Math 125 A – Fall 2013
Homework 9: Due Monday, November 24

Required Problems

Problem 1: (a) Suppose that the rule (∃P) did not have the conditional that $y \notin \text{FreeVar}(\Gamma \cup \{\exists x \varphi, \psi\})$, and give a counter-example to soundness. That is, give an example of $\Delta \subseteq \text{Sent}_L$ and $\theta \in \text{Sent}_L$ such that:
on the one hand there is a deduction of $\Delta \vdash \theta$ using such a rule without the conditional on $y$, but on the other hand $\Delta \not\models \theta$.
(b) Do the same but with the rule (∀I) instead.

Problem 2: Let $\mathcal{L} = \{0, 1, +, \times, -, (\cdot)^{-1}\}$, where $-$ and $(\cdot)^{-1}$ are unary functions. Let $\Gamma$ be the set of commutative ring axioms (look them up), plus the axioms defining $-$ and $(\cdot)^{-1}$:

$$\Gamma = \text{Ring axioms} \cup \{\forall x, y(x \times y = y \times x)\} \cup \{\forall x (x + (-x)) = 0\} \cup \{\forall x (x \neq 0 \rightarrow x \times x^{-1} = 1\} \cup \{0^{-1} = 0\}.$$  
(The last axiom $0^{-1} = 0$ is added so that the function $(\cdot)^{-1}$ is defined everywhere.) What is the term model of $\Gamma$ isomorphic to? Construct the isomorphism and prove it is an isomorphism.

Problem 3: Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\Gamma$ the theory of $\mathcal{M}$. Show that there is an embedding from the term model of $\Gamma$ to $\mathcal{M}$.