Math 125 A – Fall 2013

Homework 7: Due Friday, November 1st

Problem 1: Let $\mathcal{L} = \{\leq\}$. Let φ_{DLO} be the axiom for dense linear orderings without endpoints. That is, φ_{DLO} is the conjunction of the axioms for linear orderings, a sentence saying that for any two elements there is one in between them, and the sentence saying that for every element there is another one to the left and another one to the right.

Prove that φ_{DLO} is complete, that is that for every sentence φ, either $\varphi_{DLO} \models \varphi$ or $\varphi_{DLO} \models \neg\varphi$.

Problem 2:

Let $\mathcal{Z} = (\mathbb{Z}, 0, S, P)$, where S is the successor function, $S(n) = n + 1$, and P is the predecessor function, $P(n) = n - 1$. Let $\mathcal{Z}_2 = (\mathbb{Z}^2, \vec{0}, S_0, P_0)$, where $\vec{0} = (0, 0)$, $S_0(n, m) = (n + 1, m)$ and $P_0(n, m) = (n - 1, m)$.

Consider the embedding $h \colon \mathcal{Z} \to \mathcal{Z}_2$ given by $h(n) = (n, 0)$. Prove that h is an elementary embedding (that is, that the image of h is an elementary substructure of $\mathcal{Z}_2$).

Hint: Prove that $\mathcal{Z}_1$ is an elementary substructure of a structure isomorphic to $\mathcal{Z}_2$.

Problem 3:

Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times, \leq)$.

(a) Let $\mathcal{M}$ be such that $\mathcal{N} \preccurlyeq \mathcal{M}$ and $\mathbb{N} \subsetneq M$. Given $a, b \in M$, define the following relation. Let $a << b$ if there are infinitely many elements between a and b in M. Show that if $a << b$ then there exits $c \in M$ such that $a << c$ and $c << b$.

Hint: think of the average.

(b) Prove that there exits $\mathcal{A}$ with $\mathcal{N} \preccurlyeq \mathcal{A}$ for which there exits $a \in A$ such that for every $n \in \mathbb{N}$, $\mathcal{A} \models$“ n divides a”.

Problem 4:

(a) Suppose that $\mathcal{A} \preccurlyeq \mathcal{B}$ and not equal. Prove that the set A, as a subset of B, is not definable in $\mathcal{B}$.

(b) Let $\mathcal{C}$ be a structure such that for every $a \in C$, there is an $\mathcal{L}$-term t without variables such that $t^{\mathcal{C}} = a$. Prove that if $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{C} \equiv \mathcal{D}$, then $\mathcal{C} \preccurlyeq \mathcal{D}$.