HW6 Problem 1

Suppose $\mathcal{M}$ has n elements $a_1, \ldots a_n$

Let $\varphi(x_1 \ldots x_n) \equiv_n \Big(\bigwedge_{\substack{i,j \leq n \\ i \neq j}} x_i \neq x_j \wedge \forall z \bigvee_{i \leq n} z = x_i \wedge$

$$\bigwedge_{\substack{i,j \leq i \\ \mathcal{M} \models R(a_i, a_j)}} R(x_i, x_j) \wedge \bigwedge_{\substack{i,j \leq n \\ \mathcal{M} \models \neg R(a_i, a_j)}} \neg R(x_i, x_j) \Big)$$

where $\bigwedge_{i=1}^{k} \varphi_k$ means $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k$

Let $\sigma \equiv \exists x_1 \ldots x_m \, \varphi(x_1 \ldots x_n)$

Clearly $\mathcal{M} \models \sigma$ using a_i as x_i. So if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{N} \models \sigma$ too

Conversely, if $\mathcal{N} \models \sigma$ and $b_1 \ldots b_n \in \mathcal{N}$ are such that

$$\mathcal{N} \models \varphi(b_1 \ldots b_n)$$

The the map $a_i \mapsto b_i$ is an isomorphism

[this needs a small proof]

HW 6 Problem 2

(a) Let $p_0, p_1, p_2, \ldots$ be an enumeration of all the prime numbers

Let $h : \mathbb{Z}^\infty \to \mathbb{Q}^+$ be defined by

$$h(\langle n_0, n_1, \ldots \rangle) = \underbrace{p_0^{n_0} \cdot p_1^{n_1} \cdot p_2^{n_2} \cdots}_{\text{this is finite because the } n_i \text{ are eventually all } 0\text{'s}}$$

It is not hard to see that

$$h(n+m) = h(n) \times h(m)$$

and that h is a bijection. So h is an isomorphism

(b) For any permutation $\pi : \mathbb{N} \to \mathbb{N}$ we get an automorphism $\pi' : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ defined by $\pi'(n_0, n_1, \ldots) = \langle n_{\pi(0)}, n_{\pi(1)}, \ldots \rangle$

This induces an automorphism of $(\mathbb{Q}^+, \times)$ given by permuting the prime numbers (proof needed here)

But then there is an automorphism of $(\mathbb{Q}^+, \times)$ transposing 2 and 3. But $(2,3) \in \{(a,b) \in \mathbb{Q}^2 : a < b\}$
$(3,2) \notin$ — " —

(c) If $g : \mathbb{Q}^+ \to \mathbb{Q}^+$ is an automorphism transposing 2 and 3 then $g' : \mathbb{Q} \to \mathbb{Q}$ given by

$$g'(x) = \begin{cases} g(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -g(-x) & \text{if } x < 0 \end{cases}$$

is an automorphism of $(\mathbb{Q}, \times)$ (proof needed here) transposing 2 and 3

HW 6 Problem 3

(a) Suppose $\varphi(x)$ is s.t. for any group G, $\varphi(x)$ defines the set of torsion elements

Let $\mathcal{L}' = \mathcal{L} \cup \{c\}$

Let $\Gamma = \ell_{gr} \cup \{\varphi(c)\} \cup \{\overbrace{c*c*c*\ldots*c}^{n \text{ time}} \neq e \ : n \in \mathbb{N}\}$

$\uparrow$ group axioms

- Then Γ is not satisfiable because a model of Γ would be a group with an element c which is torsion but $c^n \neq e \ \forall n$
- Γ is finitely satisfiable, because every finite $\Gamma_0 \subseteq \Gamma$ there is an m s.t. $\Gamma_0 \subseteq \{\ell_{gr}, \varphi(c), c \neq e, c*c \neq e, \ldots, \overbrace{c*\ldots*c}^{m} \neq e\}$

But then $(\underset{}{\mathbb{Z}_{m+1}} : \underset{\overset{\uparrow}{e}}{0}, \underset{\overset{\uparrow}{*}}{+}, \underset{\overset{\uparrow}{c}}{1}) \models \Gamma_0$

This is a contradiction by compactness

(b) Suppose $Mod(\Sigma)$ = torsion groups. Let $L' = L \cup \{c\}$

Consider $\Gamma = \Sigma \cup \{\overbrace{c*\ldots*c}^{n \text{ times}} \neq e \ : n \in \mathbb{N}\}$

- Prove that Γ is not satisfiable
- Prove that Γ is finitely satisfiable

$\Big\}$ contradiction by compactness

(c) The class of torsion free group can be axiomatized by

$$\Sigma = \{\ell_{gr}\} \cup \{\forall x (x \neq e \rightarrow \overbrace{x*\ldots*x}^{n \text{ times}} \neq e) \ : n \in \mathbb{N}\}$$

But Σ is not semantically equivalent to any subset of itself.
So by the Lemma from class, $Mod(\Sigma)$ is not elementary.

HW 6 Problem 4

(a) Let $L' = L \cup \{c_0, c_1, c_2, \ldots\}$

Suppose there is $\Sigma \subseteq Sent_L$ st $Mod(\Sigma)$ = thin partial ordering

Let $\Gamma = \Sigma \cup \{c_i \nleq c_j : i, j \in N, i \neq j\}$

- Γ is not satisfiable because a model of Γ would be a model of Σ with element $c_0^M, c_1^M, \ldots$ which form an infinite antichain.
- Γ is finitely satisfiable (prove)

Then by compactness we get a contradiction

(b) Same idea. Consider $\Gamma = \Sigma \cup \{$"$c_i \leq c_j \vee c_j \leq c_i$" $i, j \in N\}$

(c) Let $L' = L \cup \{c_0, c_1\}$

Consider Γ = axiom for partial ordering $\cup$ $\varphi(c_0, c_1) \cup \{\exists x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} (c_0 < x_i \wedge x_i < c_1) \right)$

⋮

(d) Let P be finite, then any two elements are finitely apart.