^{\mathcal{M}_{\rm II}} \vDash \neg \exists y \varphi(y, x_1)\big)\}.$$

and define $C_{\mathcal{M}_{\rm I},k}^{\beta_1,\beta_2}$ analogously. As in the case $k = 1$, these sets are Σ_2^0.

($R_{\rm I}(2+k)$): For every β_1, β_2, if $(\bigstar_{k-1})(\beta_1, \beta_2)$, then $C_{\mathcal{M}_{\rm II},k}^{\beta_1,\beta_2}$ has no least element.
($R_{\rm II}(2+k)$): For every β_1, β_2, if $(\bigstar_{k-1})(\beta_1, \beta_2)$, then $C_{\mathcal{M}_{\rm I},k}^{\beta_1,\beta_2}$ has no least element.

The conditions ($R_{\rm II}3$) and ($R_{\rm II}3$) are Π_3^0: We already showed that ($\bigstar_{k-1}$) is a Π_2^0 property, and, as for $k = 1$, $C_{\mathcal{M}_{\rm II},k}^{\beta_1,\beta_2}$ having no least element is Π_3^0 because $C_{\mathcal{M}_{\rm II},k}^{\beta_1,\beta_2}$ is Σ_2^0. Also observe that if there are β_1, β_2 satisfying ($\bigstar_{k-1}$), then the player playing Th_{α_n} cannot loose by a condition ($R_{::}(2+k)$).

Lemma 5.10. *Suppose that β_1, β_2 satisfy ($\bigstar_k$). For the sake of definiteness, suppose that $\beta_1 \in \mathcal{M}$ and $\beta_2 \in \mathcal{N}$. Then*

1. *$L_\alpha \preccurlyeq_k L_{\beta_1}$ and $\mathcal{A} \preccurlyeq_k L_{\beta_2}$.*
2. *α is $(k+1)$-admissible.*
3. *There exists a descending sequence of $\mathcal{N}$ ordinals γ converging down to $On^{\mathcal{A}}$ such that $L_\gamma \preccurlyeq_k L_{\beta_2}$.*

Proof: First, we note that $\{\alpha\}$ is not Σ_k definable in L_{β_1} with parameters from L_α. Suppose otherwise. Since $\alpha \in \mathcal{M} \vDash T$, α is not Σ_n-admissible and every $\beta \in \mathcal{M}$ is countable in $\mathcal{M}$. Thus there is a Σ_n/L_α map from ω onto α (possibly with parameters). This defines in L_α, a Σ_n ordering of ω of order type α. This ordering cannot belong to $\mathcal{N}$ as it would then define its wellfounded part. In L_{β_1} we can define this ordering with a Σ_k formula using the Σ_k definition of α and quantification over L_α. Since $L_{\beta_1}^{\mathcal{M}_{\rm I}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\rm II}}$, this ordering is definable in L_{β_2} by the same formula, and hence belongs $\mathcal{N}$. Contradiction.

Now for part (1), since β_1 is (k-1)-admissible, L_{β_1} has a Σ_k-Skolem function without parameters (Lemma 3.1). Let H be the Σ_k-Skolem hull of L_α in L_{β_1}. We show that $H = L_\alpha$. Note that this implies that $L_\alpha \preccurlyeq_k L_{\beta_1}$ (as desired) and also $\mathcal{A} \preccurlyeq_k L_{\beta_2}$ since $L_{\beta_1}^{\mathcal{M}_{\rm I}} \equiv_{k,\mathcal{A}} L_{\beta_2}^{\mathcal{M}_{\rm II}}$. Suppose that $H \supsetneq L_\alpha$. Let L_γ be the Mostowski collapse of H, so $\alpha < \gamma \leq \beta_1$. Let α' be the ordinal in H that is sent to $\alpha \in L_\gamma$ by the collapse. As α' is in the image of the Σ_k Skolem function, $\{\alpha'\}$ is Σ_k definable in H with parameters from L_α . The same formula then gives a definition of $\{\alpha\}$ in L_γ as the collapse is the identity on members of L_α. Thus $\{\alpha\}$ is Σ_k definable in L_γ with parameters from L_α but clearly

$$L_\gamma^{\mathcal{M}} \equiv_{k,\mathcal{A}} H \equiv_{k,\mathcal{A}} L_{\beta_1}^{\mathcal{M}}.$$

So $\{\alpha\}$ would be Σ_k definable in $L_{\beta_1}^{\mathcal{M}}$ contradicting our first observation.

(2) Suppose α is not $(k+1)$-admissible and so (since every ordinal is countable in $\mathcal{M}$) there is a Π_k/L_α function f from ω onto α. Since $\mathcal{A} \preccurlyeq_k L_{\beta_2}$, the same formula defining f over L_α defines a function from ω onto α in L_{β_2}. But then, in $\mathcal{N}$ we could define its wellfounded part, L_α, for a contradiction.

Now since α is $(k+1)$-admissible, there are unboundedly many $\gamma < \alpha$ such that $L_\gamma \preccurlyeq_k L_\alpha$ (Lemma 3.3). For all these γ we also have $L_\gamma^{\mathcal{N}} \preccurlyeq_k L_{\beta_2}^{\mathcal{N}}$. The set of $\gamma < \beta_2$ such that $L_\gamma^{\mathcal{N}} \preccurlyeq_k L_{\beta_2}^{\mathcal{N}}$ is definable in $\mathcal{N}$. If for some $\delta \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$, the set of $\gamma < \delta$ such that $L_\gamma^{\mathcal{N}} \preccurlyeq_k L_{\beta_2}^{\mathcal{N}}$ had supremum α, then α would be definable in $\mathcal{N}$ and we know it is not. So for every $\delta \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$, there exists $\gamma < \delta$, $\gamma \in On^{\mathcal{N}} \smallsetminus \mathcal{A}$ such that $L_\gamma^{\mathcal{N}} \preccurlyeq_k L_{\beta_2}^{\mathcal{N}}$. □

Lemma 5.11. *If there is a play of our game such that for all $i \leq 2+k$, the resulting real does not satisfy any condition $(R_{\mathrm{I}}i)$ or $(R_{\mathrm{II}}i)$, then there are β_1 and β_2 satisfying $(\bigstar_k)$.*

Proof: We already showed that there are β_1 and β_2 satisfying $(\bigstar_1)$. The proof now proceeds by induction, so, without loss of generality, we assume that there exists β_1, β_2 satisfying $(\bigstar_{k-1})$. We fix such β_1, β_2.

We claim that no $\delta \in \mathcal{A}$ is in $C_{\mathcal{N},k}^{\beta_1,\beta_2}$. Consider $\delta \in \mathcal{A}$ and any S_{k-1} formula $\forall x \in z\varphi(z, \bar{y})$ and $z_2, \bar{y}_2 \in L_\delta^{\mathcal{N}} \subseteq \mathcal{A}$ such that $L_{\beta_2}^{\mathcal{N}} \vDash \forall x \in z_2\varphi(z_2, \bar{y}_2)$. Since L_{β_1} and L_{β_2} satisfy the same Σ_{k-1} formulas with parameters from $\mathcal{A}$ (by $(\bigstar_{k-1})$), $L_{\beta_1}^M \vDash \forall x \in z_1\varphi(z_1, \bar{y}_1)$, where z_1 and $\bar{y}_1$ are the images of $z_2, \bar{y}_2$ in $\mathcal{M}$. Thus $\delta \notin C_{\mathcal{N},k}^{\beta_1,\beta_2}$ by its definition.

Since the conditions $(R_{\mathrm{I}}(2+k))$ and $(R_{\mathrm{II}}(2+k))$ are not satisfied, $C_{\mathcal{N},k}^{\beta_1,\beta_2}$ has a least element δ, necessarily not in $\mathcal{A}$. By clause (3) of the previous lemma, there is a descending sequence $\delta > \gamma_0 > \gamma_1 > \gamma_2 > ...$ in $On^{\mathcal{N}}$, converging down to $\alpha = On^{\mathcal{A}}$, such that for each i $L_{\gamma_i}^{\mathcal{N}} \preccurlyeq_{k-1} L_{\beta_2}^{\mathcal{N}}$. We now argue much as for Claim 5.6. We claim that, for some i, $L_\alpha \preccurlyeq_k L_{\gamma_i}^{\mathcal{N}}$. If not, there is, for each i, a Π_{k-1} formula $\varphi_i(z)$ with parameters from $\mathcal{A}$ and a $<_L^{\mathcal{N}}$-least $z_i \in L_{\gamma_i}$ such that $L_{\gamma_i} \vDash \varphi_i(z_i)$ and $z_i \notin \mathcal{A}$. As $L_{\gamma_i}^{\mathcal{N}} \preccurlyeq_{k-1} L_{\beta_2}^{\mathcal{N}}$, $L_{\beta_2}^{\mathcal{N}} \vDash \varphi_i(z_i)$. By thinning out the sequence if necessary, we may assume that $z_i \in L_{\gamma_i}^{\mathcal{N}}$ but $z_i \notin L_{\gamma_{i+1}}^{\mathcal{N}}$. So $\{z_i : i \in \omega\}$ is an $<_L^{\mathcal{N}}$-descending sequence. Since $\gamma_i \notin C_{\mathcal{N},k}^{\beta_1,\beta_2}$, $L_{\beta_1}^{\mathcal{M}} \vDash \exists y \varphi_i(y)$. Let y_i be the $<_L^{\mathcal{M}}$-least such witness. Since $\mathcal{M}$ is wellfounded, the sequence $\{y_i : i \in \omega\}$ cannot be a $<_L^{\mathcal{M}}$-descending sequence. So, there exist $i < j$ such that $z_j <_L^{\mathcal{N}} z_i$ but $y_i <_L^{\mathcal{M}} y_j$. Therefore, $L_{\gamma_j}^{\mathcal{N}} \in L_{\gamma_j+1}^{\mathcal{N}}$is a witness for x showing the truth in $L_{\beta_2}^{\mathcal{N}}$of the S_{k-1} formula

$$\psi(x) \equiv \exists z \in x\varphi_j(z, x) \wedge \forall z \in x \neg\varphi_i(z, x)$$

while there is no such witness for the corresponding formula in $L_{\beta_1}^{\mathcal{M}}$. This contradicts our choice of δ as the least element of $C_{\mathcal{N},k}^{\beta_1,\beta_2}$. Thus we have an i such that $L_\alpha \preccurlyeq_k L_{\gamma_i}^{\mathcal{N}} \preccurlyeq_{k-1} L_{\beta_2}^{\mathcal{N}}$.

Now $L_{\gamma_i}^{\mathcal{N}}$ is Σ_{k-1} admissible by Lemma 3.4 as $L_{\beta_2}^{\mathcal{N}}$ is Σ_{k-2} admissible by $(\bigstar_{k-1})$ while α is (even) Σ_k admissible by our induction hypothesis and the previous Lemma and so $(\bigstar_k)(\alpha, \gamma_i)$ as required. □

Lemma 5.12. *The game G satisfies (1) and (2) of Lemma 1.8.*

PROOF: Since $\alpha \in \mathcal{M} \vDash Th_{\alpha_n}$, α cannot be n-admissible. We now know, by Lemma 5.10, that there are no β_1, β_2 satisfying $(\bigstar_{n-1})$. So, by Lemma 5.11, there is a $k < n$ such that either $(R_{\mathrm{I}}(2+k))$ or $(R_{\mathrm{II}}(2+k))$ holds. Suppose $(R_{\mathrm{I}}(2+k))$ is the first condition that holds and that I wins the game. Since $\forall i < 2+k$, no condition $(R_{::}i)$ holds, there are β_1, β_2 satisfying $(\bigstar_{k-1})$. So, $(R_{\mathrm{I}}(2+k))$ implies that $\mathcal{M}_{\mathrm{II}}$ is not wellfounded. An analogous argument works if $(R_{\mathrm{II}}(2+k))$ is the first condition that holds and II wins the game. □

To complete the proof of Lemma 1.8 and so of our main result, Theorem 1.2, we need to show that we can get the same results with a $\Pi^0_{3,n-1}$ game. As now defined, the game is clearly $\Pi^0_{3,2n+4}$ as there are six sets at the beginning and then $2(n-1)$ sets of the form $R_{\mathrm{I}}(2+k)$ or $R_{\mathrm{II}}(2+k)$ for $k < n$. So we need to modify the game a bit. The conditions $(R_{\mathrm{I}}0)$, $(R_{\mathrm{II}}0)$, $(R_{\mathrm{I}}1)$ and $(R_{\mathrm{II}}1)$ are Σ^0_2 and so can be absorbed into some later Π^0_3 condition as indicated below. Thus, we first prove that the rest of the game is $\Pi^0_{3,n-1}$. To decrease the number of alternations, we will change the order of the conditions. We need to make a couple observations: First, if no condition $(R_{::}i)$ for $i < 2+k$ holds, then only the player playing a nonwellfounded model can loose because of a condition $(R_{::}(2+k))$. So it does not matter which of $(R_{\mathrm{I}}(2+k))$ and $(R_{\mathrm{II}}(2+k))$ comes first in the list and we can re-order this part of the list as:

$$\begin{array}{cccccccc} ..., & (R_{\mathrm{I}}3), & & & (R_{\mathrm{I}}4), & (R_{\mathrm{I}}5), & & & (R_{\mathrm{I}}6), & ... \\ & & (R_{\mathrm{II}}3), & (R_{\mathrm{II}}4), & & & (R_{\mathrm{II}}5), & (R_{\mathrm{II}}6), & & \end{array}$$

Second, independently of the outcomes of $(R_{::}2)$, only the player playing a nonwellfounded model can loose with a condition $(R_{::}3)$, so it does not matter in which order we place these conditions either. We can now optimize the number of alternations of Π^0_3 conditions as follows:

$$\begin{array}{ccccccccccc} ..., & (R_{\mathrm{I}}2), & (R_{\mathrm{I}}3), & & & & (R_{\mathrm{I}}4), & (R_{\mathrm{I}}5), & & & (R_{\mathrm{I}}6), & ... \\ & & & (R_{\mathrm{II}}2), & (R_{\mathrm{II}}3), & (R_{\mathrm{II}}4), & & & (R_{\mathrm{II}}5), & (R_{\mathrm{II}}6), & & \end{array}$$

The first Π^0_3 condition is now $(R_{\mathrm{I}}2) \vee (R_{\mathrm{I}}3)$ which makes I win. We add in the early conditions that are at most Σ^0_2 to make our desired first condition $(R_{\mathrm{I}}0) \vee (R_{\mathrm{I}}1) \vee (R_{\mathrm{I}}2) \vee (R_{\mathrm{I}}3)$ which makes I win. The desired second condition is $(R_{\mathrm{II}}0) \vee (R_{\mathrm{II}}1) \vee (R_{\mathrm{II}}2) \vee (R_{\mathrm{II}}3) \vee (R_{\mathrm{II}}4)$ which makes II win. Then, for $i > 1$, the $(2i\text{-}1)$th condition is $(R_{\mathrm{I}}2i) \vee (R_{\mathrm{I}}(2i+1))$ which makes I win, and the $2i$th condition is $(R_{\mathrm{II}}(2i+1)) \vee (R_{\mathrm{II}}(2i+2))$ which makes II win. We end the list with $(R_{\mathrm{I}}(n+1)$ if n is even and with $(R_{\mathrm{II}}(n+1))$ if n is odd. This gives us n many conditions. We finish the description of the game by adding on the the full space 2^ω as our nth and last set. The idea here is that this prescribes a win for I when n is even (and so II failed to win at the end by providing a descending chain in $\mathcal{M}_{\mathrm{I}}$) and a win for II when n is odd (and so I failed to win at the end by providing a descending chain in $\mathcal{M}_{\mathrm{II}}$). This provision includes the conditions for a win required at the last step of our arguments and then enlarges the winning set (for one player) but remember that once we have passed through all the conditions on our list we no longer care how the game is defined. Thus, we have a $\Pi^0_{3,n-1}$ subset of 2^ω that defines our game as desired. This completes the proof of Lemma 1.8 and so of Theorem 1.2. □

6. No reversals from Σ^1_4 sentences

In this section we will show that a true (or even consistent) Σ^1_4 sentence T cannot imply Δ^1_2-CA_0 over Π^1_1-CA_0 even for β-models. At higher levels ($n \geq 2$), even with the help of Π^1_n-CA_0 or Δ^1_n-CA_0 such a T does not imply Δ^1_{n+1}-CA_0 or Π^1_n-CA_0, respectively, even for β-models. We also supply analogs for Σ^1_m sentences that are theorems of ZFC (or at least consistent with ZFC + $\mathsf{V{=}L(X)}$ for some $X \subseteq \omega$).

Our models for the failure of such implications will all be β-models with second order part of the form $\mathbb{R} \cap \mathbb{L}_\delta$. As before, we use the equivalences given by Simpson [2009, VII.5.3] between such models satisfying Π^1_{n+1}-CA_0 or Δ^1_{n+1}-CA_0 and L_δ being Σ_n nonprojectable or Σ_n admissible, respectively for n≥ 1 or $n \geq 2$, respectively. The equivalence between Σ_1 admissibility and Δ^1_2-CA_0 works when L_δ is a limit of admissibles. In fact, as long as δ is a limit of admissibles (i.e. closed under the operation of going to the next admissible) then, in L_δ, any Σ^1_{n+1} sentence is equivalent to one that is Σ_n for $n \geq 1$. (This is essentially Simpson [2009, VII.3.24].) Also note that a Σ^1_n sentence is true in L_δ if and only if it is true in (the β-model of second order arithmetic with sets in) $\mathbb{R} \cap \mathbb{L}_\delta$.

Theorem 6.1. *If T is a true Π^1_3 sentence, then there is an ordinal δ such that $L_\delta \models T$ & $\forall\gamma\exists\beta > \gamma(\beta$ is admissible$)$ but L_δ is not Σ_1 admissible and so $\mathbb{R}\cap L_\delta \nvDash \Delta^1_2$-$\mathsf{CA}_0$.*

Proof: First note that by Shoenfield absoluteness (for Σ^1_2 formulas between V and L), any true Π^1_3 sentence is true in L. Suppose T is $\forall A\exists B\forall C\Phi(A, B, C)$ where Φ is arithmetic. Define functions $f_T : \mathbb{R}^L \to ON$ and $f_a : ON \to ON$ by $f_T(A) = \mu\alpha(\exists B \in L_\alpha)\forall C\Phi(A, B, C)$ and $f_a(\alpha) = \alpha^+$, the least admissible greater than α. For any ordinal α closed under f_a, $\mathbb{R}\cap L_\alpha$ is a β-model of Π^1_1-CA Simpson [2009, VII.1.8]. If α is also closed under f_T then $L_\alpha \models T$ as well by the definition of a β-model (it is absolute for Π^1_1 formulas). Thus there are α such that $L_\alpha \models T$ & $\forall\gamma\exists\beta > \gamma(\beta$ is admissible$)$. Let δ_n be the n^{th} such ordinal and let $\delta = \cup\delta_n$. It is clear that $L_\delta \models T$ & $\forall\gamma\exists\beta > \gamma(\beta$ is admissible$)$. The sequence δ_n is clearly Σ_1 over L_δ and so L_δ is not Σ_1 admissible as required to have $\mathbb{R} \cap \mathbb{L}_\delta \nvDash \Delta^1_2$-$\mathsf{CA}_0$. □

Simpson pointed out that the above proof relativizes, of course, to any real X and so applies to Σ^1_4 sentences by working in $L(X)$ where X is the witness for the Σ^1_4 sentence.

Corollary 6.2. *If T is a true Σ^1_4 sentence (e.g. a theorem of ZFC) then T + Π^1_1-CA + Π^1_∞-TI $\nvdash$ Δ^1_2-CA_0.*

Proof: Let δ be defined in $L(X)$ as it was on L in the Theorem but relativized to X where X is the witness to the first quantifier in T. As $L_\delta(X)$ is closed under the next admissible, $\mathbb{R}\cap L_\delta(X)$ is a β-model of Π^1_1-CA_0 + Π^1_∞-TI_0 as well as T. □

Corollary 6.3. Borel-*DET* + Π^1_1-*CA* + Π^1_∞-*TI* $\nvdash$ Δ^1_2-CA_0.

Proof: **Borel**-DET is equivalent to a Π^1_3 sentence over Π^1_1-CA_0. The sentence says that for every Borel code A there is a strategy f such that, for every C, the result of applying f to C is in the set coded by A. (The only thing to point out is that being a Borel code is Π^1_1 and, assuming Π^1_1-CA_0 (which actually follows in RCA_0 from $(\Sigma^0_1 \wedge \Pi^0_1)$-DET), membership in the set coded is Δ^1_1 by Simpson [2009, V.3.3-4]). □

Indeed, if a Σ^1_4 sentence T is even consistent with ZFC then it is true in some model M of ZFC and so also in $L^M(X)$ for some $X \in M$ within which we can carry out the proof of

Theorem 6.1. Thus all the proof theoretic nonimplications of Corollary 6.2 apply to such T as well.

An often used measuring rod for determinacy assumptions as in Martin [n.d.] and Welch [2009] is associating a class Γ with the least ordinal δ such that $L_\delta \models \Gamma$-DET. The proof of Theorem 6.1 shows that, by the standards of admissibility, such ordinals are not themselves large. The following Proposition captures this version of our results. It relativizes to Σ^1_4 sentences true in L (e.g. theorems of ZFC) although not literally to all true ones.

Proposition 6.4. *If T is a true Π^1_3 sentence which implies Π^1_1-CA_0 such as Γ-DET for any of the standard classes of the Borel hierarchy from $\Sigma^0_1 \wedge \Pi^0_1$ up to the class of all Borel sets, then, the least δ such that $L \models \Gamma$-DET, while a limit of admissibles, is not itself Σ_1 admissible. Indeed, the Δ_1 projectum of δ is ω.*

Proof: Let T be $\forall x \subseteq \omega \exists y \subseteq \omega \forall z \subseteq \omega[\phi(x, y, z)]$. First note that, in L_δ, every β is countable. If not, then there is a least $\beta < \delta$ which is not countable in L_δ. In particular no new subsets of ω appear between L_β and L_δ. Thus any Π^1_∞ sentence true in L_δ is true in L_β for a contradiction to the leastness of δ. Thus every L_β for $\beta < \delta$ is coded as a subset of ω in L_δ and so the closure of $\mathbb{R} \cap \mathbb{L}_\delta$ under Π^1_1 comprehension implies that L_δ is closed under the next admissible and so is a limit of admissibles.

Next, consider the function f which takes a subset x of ω in L_δ to the least ordinal γ such that there is an $\alpha < \gamma$ with $\alpha^+ < \gamma$ as well and a subset y of ω in L_α such that $L_{\alpha^+} \models \forall z \subseteq \omega[\phi(x, y, z)]$ (or equivalently $\forall z \subseteq \omega[\phi(x, y, z)]$). As $L_\delta \models T$ and is closed under the next admissible, this function is total on L_δ and obviously Δ_0. We can then define the total Δ_0 map h taking ν to the least γ such that $f(x) \in L_\gamma$ for every $x \subseteq \omega$ in L_ν. If $\gamma_0 = \omega$ and $\gamma_{n+1} = h(\gamma_n)$ then $\gamma = \cup\gamma_n \leq \delta$. It is clear that $L_\gamma \models T$ and so $\gamma = \delta$ which is therefore, not only not admissible but actually has a total Σ_1 map from ω cofinal in it. As every $\beta < \delta$ is countable in L_δ, these countings (of the γ_n) can be combined with the cofinal map $n \mapsto \gamma_n$ to produce the desired one-one Σ_1 map from all of ω onto L_δ. □

We can actually say more than Theorem 6.1 about the reverse mathematical weakness of true Π^1_3 (or Σ^1_4) sentences. True Π^1_3 sentences T don't help to prove higher levels of comprehension.

Proposition 6.5. *If T is a true Π^1_3 sentence then, for each $n \geq 1$, there is an ordinal α which is Σ_n admissible, a limit of Σ_n admissibles, Σ_n projectable and satisfies T.*

Proof: Choose α such that L_α is the Σ_n Skolem hull of ω in L_{ω_1} under the standard Σ_n Skolem function. (Note that any Σ_n hull of L_{ω_1} is transitive since every ordinal is countable in L_{ω_1}.) Thus L_α is a Σ_n elementary substructure of $L_{\omega_1} \models T$. It is Σ_n admissible by Lemma 3.4 and is the Σ_n Skolem hull of ω in L_α as well and so Σ_n projectable into ω.

If $n \geq 2$, it is immediate that L_α satisfies the Π^1_3 formula T. For $n = 1$ we have to be a bit more careful. Suppose $T = \forall x \exists y \forall z \phi(x, y, z)$. Consider any $x \in L_\alpha$. $L_{\omega_1} \models \exists y(L_\delta \models \forall z \phi(x, y, z))$ where δ is the second admissible after x and y appear in L. Thus there is such a $y \in L_\alpha$. To conclude, note that if it is not the case that $\forall z \phi(x, y, z)$ then there would be a counterexample z inside L_δ (actually definable over the first admissible after x and y). □

Corollary 6.6. *If T is a true Π^1_3 formula then, for $n \geq 2$, Δ^1_n-CA $+T+$ Π^1_∞-TI $\nvdash \Pi^1_n$-CA_0.*

Corollary 6.7. *For $n \geq 0$, Δ^1_{n+2}-CA $+$ n-$\mathbf{\Pi}^0_3$-DET $+$ Π^1_∞-TI $\nvdash \Pi^1_{n+2}$-CA_0.*

As before, the Proposition and its first Corollary hold for true Σ^1_4 sentences by working in $L(X)$ where X is the witness for the first existential quantifier.

We complete the picture of the powerlessness of true Π^1_3 (Σ^1_4) sentences in the sense of reverse mathematics presented in Corollaries 6.2 and 6.6 by showing that they do not help Π^1_n-CA prove Δ^1_{n+1}-CA_0 even for β-models.

Proposition 6.8. *If T is a true Π^1_3 sentence then, for each $n \geq 1$, there is an ordinal α which is Σ_n nonprojectable, satisfies T but is not Σ_{n+1} admissible.*

Proof: Let γ_j be the j^{th} ordinal δ such that $L_\delta \preceq_n L_{\omega_1}$ and $L_\delta \models T$. Let $\gamma = \cup\gamma_j$ and so $L_{\gamma_j} \preceq_n L_\gamma \preceq_n L_{\omega_1}$ for each j. In addition, $L_\gamma \models T$ as for any $x \in L_\gamma$, $x \in L_{\gamma_j}$ for some j and the witness for x verifying T in L_{γ_j} is one in L_γ since, as Σ_1 elementary substructures of L_{ω_1}, both are β-models. In particular, L_γ is Σ_n admissible by Lemma 3.4.

First note that L_γ is not Σ_n projectable. The point here is that if $\beta < \gamma$ and θ defines a Σ_n subset of β over L_γ then for some j, γ_j is larger than β and all parameters in θ are also in L_{γ_j}. For such a j, the Σ_n definition θ defines the same subset of β over L_{γ_j} as over L_γ. The subset is then in L_γ as required.

Finally, γ is not Σ_{n+1} admissible as the sequence γ_j is definable by recursion over L_γ by a Σ_{n+1} formula. γ_j is the j^{th} ordinal δ such that $L_\delta \preceq_n L_\gamma$ (as $L_\gamma \preceq_n L_{\omega_1}$ and so this condition is the same as $L_\delta \preceq_n L_{\omega_1}$) and $L_\delta \models T$. To say that $L_\delta \preceq_n L_\gamma$ is to say that for every $c \in L_\delta$ and every Σ_n formula $\phi(cx)$, $L_\gamma \models \phi(c) \rightarrow L_\delta \models \phi(c)$. By the Σ_n admissibility of L_γ this is equivalent to a Π_n formula and so its negation to a Σ_n one. Thus the sequence γ_j is Σ_{n+1} as required. □

Corollary 6.9. *If T is a true Π^1_3 formula then, for $n \geq 2$, Π^1_n-CA + T+ Π^1_∞-TI $\nvdash \Delta^1_{n+1}$-CA_0.*

Once again this Proposition and Corollary generalize to true Σ^1_4 sentences by relativization. More generally, the results of this section can be suitably generalized to Σ^1_m formulas T which are theorems of ZFC or at least consistent with ZFC + $\mathsf{V{=}L(X)}$ for some subset X of ω as they are then true in a model M of $V = L(X)$ and so in $L_{(\omega_1)^M}(X)$. The proofs are essentially the same beginning with the fact that T is true in $L_{\omega_1}(X)$ (of a model M).

Corollary 6.10. *If, for $m > 4$, T is a Σ^1_m theorem of ZFC (or is even consistent with ZFC $+V = L(X)$ for some subset X of ω) and $n \geq m-2$, then Π^1_{n-1}-CA + T+ Π^1_∞-$\mathsf{TI}_0 \nvdash \Delta^1_n$-$\mathsf{CA}_0$ and Δ^1_n-CA $+T$+ Π^1_∞-$\mathsf{TI}_0 \nvdash \Pi^1_n$-$\mathsf{CA}_0$. The proofs follow those of Theorem 6.1 (with Corollary 6.2) and Proposition 6.5 (with Corollary 6.6), respectively.*

We close by pointing out that the results of this section are optimal in terms of the quantifier complexity of theorems that cannot imply Π^1_n-CA_0 or even Δ^1_n-CA_0.

Remark 6.11. For $n \geq 1$. there is a Π^1_{n+2} theorem T of second order arithmetic that implies Π^1_n-CA_0 (and so Δ^1_n-CA_0) over RCA_0. The sentence T says that for every Π^1_n formula $\forall Z\psi(n, Z, A)$ and for every choice of parameter A, there is an X such that $\forall n(n \in X^{[0]} \rightarrow \forall Z\psi(n, Z, A))$ & $\forall n(n \notin X^{[0]} \rightarrow \neg\psi(n, X^{[n+1]}, A))$.

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