Computability theoretic classifications for classes of structures

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Abstract. In this paper, we survey recent work in the study of classes of structures from the viewpoint of computability theory. We consider different ways of classifying classes of structures in terms of their global properties, and see how those affect the structures inside the class. On one extreme, we have the classes that are Σ-small. These are the classes which realize only countably many $\exists$-types, and are characterized by having tame computability theoretic behavior. On the opposite end, we look at various notions of completeness for classes which imply that all possible behaviors occur among their structures. We introduce a new notion of completeness, that of being on top for effective-bi-interpretability, which is stronger and more structurally oriented than the previously proposed notions.

Mathematics Subject Classification (2010). Primary 03D45; Secondary 03C57.

Keywords. Sigma small classes, back-and-forth relations, rice relations, low property, bf-ordinal, effective-bi-interpretability.

1. Introduction

In this paper, we survey recent work on the study of classes of structures from the viewpoint of computability theory. By classes of structures we mean classes like the one of fields or of p-groups or of linear orderings. Our general objective is to consider global properties of the classes and derive properties about their individual structures.

Computable structure theory is an area inside computability theory and logic that is concerned with the computable aspects of mathematical objects and constructions. In particular, we are interested in the interplay between structure and complexity, or in other words, in understanding how the algebraic properties of a structure interact with its computational properties. For instance, we ask questions like the following: What kind of information can be encoded into an isomorphism

*The author was partially supported the Packard Fellowship. The author would like to thank Matthew Harrison-Trainor and Noah Schweber for useful comments on an earlier draft of this paper.

[0] Saved: Submitted on March 28, 2014
Compiled: March 28, 2014

type of a structure? How difficult is it to represent a certain structure? How difficult is it to recognize it?

When we consider classes of structures, there are two ends of the spectrum. On the one end are the classes which have some global property restricting the behavior of their structures. On the other end are the classes which are complete in the sense that they allow all possible behaviors to happen. Let us say a bit more about these two extremes.

Tame classes. In Section 2, we will review some concepts we will need later. One notion of simplicity is that of a class having a bound on the Scott rank of its structures. These classes are not necessarily that simple from a computational viewpoint, and much less from the viewpoint of Borel equivalence relations. However, for natural classes this bound tends to be quite low, which makes them easier to analyze. The Scott rank of a structure is related to the number of Turing jumps necessary to fully understand it, and hence, the lower the Scott rank, the more manageable the structure.

A second notion of simplicity, one we believe is the most relevant to computability theory, is that of Σ-smallness, or actually effective Σ-smallness. This notion is studied in Section 3. The author started studying such classes in [Mon10b],[1] although the term "Σ-small" is new.

Definition 1.1. A class of structures $\mathbb{K}$ is Σ-*small* if it realizes countably many $\exists$-types, that is, if the set

$$\{\exists\text{-}tp_{\mathcal{A}}(\bar{a}) : \mathcal{A} \in \mathbb{K}, \bar{a} \in \mathcal{A}^{<\omega}\}$$

is countable, where

$$\exists\text{-}tp_{\mathcal{A}}(\bar{a}) = \{\varphi : \varphi(\bar{x}) \text{ is a first order existential formula with } \mathcal{A} \models \varphi(\bar{a})\}.$$

We remark that knowing the $\exists$-type of $\bar{a}$ is equivalent to knowing what finite sub-structures we can find in $\mathcal{A}$ extending $\bar{a}$. This is not entirely correct when $\mathcal{L}$ is infinite, where we need to consider sub-structures which only mention only a finite number of the symbols in $\mathcal{L}$. Another remark is that the types above are without parameters.

We start Section 3 by developing the effective version of this notion. The effectiveness assumption is not that strong, as it holds of all the examples we have analyzed. A large list of examples can be found in Subsection 3.1. We then study the role of Σ-small classes in many topics that have been widely studied in computable structure theory: Richter's extendibility condition, jumps of structures, the low property, the categoricity property and the Turing ordinal. As more evidence towards its naturalness, we will see in Theorems 3.14 and 3.15 how Σ-smallness induces a strong dichotomy on classes.

[1] In [Mon10b] we used the phrase "$\mathbb{K}$ has a computable 1-back-and-forth structure" for what we now say "$\mathbb{K}$ is effectively Σ-small."

Complete classes. In Section 4, we review various ways of mapping structures from one class into another. For each of these reducibilities we have classes that are *on top* in the sense that all other classes can be reduced to it. We start with the well-known notion of Borel reducibility, and then move on to effective reducibility and Turing-computable reducibility, and to classes that are complete in the sense of Hirschfeldt, Khoussainov, Shore and Slinko.

In Section 5, we develop a new and stronger notion of reducibility based on the idea of *effective-bi-interpretability* between structures. We do not know that much about this new notion, but we do show that it preservers even more computational properties than all the previous reducibilities.

Properties on a cone. There are many properties in computability theory which tend to behave nicely when we have a nice natural class of structures, but that do not in general. One can often build strange and unnatural classes of structures where these nice behaviors do not occur. In this paper, we are interested in properties that hold of natural classes. Since we cannot quantify over "all natural classes," we often use the technical device of considering *properties on a cone*.

When we have a computability theoretic property P, we can often consider its relativization P^X for a given oracle $X \in 2^\omega$. We then consider the properties *relatively-P*, which means that P^X holds for all X, and P *on a cone*, which means that there is a $Z \in 2^\omega$ such that P^X holds for all $X \geq_T Z$. When we have a proof that a natural property P holds (or does not hold) when applied to a natural object, this proof almost always relativizes. Thus, we have a proof of relatively-P, and in particular, of P on a cone. So, if our objects are natural, we should not care whether we are using P, relatively-P, or P on a cone. However, due to the unnatural examples, many results can only be proved in general if we consider the properties *on a cone*. Such results usually call for a further analysis of its degree of effectiveness – we will not concentrate on this here.

Disclaimer. This paper does not pretend to be exhaustive. What it attempts is to convey the author's viewpoint, unifying many ideas that have been floating around for a while. The choice of topics and how much attention they receive is purely motivated by the author's taste, the author's own work, and the new ideas the author wants to develop.

Background and notation.

We only consider countable structures throughout, so "structure" means "countable structure." We only consider relational languages, as we do not lose any generality for our purposes. The languages we consider are all computable: that is, if $\mathcal{L}$ consists of relations R_i for $i \in I$, where $I \subseteq \omega$ and R_i has arity $a(i)$, the function $a\colon I \to \omega$ is computable. (This only matters when $\mathcal{L}$ is infinite.)

A *presentation* of a structure $\mathcal{A}$, or a *copy* of $\mathcal{A}$, is just a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ whose domain is a subset of ω. This allows us to use everything we know about computable functions on ω to study $\mathcal{B}$. Given a presentation $\mathcal{A} = (A; R_i^{\mathcal{A}}, i \in I)$, with $A \subseteq \omega$, we let

$$D(\mathcal{A}) \quad = \quad A \oplus \bigoplus_{i \in I} R_i^{\mathcal{A}} \quad \subseteq \quad \omega \sqcup \bigsqcup_{i \in I} \omega^{a(i)}.$$

Via standard coding, we then think of $D(\mathcal{A})$ as a subset of ω, or equivalently a sequence in 2^ω. Note that $D(\mathcal{A})$ is essentially the atomic diagram of $\mathcal{A}$. When we say that the presentation $\mathcal{A}$ computes X, or is computable in Y, we mean that $D(\mathcal{A})$ computes X or is computable in Y. By a *class of $\mathcal{L}$-structures* we mean a set $\mathbb{K}$ of presentations of $\mathcal{L}$-structures which is closed under isomorphism. We often think of $\mathbb{K}$ and of $\{D(\mathcal{A}) : \mathcal{A} \in \mathbb{K}\} \subseteq 2^\omega$ as the same thing, and hence treat $\mathbb{K}$ as a class of reals.

We will often consider the infinity language $\mathcal{L}_{\omega_1,\omega}$, where countably infinite conjunctions and disjunctions are allowed, and its computable version, where these conjunctions and disjunctions must be computable. See [AK00, Sections 6 and 7]. We use $\Sigma_\alpha^{\mathtt{in}}$ to denote the infinitary Σ_α formulas and $\Sigma_\alpha^{\mathtt{c}}$ to denote the computably infinitary Σ_α formulas. For the *Scott rank* of a structure $\mathcal{A}$ we use the following definition: $SR(\mathcal{A})$ is the least α such that every automorphism orbit in $\mathcal{A}$ is $\Sigma_\alpha^{\mathtt{in}}$-definable without parameters. There are various definitions of Scott rank in the literature that give slightly different values (see [AK00, Section 6.7]). The reason we prefer ours is that it matches better with other complexity measures used in computability theory and descriptive set theory (see [Monc]).

2. Axiomatization and the isomorphism problem

The first measure of the complexity for a class is in terms of its complexity as a set of reals. This is directly connected with the complexity of the class in terms of its axiomatizations:

Theorem 2.1. *(Lopez-Escobar [LE65]) Let $\mathbb{K} \subseteq 2^\omega$ be a class of presentations of structures closed under isomorphisms. Then $\mathbb{K}$ is $\boldsymbol{\Sigma}_\alpha^0$ in the Borel hierarchy if and only if $\mathbb{K}$ is axiomatizable by an infinitary $\Sigma_\alpha^{\mathtt{in}}$ sentence.*

The lightface version of this theorem is also true: $\mathbb{K}$ is lightface Σ_α^0 if and only if it is axiomatizable by a computably infinitary $\Sigma_\alpha^{\mathtt{c}}$ formula [VB07].

Not all nice classes of structures are $\mathcal{L}_{\omega_1,\omega}$-axiomatizable, or equivalently Borel, as for instance the class of ordinals, which is Π_1^1-complete. We, however, are mostly interested in $\mathcal{L}_{\omega_1,\omega}$-axiomatizable classes. Most of the natural classes we consider are actually $\Pi_2^{\mathtt{c}}$-axiomatizable, so we will sometimes make this assumption when we prove general results.

The second measure of complexity is the difficulty in telling apart different structures in $\mathbb{K}$. This is captured by the set

$$\{\langle D(\mathcal{A}), D(\mathcal{B})\rangle : \mathcal{A}, \mathcal{B} \in \mathbb{K}, \mathcal{A} \cong \mathcal{B}\} \quad \subseteq \quad (2^\omega)^2,$$

usually called the *isomorphism problem* for $\mathbb{K}$. For a Borel class of structures, this set is $\boldsymbol{\Sigma}_1^1$. For some classes, like linear orderings, this problem is Σ_1^1-complete.

For other classes, this problem is quite simple, like $\mathbb{Q}$-vector spaces for which it is Π^0_3-complete. If we assume $\mathsf{ZFC}+\forall X\ (X^\sharp$ exists), then non-Borel isomorphism problem must be $\mathbf{\Sigma}^1_1$-complete. This follows from Wadge's theorem (see [Mos80, Lemma 7D.3]), as every set that is not $\mathbf{\Pi}^1_1$, is $\mathbf{\Sigma}^1_1$-hard.

Theorem 2.2. *([BK96, Corollary 7.14]) Let $\mathbb{K}$ be an $\mathcal{L}_{\omega_1,\omega}$-axiomatizable class. The following are equivalent:*
(1) The isomorphism problem for $\mathbb{K}$ is Borel.
(2) $\mathbb{K}$ has bounded Scott rank.

When we say that $\mathbb{K}$ has *bounded Scott rank* we mean $\{SR(\mathcal{A}) : \mathcal{A} \in \mathbb{K}\}$ has a supremum $\beta < \omega_1$. For example, the classes of $\mathbb{Q}$-vector spaces and of algebraically closed fields have bound 2 on their Scott ranks. The classes of equivalence structures and of torsion free abelian groups of finite rank have bound 3.

A simple remark is that if $\mathbb{K}$ has countably many structures (up to isomorphism, of course), it has bounded Scott rank. It follows from the model-theoretic Martin's conjecture that if $\mathbb{K}$ has a first order axiomatization and countably many structures, then $\omega + \omega$ is a bound for the Scott ranks of the structures in $\mathbb{K}$ (see [Gao01] for the statement of Martin's conjecture). For more on how high this bound can be see [Mar90, Sac07].

3. Σ-Small classes of structures

In this section, we see how the notion of Σ-small class connects with a lot of well-known concepts in computable structure theory.

Before looking at examples among familiar classes, let us introduce the effective version of this definition. If $\mathbb{K}$ is a natural class of structures and it is Σ-small, we have a natural countable collection of ∃-types. It is then reasonable to expect that one can list, compare and manipulate these types. An effectively Σ-small class is one where we can do this computably:

Definition 3.1. A Σ-small class $\mathbb{K}$ is *effectively Σ-small* if there is a computable list $\{p_i : i \in \omega\}$ of computable ∃-types listing all the ∃-types realized in $\mathbb{K}$ without repetitions, where the operations of erasing and permuting variables are computable, and deciding inclusion of ∃-types is also computable.[2]

3.1. Examples.

All the classes of structures below are Σ-small. It is worth remarking that most of the examples mentioned below have been attractive to computability theorists for a long time because they enjoy nice computability properties other classes do not.

Vector spaces (over a fixed computable field F). If $\mathbb{K}$ has only countably many structures, it is clearly Σ-small. Proving that F-vector spaces are effectively Σ-small requires understanding their ∃-types, which is not hard to do.

[2] The exact list of properties that are required for a class to be *effectively* Σ-small is currently work in progress, and so far they are motivated from what we see in applications.

Algebraically closed fields. Same as above.

Differentially closed fields of characteristic 0 (DCF_0). They are Σ-small because they are ω-stable. The class of models of an ω-stable theory is always Σ-small, as even using countably many parameters and full first-order types, there are still countably many types. It has not been verified whether DCF_0 is effectively Σ-small or not.

Abelian p-groups. That Abelian p-groups are effectively Σ-small follows from work of Khisamiev [Khi04].

Equivalence structures. We refer the reader to [Mon10b, Section 4.2] for an analysis of the $\exists$-types on equivalence structures.

Trees (as partial orderings). By 'trees' we mean downward closed subsets of $\omega^{<\omega}$. That they are effectively Σ-small in the language of partial orderings follows from Richter's work [Ric81]. Let us remark that a key tool in her proof is Kruskal's theorem [Kru60] on the well-quasi-ordering of finite trees.

Trees of finite height (as graphs). The proof is like the case above using Kruskal's theorem for finite trees of a fixed height.

Linear orderings. All an $\exists$-type can say about a tuple $\bar{a} = \langle a_0, ..., a_{k-1}\rangle$ is the order among the elements of the tuple and, for each $n \in \omega$ and each $i, j < |\bar{a}|$, whether there are at least n elements between a_i and a_j. Thus, existential types are determined by the number of elements between the elements of the tuple, and hence there are countably many of them. One can also use this to prove they are effectively Σ-small.

Linear orderings with an added relation for adjacency. When we add the adjacency relation, the $\exists$-types get a bit more complicated, but they are still effective and countable (see [Mon10b, Section 4.1]).

Boolean algebras. All an $\exists$-type can say about a tuple is how many elements are below each Boolean combination of the elements of the tuple. These are, again, not that difficult to analyze.

Boolean algebras with an added relation that identifies atoms. What makes Boolean algebras particularly interesting is that they remain Σ-small even if we add to them any $\Sigma^{\mathtt{in}}_{<\omega}$ relation. For instance, we can add all of the relations used by Knight and Sob [KS00] (atom, atomless, infinite, atomic, 1-atomic, atominf, $\sim$-inf, Int$(\omega + \eta)$, infatomicless, 1-atomless, and nomax-atomless) and they remain effectively Σ-small. An in-depth analysis of the $\Sigma^{\mathtt{in}}_n$-types of Boolean algebras was done by Harris and Montalbán in [HM12]. The fact that there are countably many of them uses key ideas from work of Flum and Ziegler [FZ80].

Generalized Boolean algebras. These are distributive lattices with 0 and where every interval $[a, b]$ is a Boolean algebra. They are usually known in Russia as Ershov algebras. That they are effectively Σ-small follows from work of Khisamiev [Khi04].

3.2. Richter's computable extendibility condition. In her Ph.D. thesis[3] [Ric77], Linda Richter introduced the computable extendibility condition in order to show that there are structures that do not have Turing degree as defined by Jockusch. (A structure $\mathcal{A}$ has Turing degree $\mathbf{x}$ if $\mathbf{x}$ computes a copy of $\mathcal{A}$, and every copy of $\mathcal{A}$ computes $\mathbf{x}$.)

Definition 3.2. [Ric81, Section 3] A structure $\mathcal{A}$ has the *computable extendibility condition* if each $\exists$-type realized in $\mathcal{A}$ is computable. A structure $\mathcal{A}$ has the *c.e. extendibility condition* if each $\exists$-type realized in $\mathcal{A}$ is c.e.

Richter's original definition was not in terms of types but in terms of finite structures extending a fixed tuple. As we mentioned right after Definition 1.1, these formulations are equivalent. The c.e. extendibility condition was not considered by Richter, but we include it here because it makes Theorem 3.3 below more rounded. In Russia, structures with the c.e. extendibility condition are said to be *locally constructivizable*.

Of course, if $\mathbb{K}$ is effectively Σ-small, then every structure $\mathcal{A}$ in $\mathbb{K}$ satisfies the computable extendibility condition. On the other hand, if every structure in $\mathbb{K}$ satisfies the c.e. extendibility condition, then $\mathbb{K}$ is Σ-small, because there are only countably many c.e. sets. Furthermore, the proofs in the literature that linear orderings, Boolean algebras and trees (as posets) in [Ric77, Ric81] and p-groups and generalized Boolean algebras in [Khi04] satisfy the computable extendibility condition are essentially proofs that these classes are effectively Σ-small.

The reason Richter introduced this notion is to prove the following theorem and its corollary below.

Theorem 3.3. *(Essentially Richter) Let $\mathcal{A}$ be any structure. The following are equivalent:*

(1) $\mathcal{A}$ has the c.e. extendibility condition.
(2) Every set $X \subseteq \omega$ which is c.e. in every presentation of $\mathcal{A}$ is already c.e.

Proof. That (1) implies (2) is essentially the same proof as [Ric81, Theorem 3.1]. For the other direction, notice that every $\exists$-type realized in $\mathcal{A}$ is c.e. in every presentation of $\mathcal{A}$. □

Corollary 3.4. *If $\mathcal{A}$ has the c.e. extendibility condition and has Turing degree $\mathbf{x}$, then $\mathbf{x} = \mathbf{0}$.*

We can read Theorem 3.3 as saying that structures in an effectively Σ-small class cannot directly encode any non-trivial information. In a sense, Σ-smallness is not only a sufficient, but also a necessary condition for this to be the case. The following theorem shows that if $\mathbb{K}$ is not Σ-small, quite the opposite happens: every real is coded by some structure in $\mathbb{K}$ in a left-c.e. way. We recall that $X \in 2^\omega$ is

[3] directed by Carl Jockusch

left-c.e. in $A \in 2^\omega$ if the set $\{\sigma \in 2^{<\omega} : \sigma \leq_{lex} X\}$ is c.e. in A, or equivalently, if there is an A-computable approximation to X from the left.

Theorem 3.5. *[Mon10b, Theorem 3.1] Let $\mathbb{K}$ be an $\mathcal{L}_{\omega_1,\omega}$-axiomatizable class of structures. The following are equivalent:*

(1) $\mathbb{K}$ is not Σ-small.

(2) Relative to every oracle on a cone, the following holds: For every $Y \in 2^\omega$, there is a structure $\mathcal{A} \in \mathbb{K}$ such that Y is left-c.e. in every copy of $\mathcal{A}$.

Part (2) of the theorem can be strengthened by changing "left-c.e." to just "c.e." in most cases. However, an example where left-c.e.ness is required is constructed in [Mon10a, Section 2.2].

3.3. Complete sets of r.i.c.e. relations. Another advantage of Σ-small classes is that they have nice structural jumps. We will see below how, when we have a Σ-small class $\mathbb{K}$, we usually have a nice and simple set of relations that give us all the structural information about the jump of the structures in $\mathbb{K}$. Understanding these complete sets of relations is usually very useful in applications.

Before talking about the jump, we need a notion of c.e.-ness among the relations on a structure. We will then look at complete relations among these and use them to define the jump of a structure. An equivalent notion of jump for a structure was originally defined by I. Soskov [Bal06], although he used a very different format. The definition we use here is in the spirit of that introduced in [Mon09] (see [Mon12, Definition 1.2] for more historical remarks).

Definition 3.6. A relation $R \subseteq \mathcal{A}^{<\omega}$ is *relatively intrinsically computably enumerable (r.i.c.e.)* if, on every copy $(\mathcal{B}, R^\mathcal{B})$ of $(\mathcal{A}, R)$, we have that $R^\mathcal{B}$ (viewed as a subset of $\omega^{<\omega}$) is c.e. in $D(\mathcal{B})$.

Example 3.7. Over a $\mathbb{Q}$-vector space, the relation of linear dependence is r.i.c.e.; Over a ring, the relation that holds of $(r_0, ..., r_k)$ if the polynomial $r_0+r_1x+...+r_kx^k$ has a root is r.i.c.e.

This definition gives a notion of c.e.-ness that we can use to define other standard concepts from computability theory on the subsets of $\mathcal{A}^{<\omega}$.

Definition 3.8. A relation $R \subseteq \mathcal{A}^{<\omega}$ is *relatively intrinsically (r.i.) computable* if it and its complement are both r.i.c.e. R is *r.i. computable in* $Q \subseteq \mathcal{A}^{<\omega}$ if R is r.i. computable in $(\mathcal{A}, Q)$. A partial function $f : \mathcal{A}^{<\omega} \rightharpoonup \mathcal{A}^{<\omega}$ is *partial r.i. computable* if its graph is a r.i.c.e. subset of $(\mathcal{A}^{<\omega})^2$.

Remark 3.9. The use of subsets of $\mathcal{A}^{<\omega}$ not only allows us to consider sequences of subsets of $\mathcal{A}^n$ for all n uniformly, but essentially all finite objects that can be built over $\mathcal{A}$. For instance, given $Q \subseteq (\mathcal{A}^{<\omega})^2$, let us define $R \subseteq \mathcal{A}^{<\omega}$ by $\bar{b} \in R$ iff $|\bar{b}| = \langle n, m\rangle$ for some $n, m \in \omega$ and $((b_0, ..., b_{n-1}), (b_n, ..., b_{n+m-1})) \in Q$. We then have that Q is r.i.c.e. (as in Definition 3.6 but for subsets of $(\mathcal{A}^{<\omega})^2$) if and only if R is r.i.c.e. In a similar way, we can code subsets of $(\mathcal{A}^{<\omega})^{<\omega}$ by subsets of $\mathcal{A}^{<\omega}$. Given $R, Q \subseteq \mathcal{A}^{<\omega}$, we define $R \oplus Q$ by $\bar{b} \in R \oplus Q$ if either $|\bar{b}| = 2n$ and $\bar{b} \restriction n \in R$ or $|\bar{b}| = 2n + 1$ and $\bar{b} \restriction n \in Q$. We can also encode a set $X \subseteq \omega$ by a set $\vec{X} \subset \mathcal{A}^{<\omega}$

by letting $\bar{b} \in \vec{X}$ if and only if $|\bar{b}| \in X$. With a bit more work, we can encode any subset of $HF(\mathcal{A})$ (the hereditarily finite extension of $\mathcal{A}$) as a subset of $\mathcal{A}^{<\omega}$ (see [Mon12, Section]).

R.i.c.e. relations can be characterized in a purely syntactic way, without referring to the different copies of the structure, using $\Sigma_1^{\mathtt{c}}$ formulas. Recall that a $\Sigma^{\mathtt{c}}$ formula is just a computable disjunction of $\exists$-formulas over a finite set of free variables.

Theorem 3.10 (Ash, Knight, Manasse, Slaman [AKMS89]; Chisholm [Chi90]). *Let $\mathcal{A}$ be a structure, and $R \subseteq A^{<\omega}$ a relation on it. The following are equivalent:*

(1) R is r.i.c.e.
(2) R is uniformly definable by $\Sigma_1^{\mathtt{c}}$ formulas with parameters from $\mathcal{A}$. That is, there is a tuple $\bar{a} \in \mathcal{A}^{<\omega}$ and a computable sequence of $\Sigma_1^{\mathtt{c}}$ formulas $\varphi_i(x_1, ..., x_{|\bar{a}|}, y_1, ..., y_i)$, for $i \in \omega$, such that

$$(\forall \bar{b} \in \mathcal{A}^{<\omega}) \quad \bar{b} \in R \iff \mathcal{A} \models \varphi_{|\bar{b}|}(\bar{a}, \bar{b}).$$

Definition 3.11. A relation $R \subseteq \mathcal{A}^{<\omega}$ is *r.i.c.e. complete in* $\mathcal{A}$ if it is r.i.c.e. and every other r.i.c.e. relation $Q \subseteq \mathcal{A}^{<\omega}$ is r.i. computable from R.

R.i.c.e. complete relations always exist. For instance, we can consider the analog of Kleene's predicate K: If we let $\varphi_{i,j}(y_1, ..., y_j)$ be the ith $\Sigma_1^{\mathtt{c}}$ formula of arity j, then the relation $\vec{K}^{\mathcal{A}} \subseteq \mathcal{A}^{<\omega} \times \omega$ defined by

$$(\bar{b}, i) \in \vec{K}^{\mathcal{A}} \iff \mathcal{A} \models \varphi_{i,|\bar{b}|}(\bar{b})$$

is r.i.c.e. complete.

Definition 3.12. [Mon12] We define *the jump of* $\mathcal{A}$ to be the structure $\mathcal{A}' = (\mathcal{A}, \vec{K}^{\mathcal{A}})$. Given a class $\mathbb{K}$, we let $\mathbb{K}' = \{\mathcal{A}' : \mathcal{A} \in \mathbb{K}\}$.

If $X \subseteq \omega$ is a c.e. set, then $\vec{X} \subseteq \mathcal{A}^{<\omega}$ (as in Remark 3.9) is clearly r.i.c.e. It follows that $\vec{0'}$ must be r.i. computable in every r.i.c.e. complete relation. On some structures, like $(\omega; 0, 1, +)$, $\vec{0'}$ is r.i.c.e. complete, but this is always not the case. If $\mathcal{A}$ is a linear ordering, then *co-Adj* $\oplus \vec{0'}$ is r.i.c.e. complete, where *co-Adj* is the complement of the adjacency relation. For most linear orderings *co-Adj* is not r.i. computable from $\vec{0'}$.

Definition 3.13. We say that R is *structurally r.i.c.e. complete* if $R \oplus \vec{0'}$ is r.i.c.e. complete. We then say that $(\mathcal{A}, R)$ is *a structural jump* of $\mathcal{A}$.

So, for a linear ordering L, $(\mathcal{L}, \textit{co-Adj})$ is a structural jump.

The author showed in [Mon10b] that if $\mathbb{K}$ is Σ-small, there is a countable sequence of $\Sigma_1^{\mathtt{in}}$ formulas which define a structurally complete r.i.c.e. relation in all the structures in $\mathbb{K}$ relative to every oracle on a cone.

Theorem 3.14. *([Mon10b]) Let $\mathbb{K}$ be effectively Σ-small, and let $\{p_i : i \in \omega\}$ be a computable list of all the $\exists$-types realized in $\mathbb{K}$. Then, the $\Sigma^{\mathtt{c}}_1$ formulas*

$$\varphi_i \equiv \bigvee \{\psi : \psi \text{ is an } \exists\text{-formula, and } \psi \notin p_i\},$$

for $i \in \omega$ define a structurally r.i.c.e. complete relation on all structures in $\mathbb{K}$.

In most natural examples, we can find simpler structurally r.i.c.e. complete relations than the one given by Theorem 3.14. For instance, on $\mathbb{Q}$-vector spaces and algebraically closed fields, the relations of linear dependence and algebraic dependence are structurally r.i.c.e. complete, and on Boolean algebras, the not-atom relation is structurally r.i.c.e. complete.

It is also shown in [Mon10b] that this is not the case when $\mathbb{K}$ is not Σ-small: there is no sequence of formulas which works for all structures simultaneously.

Theorem 3.15. *If $\mathbb{K}$ is not Σ-small, there is no computable sequence of $\Sigma^{\mathtt{c}}_1$-formulas defining a structurally r.i.c.e. complete relation simultaneously on all structures in $\mathbb{K}$.*

3.4. The low property. The low property has been studied for various classes in the last couple of decades. Only recently has it been looked at a general setting.

Definition 3.16. A class $\mathbb{K}$ has the *low property* if every low presentation $\mathcal{A} \in \mathbb{K}$ has a computable copy.

We recall that a set $X \subseteq \omega$ is *low* if X' is computable from $0'$. A presentation $\mathcal{A}$ is *low* if $D(\mathcal{A})$ is.

Jockusch and Soare [JS91] proved that the class of linear orderings does not have the low property, that is, that there is a low linear ordering without a computable copy. Downey and Jockusch [DJ94] proved that the class of Boolean algebras has the low property. In that paper, they asked the following question, that is still open despite the efforts of various researches:

Question 1. Does every low_n Boolean algebras have a computable copy?

Some partial results are known. Thurber [Thu95] proved that Boolean algebras have the low_2 property and Knight and Stob [KS00] the low_4 property. The low_5 property is still open. Harris and Montalbán showed that the difficulty at level 5 is not just that it needs one more jump, but a qualitatively new behavior: to show this behavioral difference is essential, they produced a low_5 Boolean algebra not $0^{(7)}$-isomorphic to any computable one – for $n = 1, 2, 3, 4$, it was known that every low_n Boolean algebra is $0^{(n+2)}$-isomorphic to a computable one.

Let us review some of the other examples. The class of equivalence structures does not have the low property. However, the class of equivalence structures with infinitely many infinite equivalence classes has the low property, as it follows from [CCHM06, Lemmas 2.2.(c) and 2.3]. Even if linear orderings do not have the low property, some sub-classes do. For instance, the class of all linear orderings where all elements have successor and predecessors does. More examples can be found in [ATF09, Fro10, Fro12]. The class of linear orderings with only finitely many descending sequences (up to equivalence) was proved to have the low$_n$ property for all n by Kach and Montalbán [KM11] (where two sequences are equivalent if they determine the same cut). It is open whether scattered linear orderings have the low property. The class of ordinals not only has the low property, but the low$_\alpha$-property for all computable ordinals α. Classes that have the low$_\alpha$-property for all $\alpha < \omega_1^{CK}$ are said to satisfy *hyperarithmetic-is-recursive*, that is, every hyperarithmetic structure in the class has a computable copy. We will get back to these classes in Theorem 3.26.

All the examples of classes with the low property we know are Σ-small. This happens for a reason:

Theorem 3.17. *Let $\mathbb{K}$ be a $\Pi_2^{\mathtt{in}}$-class. If $\mathbb{K}$ has the low property on a cone, then $\mathbb{K}$ is Σ-small.*

Sketch of the proof. It follows from the author's construction in [Mon10b, Lemma 2.9 and Theorem 3.1] that if $\mathbb{K}$ is not Σ-small, there is an oracle relative to which the following happens: For every $X \in 2^\omega$ and Y c.e. in X, there is a structure $\mathcal{A}$ in $\mathbb{K}$ computable from X and such that Y is left-c.e. in every copy of $\mathcal{A}$. All we need to do now is observe that there is a set that is c.e. over a low set but not left-c.e. For instance, Chaitin's Ω relativized to low 1-random real R is c.e. in R and, since it is 2-random, is not of c.e. degree. Thus $\{\sigma \in 2^{<\omega} : \sigma \leq_{lex} \Omega^R\}$ is c.e. in R but not left-c.e (because left-c.e. sets have c.e. degree). □

3.5. Listable classes.

The author started looking at this property with the intention of characterizing the low property.

Definition 3.18. A class $\mathbb{K}$ is *listable* if there exists a Turing functional which, for every oracle X, produces an X-computable sequence of structures listing all the X-computable structures in $\mathbb{K}$ (allowing repetitions).

This definition appeared first in [Mon13b], but the underlying idea of considering classes whose computable models can be listed computably is much older. However, the uniformity in Definition 3.18 is needed to get the consequences we want. Nurtazin [Nur74], almost four decades ago, gave a sufficient condition for a class of structures to be listable which includes the classes of linear orderings, Boolean algebras, equivalence structures, Abelian p-groups, and algebraic fields of characteristic p. Nurtazin's result says that, if there exists a computable structure in the class such that any other structure can be embedded into it, and such that any subset of that structure generates a structure in the class, then the class is listable (see [GN02, Theorem 5.1]). Nurtazin's condition is not a necessary condition for a class to be listable, and for many of the cases we are interested in, it is too strong.

The more general way of proving that a class is listable is by a priority argument, where one monitors all computable functions and tries to list the ones that code structures in the class. In [Mon13b], the author developed a game, $\mathsf{G}^\infty(\mathbb{K})$, that captures the combinatorial argument behind these constructions. This game is played by two players, C and D. Along the game, player C builds an infinite list of structures in $\mathbb{K}$ via finite approximations, adding a new element to each structure in each move. Player D, however, builds only one structure and he is allowed to wait and only add elements to his structure when he thinks it is worth it. Player C's goal is to get one of his structures to be isomorphic to D's structure (i.e. "to copy"), while D's goal is to diagonalize. See [Mon13b] for a more detailed definition and for other related games. The classes $\mathbb{K}$ for which C has a winning strategy are said to be ∞-*copyable*.

The connection between listability, the low property, and the games was unexpected.

Theorem 3.19. *([Mon13b]) Let $\mathbb{K}$ be a $\Pi^{\mathtt{in}}_2$ class of structures. The following are equivalent:*
(1) $\mathbb{K}$ has the low property on a cone.
(2) $\mathbb{K}$ is Σ-small and $\mathbb{K}'$ is listable on a cone.
(3) $\mathbb{K}$ is Σ-small and $\mathbb{K}'$ is ∞-copyable.

3.6. Computable categoricity. The objective of this subsection is to argue that computable categoricity is easier to analyze on Σ-small classes.

The notion of computable categoricity has been studied intensively for the past few decades. A feature of computable structure theory is that computational properties of presentations need not be invariant under isomorphism, and instead they are invariant under computable isomorphisms. In other words, a structure can have two isomorphic computable presentations which have different computational properties. For instance, there are computable presentations of the countable, infinite-dimensional $\mathbb{Q}$-vector space, $\mathbb{Q}^\infty$, where all the finite-dimensional subspaces are computable, and computable presentations of $\mathbb{Q}^\infty$ where no finite-dimensional subspace is computable (see [DHK$^+$07]). The computably categorical structures are exactly the ones where this does not happen:

Definition 3.20. A computable structure $\mathcal{A}$ is *computably categorical* if between any two computable copies of $\mathcal{A}$ there is a computable isomorphism.

There are many results classifying the computably categorical structures within certain classes. A linear order is computably categorical if and only if it has finitely many adjacencies (Dzgoev and Goncharov [GD80]); a Boolean algebra is computably categorical if and only if it has finitely many atoms (Goncharov, and independently La Roche [LR78]); a $\mathbb{Q}$-vector space is computably categorical if and only if it has finite dimension; a p-group is computably categorical if and only if it can be written in one of the following forms: (i) $(\mathbb{Z}(p^\infty))^\ell \oplus G$ for $\ell \in \omega \cup \{\infty\}$ and G finite, or (ii) $(\mathbb{Z}(p^\infty))^n \oplus (\mathbb{Z}_{p^k})^\infty \oplus G$ where G is finite, and $n, k \in \omega$ (Goncharov [Gon80] and Smith [Smi81]); a tree of finite height is computably categorical if and only if it is of finite type (Lempp, McCoy, R. Miller, and Solomon [LMMS05]); and so on.

There are also many classes where computably categoricity is quite difficult to describe. Indeed, it was recently proved by Downey, Kach, Lempp, Lewis, Montalbán and Turetsky [DKL$^+$] that the index set of computable categorical structures is Π_1^1-complete. The relativized notion is, however, much better behaved and usually easier to characterize (recall the notions of "relatively P" and "P on a cone" from Section 1). A nice characterization of the relatively computably categorical structures was given by Goncharov [Gon77]: they are exactly the atomic models, over a finite set of parameters, where all the types are generated by $\exists$-formulas, and there is a c.e. listing of those formulas. The author [Monc] has recently found that there is an even nicer characterization of the structures which are computably categorical on a cone. They are exactly the ones that have a $\Sigma_3^{\mathtt{in}}$ Scott sentence. The classes where we have the best hope of characterizing computable categoricity are the ones where the three notions of computable categoricity – plain, relative, and on a cone – coincide.

Definition 3.21. We say that $\mathbb{K}$ has the *categoricity property* if every computably categorical structure in $\mathbb{K}$ is relatively computably categorical.

$\mathbb{Q}$-vectors spaces, algebraically closed fields, Boolean algebras, linear orderings, equivalence structures, trees (as posets), ordered abelian groups, and p-groups all have the categoricity property.

Conjecture 1. Every Σ-small $\Pi_2^{\mathtt{in}}$-class $\mathbb{K}$ satisfies the categoricity property on a cone.

One thing we know is that Σ-small $\Pi_2^{\mathtt{in}}$-classes always contain structures which are categorical on a cone [Monc].

3.7. The back-and-forth ordinal.

It is not hard to observe that $\Sigma_2^{\mathtt{c}}$ types over a structure $\mathcal{A}$ are equivalent to $\exists$-types over $\mathcal{A}'$, and $\Sigma_3^{\mathtt{c}}$-types to $\exists$-types over $\mathcal{A}''$. One can use this to get, for instance, that a structure is Δ_2^0-categorical on a cone if and only if it is $\mathcal{A}'$ is computably categorical on a cone; or that a a class $\mathbb{K}$ has the low$_2$ property on a cone if and only if both $\mathbb{K}$ and $\mathbb{K}'$ have the low property on a cone. Thus, understanding $\mathbb{K}'$ can be helpful for the understanding of these higher-level properties. If we have an effectively Σ-small class $\mathbb{K}$, we have a nice and uniform notion of jump among the structures in $\mathbb{K}$ (Theorem 3.14). We then want to know if $\mathbb{K}'$ is effectively Σ-small. If it is, we can then consider $\mathbb{K}''$ and ask if it is Σ-small.

Definition 3.22. $\mathbb{K}$ is $\Sigma_\alpha^{\mathtt{in}}$-small if it realizes countably many $\Sigma_\alpha^{\mathtt{in}}$ types.

The effective notion of $\Sigma_n^{\mathtt{in}}$-smallness is considered in [Mon10b, Mon13a] under the name "effective n-back-and-forth structure." We omit the definition here. In [Mon10b], we proved that on $\Sigma_n^{\mathtt{in}}$-small classes have nice $\Sigma_n^{\mathtt{c}}$-complete relations and that no set can be coded in the $(n-1)$st jump of their structures. An opposite behaviour happens, on a cone, for structures that are not $\Sigma_n^{\mathtt{in}}$-small. Thus, for a class $\mathbb{K}$, there is a qualitative jump in behavior from the α's at which $\mathbb{K}$ is $\Sigma_\alpha^{\mathtt{in}}$-small to the ones where it is not.

Definition 3.23. The *bf-ordinal* of a class of structures $\mathbb{K}$ is the least $\alpha \in \omega_1 + 1$ such that $\mathbb{K}$ is not $\Sigma_\alpha^{\mathtt{in}}$-small, and we let it be ∞ if there is no such α.

The notion of bf-ordinal is quite close to the notion of "Turing-ordinal" introduced by Jockusch and Soare [JS94].

Definition 3.24. A structure $\mathcal{A}$ has *βth jump Turing degree* $\mathbf{x}$ if the βth jump of every copy of $\mathcal{A}$ computes $\mathbf{x}$, and $\mathbf{x}$ computes the βth jump of some copy of $\mathcal{A}$.

A class $\mathbb{K}$ has *Turing-ordinal* τ if for all $\beta < \tau$, whenever a structure in $\mathbb{K}$ has βth-jump Turing degree, it is $0^{(\beta)}$, while for all $\mathbf{x} \geq_T 0^{(\tau)}$, there is a structure in $\mathbb{K}$ with τth-jump Truing degree $\mathbf{x}$.

Theorem 3.25. *Let $\mathbb{K}$ be a $\Pi^{\mathtt{in}}_2$-class with bf-ordinal $\tau < \omega_1$, and suppose that it has Turing ordinal on a cone. Then, on a cone, $\mathbb{K}$ has Turing ordinal either τ or $\tau + 1$.*

Sketch of the proof. This sketches assumes familiarity with either [Mon10b, Proof of theorem 3.1] or [Mon13a, Section 5]. Since for all $\gamma < \tau$, $\mathbb{K}$ is $\Sigma^{\mathtt{in}}_\gamma$-small, on a cone the greatest lower bound of each degree spectrum of structures in $\mathbb{K}$ is $0^{(\gamma)}$. On the other hand, since $\mathbb{K}$ is not $\Sigma^{\mathtt{in}}_\tau$-small, there is a perfect tree T of $\exists$-types over the language $\mathcal{L}_\tau$ as in [Mon10b, Proof of theorem 3.1] and also [Mon13a, Section 5]. Let us relativize to that tree, and hence assume T is computable. For every X above $0'$, there is a 1-generic G such that $G \oplus 0' \equiv_T G' \equiv_T X$. Then, there is a structure $\mathcal{A}_\tau \in \mathbb{K}$ computable in G realizing the $\exists$-type $T[G]$ (by [Mon10b, Lemma 2.9]). So X computes the jump of some copy of $\mathcal{A}_\tau$. On the other hand, G is left-c.e. in every copy of $\mathcal{A}_\tau$. So, the jump of every copy of $\mathcal{A}'_\tau$ computes G and $0'$, and hence X. It follows that $\mathcal{A}$ has $\tau+1$-jump degree X.

Thus, the Turing ordinal on a cone of $\mathbb{K}$ is at least τ and at most $\tau + 1$. □

It is not hard to see that the bf-ordinal of $\mathbb{K}$ is ∞ if and only if it has countably many structures. If $\mathbb{K}$ is an $\mathcal{L}_{\omega_1,\omega}$-axiomatizable class, it follows from Morley's proof [Mor70] that, for every $\alpha < \omega_1$, the number of $\Sigma^{\mathtt{in}}_\alpha$ types it realizes is either countable or continuum. Therefore, the bf-ordinal of $\mathbb{K}$ is less than ω_1 if and only if $\mathbb{K}$ has continuum many countable models. In the remaining case, when the bf-ordinal of $\mathbb{K}$ is ω_1, $\mathbb{K}$ must then have $\aleph_1$ many countable models. Vaught's conjecture [Vau61] claims that this last case never occurs. Thus, if $\mathbb{K}$ is $\mathcal{L}_{\omega_1,\omega}$-axiomatizable and its bf-ordinal is ω_1, we say that $\mathbb{K}$ is a *counterexample to Vaught's conjecture*.

We actually do not know any example of a $\Pi^{\mathtt{in}}_\alpha$-class $\mathbb{K}$ whose bf-ordinal is greater than $\alpha + \omega$ unless it is ∞. The closest example we know is the class of Boolean algebras which is $\forall_2$ and has bf-ordinal ω.

Question 2. Is there a $\Pi^{\mathtt{in}}_2$ class $\mathbb{K}$ with $2^{\aleph_0}$ many models whose bf-ordinal is greater than ω?

Let us remark that question 2 asks about a strengthening of Vaught's conjecture in the opposite direction as Martin's conjecture. Martin's model-theoretic conjecture is about complete first order theories which have less than $2^{\aleph_0}$ many countable models, and implies that they all have bounded Scott rank by at most $\omega + \omega$. Wagner had proposed a strengthening of Martin's conjecture which included theories with $2^{\aleph_0}$ many countable models, which turned out to be false (Gao [Gao01]).

The author found the following connection between these examples and the iterates of the low-property mentioned above. Recall that $\mathbb{K}$ satisfies "hyperarithmetic-is-recursive" if it has the low$_\alpha$ property for all $\alpha < \omega_1^{CK}$, which is equivalent to saying that every hyperarithmetic structure in $\mathbb{K}$ has a computable copy.

Theorem 3.26. *(ZFC+$\forall X$ ($X^\sharp$ exists)) Let $\mathbb{K}$ be an $\mathcal{L}_{\omega_1,\omega}$-axiomatizable class with uncountably many models. The following are equivalent:*
(1) $\mathbb{K}$ is a counterexample to Vaught's conjecture.
(2) $\mathbb{K}$ satisfies hyperarithmetic-is-recursive relative to all oracles on a cone.

The proof in [Mon13a] used projective determinacy, but this was then improved to $\forall X$ ($X^\sharp$ exists) in [Mona]. Furthermore, in [Mona] the result above is extended to all analytic equivalence classes E: E has $\aleph_1$ equivalence classes if and only if it satisfies hyperarithmetic-is-recursive on a cone non-trivially.

The main theorem of [Mon13a] is actually stronger than Theorem 3.26. Assuming $\mathbf{\Sigma}_2^1$-determinacy and relative to all oracles on a cone, $\mathbb{K}$ has the *low-for-ω_1 property*, that is, if a structure in $\mathbb{K}$ has a presentation that is low-for-ω_1, then it has a computable copy. (We recall that X is low-for-ω_1 if $\omega_1^X = \omega_1^{CK}$.)

4. Comparing the complexity of classes

Reducibilities between classes allow us to classify structures in one class in terms of structures in another class. With this in mind, Friedman and Stanley [FS89] defined the notion of Borel reducibility. Since then, the study of Borel reducibility on arbitrary Borel and analytic equivalence relations has been extremely active in descriptive set theory. We concentrate here on the isomorphism relation.

Definition 4.1. (H. Friedman and L. Stanley [FS89]) A class of structures $\mathbb{K}$ is *Borel reducible* to a class $\mathbb{S}$, and we write $\mathbb{K} \leq_B \mathbb{S}$, if there is a Borel function $f\colon 2^\omega \to 2^\omega$ that maps presentations of structures in $\mathbb{K}$ to structures in $\mathbb{S}$ and preserves isomorphism. That is, for all $\mathcal{A} \in \mathbb{K}$, $f(D(\mathcal{A})) = D(\mathcal{B})$ for some $\mathcal{B} \in \mathbb{S}$, and if $\tilde{\mathcal{A}} \in \mathbb{K}$ with $f(D(\tilde{\mathcal{A}})) = D(\tilde{B})$, then

$$\mathcal{A} \cong \tilde{\mathcal{A}} \iff \mathcal{B} \cong \tilde{\mathcal{B}}.$$

(Recall that $D(\mathcal{A})$ is the atomic diagram of $\mathcal{A}$ coded as a subset of ω.)

A class $\mathbb{K}$ is *on top for Borel reducibility* if for every language $\mathcal{L}$, the class of $\mathcal{L}$-structures is Borel-reducible to $\mathbb{K}$.[4]

They first observed that it is enough to use the language with only one binary relation (i.e. directed graphs) in the definition above. Then, they built Borel reductions to show that the classes of trees, linear orderings, 2-step nilpotent groups

[4]In the literature these classes are sometimes called *Borel complete*, but we want to avoid that notation here. The reason is that when we say that $\mathbb{K}$ is $\mathbf{\Sigma}_1^1$-complete we mean that there is a continuous reduction from any $\mathbf{\Sigma}_1^1$ subset of 2^ω to the isomorphism problem of $\mathbb{K}$ as a set, and not as an equivalence relation. Reductions that preserve equivalence relations are quite different.

and fields are all on top for Borel reducibility. Camerlo and Gao [CG01] added Boolean algebras to that list. Friedman and Stanley observed that if a class is on top, then its isomorphism problem must be $\mathbf{\Sigma}^1_1$-complete, giving them a whole range of examples which are not on top for Borel reducibility. Torsion abelian groups are an interesting class: their isomorphism problem is Σ^1_1-complete, but they are not on top for Borel reducibility [FS89, Theorem 5]. (The reason is that their isomorphism problem can be reduced to countable subsets of ordinals via de Ulm invariants in a constructible way, and hence E_0 does not reduce to it.) Whether torsion-free abelian groups are on top was stated as an open question then and remains so. Since then, Downey and Montalbán showed that their isomorphism problem is Σ^1_1-complete, using ideas of Hjorth [Hjo02]. It is also open if abelian groups are on top.

In this paper, we are interested in effective versions of this reducibility.

4.1. Effective reducibility. One way of effectivizing the notion of Borel reducibility is by considering computable reductions that act on indices of computable structures. There has been some recent interest on this reducibility which has turned out to be much more interesting than expected [FF09, FFH$^+$12, CHM12, Monb].

Definition 4.2. We say that a class of structures $\mathbb{K}$ is *effectively reducible* to a class $\mathbb{S}$ if there is a computable function $f\colon \omega \to \omega$ which maps indices of computable structures in $\mathbb{K}$ to indices of computable structures in $\mathbb{S}$ preserving isomorphism. A class of structures $\mathbb{K}$ is said to be *on top for effective reducibility* if for any computable language $\mathcal{L}$, the class of $\mathcal{L}$-structures effectively reduces to it.

(One could also consider hyperarithmetic reductions, but the author has recently shown that, on a cone, being on top for effective reducibility is equivalent to being on top for hyperarithmetic reducibility [Monb, Theorem 1.6], and hence does not make a difference for natural classes.)

E. Fokina, S. Friedman, V. Harizanov, J. Knight, C. MaCoy and A. Montalbán [FFH$^+$12] gave proofs that linear orderings, trees, fields, p-groups and torsion-free abelian groups are all on top. Note that this is different from the Borel-reducibility case where p-groups are not on top, and where it is open if abelian groups are.

It is not hard to see that if a class is on top, its isomorphism-index-set,

$$E(\mathbb{K}) = \{\langle n,m\rangle \in \omega^2 :\ n \text{ and } m \text{ are indices for isomorphic computable structures in } \mathbb{K}\},$$

must be Σ^1_1-complete. Thus, $\mathbb{Q}$-vector spaces, equivalence structures, torsion-free abelian groups of finite rank, etc. cannot be on top because they have arithmetic isomorphism problems. So far, this is the only way we know to produce examples of classes which are not on top.

Definition 4.3. A class $\mathbb{K}$ is *intermediate for effective reducibility* if it is not on top for effective reducibility, and its isomorphism-index-set is not hyperarithmetic.

No specific example of an intermediate class is known. Becker [Bec], and independently Knight and Montalbán [unpublished], showed that such a class of structures exists under the assumption that Vaught's conjecture fails (relative to some oracle). The question now is whether such examples can be built without using a counterexample to Vaught's conjecture:

Question 3. Are the following statements equivalent?

- Vaught's conjecture.
- No $L_{\omega_1,\omega}$-axiomatizable class of structures is intermediate for effective reducibility, relative to every oracle on a cone.

Recent work by the author [Monb] gives a partial reversal, showing that the second statement follows from a strengthening of Vaught's conjecture (which might turn out to be equivalent to Vaught's conjecture too).

4.2. Turing-computable reducibility. The notion of Turing computable reducibility between classes of structures was introduced by Calvert, Cummins, Knight and S. Miller [CCKM04]. It is defined exactly as Borel reducibility (Definition 4.1) except that the function f is required to be a computable operator.

Definition 4.4. A class $\mathbb{K}$ is *Turing computable reducible (tc-reducible)* to $\mathbb{S}$, and we write $\mathbb{K} \leq_{tc} \mathbb{S}$, if there is a Turing operator Φ such that for every presentation $\mathcal{A} \in \mathbb{K}$, $\Phi^{D(\mathcal{A})}$ is the characteristic function of $D(\mathcal{B})$ for some $\mathcal{B} \in \mathbb{S}$ in a way that, if also $\Phi^{D(\tilde{\mathcal{A}})} = D(\tilde{\mathcal{B}})$, then

$$\mathcal{A} \cong \tilde{\mathcal{A}} \iff \mathcal{B} \cong \tilde{\mathcal{B}}.$$

So, instead of working on indices, these operators act on the atomic diagrams given as reals. This makes more of a difference than it seems. It is not hard to see that tc-reducibility implies effective reducibility. This implication does not reverse, as tc-reducibility also implies Borel-reducibility, which is not implied by effective-reducibility (e.g. abelian p-groups).

A class $\mathbb{K}$ is then *on top for tc-reducibility* if for every computable language $\mathcal{L}$, the class of $\mathcal{L}$-structures tc-reduces to $\mathbb{K}$. All the reducibilities produced in [FS89] are not just Borel but also effective, showing that trees, linear orderings, nilpotent groups and fields are actually on top for tc-reducibility. However, the fact that tc-reduction is finer than Borel reduction allows it to get finer comparabilities between certain classes of structures. For instance, any two classes of structures with countably infinitely many models are Borel-equivalent – this is not the case for tc-reductions, and an interesting structure can be found among these (see [KMVB07]). There are even classes of finite structures that are not trivial under tc-reducibility. But the most interesting fact about tc-reducibility is that it preserves the back-and-forth structure:

Theorem 4.5 (Pull Back theorem). *(Knight, S. Miller and Vanden Boom [KMVB07]) Let Φ be a Turing computable embedding from $\mathbb{K}$ to $\mathbb{S}$. Then, for every $\Pi^{\mathtt{c}}_\alpha$-formula φ, there is a $\Pi^{\mathtt{c}}_\alpha$ formula φ^* such that for all $\mathcal{A} \in \mathbb{K}$,*

$$\mathcal{A} \models \varphi^* \iff \Phi(\mathcal{A}) \models \varphi$$

(where $\Phi(\mathcal{A})$ is the presentation $\mathcal{B}$ such that $\Phi^{D(\mathcal{A})} = D(\mathcal{B})$). It follows that if $\mathcal{A} \leq_\alpha \tilde{\mathcal{A}}$, then $\Phi(\mathcal{A}) \leq_\alpha \Phi(\mathcal{B})$ (where $\leq_\alpha$ is the α-back-and-forth relation as in [AK00, Chapter 15]).

This theorem allowed Knight, S. Miller and Vanden Boom to characterize the classes $\mathbb{K}$ such that $\mathbb{K} \leq_{tc} \mathbb{S}$ for certain fixed classes $\mathbb{S}$, like $\mathbb{Q}$-vector spaces.

4.3. Completeness for degree spectrum.

A stronger notion of completeness was analyzed by Hirschfeldt, Khoussainov, Shore and Slinko [HKSS02]. The idea is that these complete classes of structures contains structures exhibiting all the possible computability theoretic behaviors that structures can have. Their objective was to show that certain nice classes of structures are indeed complete in this sense. Their definition is rather cumbersome, but we include it here for completeness.

Definition 4.6. [HKSS02, Definition 1.21] A class of structures $\mathbb{K}$ is *complete with respect to degree spectra of nontrivial structures, effective dimensions, expansion by constants, and degree spectra of relations* (which we will write as HKSS-complete) if for every non-trivial structure $\mathcal{G}$ over a computable language $\mathcal{L}$, there is a structure $\mathcal{A} \in \mathbb{K}$ with the following properties:

(1) $DgSp(\mathcal{A}) = DgSp(\mathcal{G})$.
(2) If $\mathcal{G}$ is computably presentable, then the following holds:
  (i) For any degree $\mathbf{d}$, $\mathcal{A}$ has the same $\mathbf{d}$-computable dimension as $\mathcal{G}$.
  (ii) If $x \in G$, there is an $a \in A$ such that $(\mathcal{A}, a)$ has the same computable dimension as $(\mathcal{G}, x)$.
  (iii) If $S \subseteq G$, there exists $U \subseteq A$ such that $DgSp_{\mathcal{A}}(U) = DgSp_{\mathcal{G}}(S)$ and if S is intrinsically c.e., then so is U.

We recall that the *degree spectrum* of a structure $\mathcal{A}$ is $DgSp(\mathcal{A}) = \{X \in 2^\omega : X \text{ computes a copy of } \mathcal{A}\}$.

They did not talk about a reducibility, but their notion can be easily be made into a reduction.

They showed that undirected graphs, partial orderings, lattices, integral domains of arbitrary characteristic (and in particular rings), commutative semigroups, and 2-step nilpotent groups are all HKSS-complete. Σ-small classes cannot be HKSS-complete. This is because the degree spectrum of a structure in a Σ-small class can never be the upper cone over a base which is higher than the complexity of all the $\exists$-types realized in the structure.

We suspect that if a nice class is not on top for tc-reducibility, it should not be HKSS-complete either.

5. Complete classes for effective-bi-interpretability

In this section, we introduce a notion of completeness much stronger than the ones above. This new notion is more structural, as its definition does not involve presentations of structures. Its main attraction is that it preserves a whole range of computational properties. The notion as defined here is new, although it is composed of a few already-well-known concepts. In [HKSS02], one can already see the idea of having interpretations which are somewhat effective. However, the properties they require use presentations and are not as clean cut or as general as the one here.

We start by introducing the notion of *effective-bi-interpretability* which is a variation of the classical model theoretic notion of bi-interpretability.

5.1. Effective-bi-interpretability. Before looking at effective-bi-interpretability, let us consider effective-interpretability in just one direction. Informally, a structure $\mathcal{A}$ is *effectively-interpretable* in a structure $\mathcal{B}$ if there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ as in model theory, but where the domain of the interpretation is allowed to be a subset of $\mathcal{B}^{<\omega}$, and where all sets in the interpretation are required to be uniformly r.i. computable, except for the domain which is allowed to be uniformly r.i.c.e.[5]

Before giving the formal definition, we need to review one more concept. A relation R on $\mathcal{A}^{<\omega}$ is said to be *uniformly r.i.c.e.* if there is a c.e. operator W such that for every copy $(\mathcal{B}, R^{\mathcal{B}})$ or $(\mathcal{A}, R)$, $R^B = W^{D(\mathcal{B})}$. These are exactly the Σ^c_1-definable relations without parameters. We can then extend this definition and define *uniformly r.i. computable* in the obvious way.

Definition 5.1. Let $\mathcal{A}$ be an $\mathcal{L}$-structure, and $\mathcal{B}$ be any structure. Let us assume that $\mathcal{L}$ is a relational language $\mathcal{L} = \{P_0, P_1, P_2, ...\}$ where P_i has arity $a(i)$; so $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, ...)$ and $P_i^{\mathcal{A}} \subseteq A^{a(i)}$.

We say that $\mathcal{A}$ is *effectively-interpretable* in $\mathcal{B}$ if, in $\mathcal{B}$, there is

- a uniformly r.i.c.e. set $D_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$ (the domain of the interpretation),
- a uniformly r.i. computable relation $\eta \subseteq \mathcal{B}^{<\omega} \times \mathcal{B}^{<\omega}$ which is an equivalence relation on $D_{\mathcal{A}}^{\mathcal{B}}$ (interpreting equality),
- a uniformly r.i. computable sequence of relations $R_i \subseteq (B^{<\omega})^{a(i)}$, closed under the equivalence η within $D_{\mathcal{A}}^{\mathcal{B}}$ (interpreting the relations P_i),
- and a function $f_{\mathcal{A}}^{\mathcal{B}} \colon D_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ which induces an isomorphism:

$$(D_{\mathcal{A}}^{\mathcal{B}}/\eta; R_0, R_1, ...) \cong (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, ...).$$

Let us clarify that: The sets R_i do not need to be subsets of $(D_{\mathcal{A}}^{\mathcal{B}})^{a(i)}$, and, when we refer to the structure $(D_{\mathcal{A}}^{\mathcal{B}}/\eta; R_0, R_1, ...)$, we of course mean $(D_{\mathcal{A}}^{\mathcal{B}}/\eta; (R_0 \cap (D_{\mathcal{A}}^{\mathcal{B}})^{a(0)})/\eta, (R_1 \cap (D_{\mathcal{A}}^{\mathcal{B}})^{a(1)})/\eta, ...)$. By *uniformly r.i. computable sequence* we mean that $\bigoplus_i R_{i\in I}$ is uniformly r.i. computable.

If we add parameters, this notion is equivalent to that of Σ-definability introduced by Ershov [Ers96] and widely studied in Russia. Ershov's definition is quite different in format: it uses $HF(\mathcal{B})$ instead of $\mathcal{B}^{<\omega}$, and $\exists$-definable sets with parameters instead of uniformly r.i.c.e. ones. (It is known that r.i.c.e. subsets of $\mathcal{B}^{<\omega}$ are

[5]We remark that this definition is slightly different from what the author called effective-interpretability in [Mon13c, Definition 1.7], as we now allow the domain to be a subset of $\mathcal{B}^{<\omega}$ rather than $\mathcal{B}^n$ for some n, and we do not allow parameters in the definitions.

equivalent to Σ-definable (with parameters) subsets of $HF(\mathcal{B})$; see [Mon12, Section 4].) Another well-known notion is that of Σ-*equivalence* between two structures, which just means that the structures are Σ-definable in each other. This is, indeed, quite a strong notion of equivalence, but the one we consider below is stronger, as we also require the composition of the isomorphisms to be computable in the respective structures. Here is the formal definition:

Definition 5.2. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively-bi-interpretable* if there are effective-interpretations of each structure in the other as in Definition 5.1 such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \tilde{f}_{\mathcal{A}}^{\mathcal{B}} \colon D_{\mathcal{B}}^{(D_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \tilde{f}_{\mathcal{B}}^{\mathcal{A}} \colon D_{\mathcal{A}}^{(D_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A}$$

are uniformly r.i. computable in $\mathcal{B}$ and $\mathcal{A}$ respectively. (Here $\tilde{f}_{\mathcal{A}}^{\mathcal{B}} \colon (D_{\mathcal{A}}^{\mathcal{B}})^{<\omega} \to \mathcal{A}^{<\omega}$ is the obvious extension of $f_{\mathcal{A}}^{\mathcal{B}} \colon D_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$.)

In the next lemma, we see how effective-bi-interpretability preserves most computability theoretic properties.

Lemma 5.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be effectively-bi-interpretable.*

(1) $\mathcal{A}$ and $\mathcal{B}$ have the same degree spectrum.
(2) $\mathcal{A}$ is computably categorical if and only if $\mathcal{B}$ is.
(3) $\mathcal{A}$ and $\mathcal{B}$ have the same computable dimension.
(4) $\mathcal{A}$ is rigid if and only if $\mathcal{B}$ is.
(5) $\mathcal{A}$ and $\mathcal{B}$ have the same Scott rank.
(6) For every $\bar{a} \in \mathcal{A}^{<\omega}$, there is a $\bar{b} \in \mathcal{B}^{<\omega}$ such that $(\mathcal{A}, \bar{a})$ and $(\mathcal{B}, \bar{b})$ have the same computable dimension, and vice-versa.
(7) For every $R \subseteq \mathcal{A}^{<\omega}$, there is a $Q \subseteq \mathcal{B}^{<\omega}$ which has the same degree spectrum, and vice-versa.
(8) $\mathcal{A}$ has the c.e. extendibility condition if and only if $\mathcal{B}$ does.
(9) The index sets of $\mathcal{A}$ and $\mathcal{B}$ are Turing equivalent, assuming $\mathcal{A}$ and $\mathcal{B}$ are infinite structures.
(10) The jumps of $\mathcal{A}$ and $\mathcal{B}$ are effectively-bi-interpretable.

(Of course, items (2)-(10) assume $\mathcal{A}$ and $\mathcal{B}$ are computable.)

Sketch of the proof. Throughout this proof, assume that $\mathcal{A}$ is the presentation that is coded inside $\mathcal{B}^{<\omega}$, i.e. with domain $D_{\mathcal{A}}^{\mathcal{B}}$, and $\tilde{\mathcal{B}}$ is the copy of $\mathcal{B}$ coded inside $\mathcal{A}^{<\omega}$, i.e. with domain $D_{\mathcal{B}}^{\mathcal{A}} = D_{B}^{\mathcal{D}_{\mathcal{A}}^{\mathcal{B}}}$. We let f be the isomorphism from $\tilde{\mathcal{B}}$ to $\mathcal{B}$.

For part (1), just observe that via the Σ-interpretation, given a copy of $\mathcal{B}$, we can enumerate a copy of $\mathcal{A}$, and hence compute one.

For part (2), we need the following observation: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be copies of $\mathcal{B}$, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the presentations of $\mathcal{A}$ coded inside $\mathcal{B}_1^{<\omega}$ and $\mathcal{B}_2^{<\omega}$ respectively. The point we need to make here is that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are computably isomorphic, then so are $\mathcal{B}_1$ and $\mathcal{B}_2$: A computable isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ induces a computable isomorphism between $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$, which each are computably-isomorphic to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Thus, if $\mathcal{A}$ is computably categorical, so is

$\mathcal{B}$. For (3), we have that if $\mathcal{B}$ has k non-computably isomorphic copies $\mathcal{B}_1, ..., \mathcal{B}_k$, then the respective structures $\mathcal{A}_1, ..., \mathcal{A}_k$ cannot be computably isomorphic either. So the effective dimension of $\mathcal{A}$ is at least that of $\mathcal{B}$, and hence, by symmetry, they must be equal.

For part (4), suppose $\mathcal{B}$ is not rigid. Let h be a nontrivial automorphism of $\mathcal{B}$. It then induces an automorphism of $\mathcal{B}^{<\omega}$, which then induces an automorphism g of $\mathcal{A}$, which then induces an automorphism h_1 of $\tilde{\mathcal{B}}$. Since f is invariant, we have $h \circ f = f \circ h_1$ and, since h is nontrivial, h_1 is not trivial either. It follows that the automorphism g of $\mathcal{A}$ cannot be trivial either.

For part (5), suppose that $SR(\mathcal{A}) = \alpha$, that is, that every automorphism orbit in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_\alpha$ definable. Take a tuple $\bar{b} \in \mathcal{B}^{<\omega}$; we will show its orbit is also $\Sigma^{\mathtt{in}}_\alpha$ definable. Let $\bar{c} \in \tilde{\mathcal{B}}^{<\omega} \subseteq \mathcal{B}^{<\omega}$ be such that $f(\bar{c}) = \bar{b}$. The orbit of $\bar{c}$ is $\Sigma^{\mathtt{in}}_\alpha$ definable inside $\mathcal{A}$, and since $\mathcal{A}$ is $\Sigma^{\mathtt{c}}_1$-definable in $\mathcal{B}$, the orbit of $\bar{c}$ is also $\Sigma^{\mathtt{in}}_\alpha$ definable in $\mathcal{B}$. Since f is $\Sigma^{\mathtt{c}}_1$-definable in $\mathcal{B}$, the orbit of $\bar{b}$ is also $\Sigma^{\mathtt{in}}_\alpha$ definable. It follows that $SR(\mathcal{B}) \leq \alpha$, and, by symmetry, that $SR(\mathcal{B}) = \alpha$.

For part (6), think of $\bar{a}$ as a tuple in $(D^B_{\mathcal{A}})^{<\omega} \subseteq \mathcal{B}^{<\omega}$ and call it $\bar{b}$. It is not hard to show that $(\mathcal{A}, \bar{a})$ and $(\mathcal{B}, \bar{b})$ are effectively-bi-interpretable.

For part (7), think of R as a subset of $(D^B_{\mathcal{A}})^{<\omega} \subseteq \mathcal{B}^{<\omega}$ and call it Q. Clearly, for every copy of $\mathcal{B}$, R and Q have the same degree. Conversely, for each copy of $\mathcal{A}$, if we look at the copy of $\mathcal{B}$ inside and then at the one of $\mathcal{A}$ inside it, we get that R and Q have the same degree too.

For part (8), all we have to notice is that each $\exists$-type in $\mathcal{A}$ is 1-1 reducible to a $\Sigma^{\mathtt{c}}_1$-type in $\mathcal{B}$, and vice-versa.

For part (9), given an index of a structure that we want to know if it is isomorphic to $\mathcal{B}$, we can produce an index for the structure that is then supposed to be isomorphic to $\mathcal{A}$. If it is not, then we know the original structure was not isomorphic to $\mathcal{B}$. If it is, we need to check that the bi-interpretability does produce an isomorphism, which $0''$ can check. One has to notice that all index sets compute $0''$, as their domain must be infinite.

Last, for part (10), it is not hard to interpret the complete r.i.c.e. relations from one structure into the other by interpreting $\Sigma^{\mathtt{c}}_1$-formulas in one by $\Sigma^{\mathtt{c}}_1$ formulas in the other. □

5.2. Reduction via effective-bi-interpretability. As we mentioned before, uniform r.i.c.e. sets are $\Sigma^{\mathtt{c}}_1$ definable. So, an effective-bi-interpretation is given by a list of $\Sigma^{\mathtt{c}}_1$ formulas defining all the relations involved. When we fix these formulas, we obtain a map from one kind of structure into another (which might not always define a bi-interpretation). We can use this to define a reducibility between classes:

Definition 5.4. A class $\mathbb{K}$ is *reducible to* $\mathbb{S}$ *via effective-bi-interpretability* if there are $\Sigma^{\mathtt{c}}_1$ formulas such that for every $\mathcal{A} \in \mathbb{K}$, there is a $\mathcal{B} \in \mathbb{S}$ such that $\mathcal{A}$ and $\mathcal{B}$ are effectively-bi-interpretable using those formulas. A class $\mathbb{K}$ is *on top for effective-bi-interpretability* if for every computable language $\mathcal{L}$, the class of $\mathcal{L}$-structures is reducible to $\mathbb{K}$ via effective-bi-interpretability.

Not much is known about this definition. Classes that are Σ-small are not on top for effective-bi-interpretability for the same reason they are not HKSS-complete. Classes that have bounded Scott rank cannot be on top because effective-bi-interpretability preserves Scott ranks. Using the interpretations defined by Hirschfeldt, Khoussainov, Shore and Slinko [HKSS02], we get the following: undirected graphs, partial orderings, and lattices are on top for effective-bi-interpretability; if we add a finite set of constants to the languages of integral domains, commutative semigroups, or 2-step nilpotent groups, they become on top for effective-bi-interpretability too. A recent result by R. Miller, J. Park, B. Poonen, H. Schoutens, and A. Shlapentokh [MPP+] shows that fields are on top for effective-bi-interpretability.

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