Math 125 A – Fall 2013

Homework 6: Due Friday, October 25

Problem 1: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there exists $\sigma \in Sent_{\mathcal{L}}$ such that for all $\mathcal{L}$-structures $\mathcal{N}$, we have

$$\mathcal{N} \models \sigma \text{ if and only if } \mathcal{M} \cong \mathcal{N}$$

(This problem easily generalizes to any finite language $\mathcal{L}$. In particular, if $\mathcal{L}$ is a finite language and $\mathcal{M}$ is finite $\mathcal{L}$-structure, then for any $\mathcal{L}$-structure $\mathcal{N}$, we have $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M} \cong \mathcal{N}$.)

Problem 2:

Let $\mathbb{Z}^\infty$ be the set of infinite sequences $\langle n_0, n_1, n_2, ...\rangle$ of integers that are equal to zero from some point on (i.e. there exists k such that for all $j \geq k$, $n_j = 0$). Given two sequences $\mathbf{n} = \langle n_0, n_1, n_2, ...\rangle$ and $\mathbf{m} = \langle m_0, m_1, m_2, ...\rangle$ we define a new sequence $\mathbf{n+m} = \langle n_0 + m_0, n_1 + m_1, n_2 + m_2, ...\rangle$.

Let $\mathbb{Q}^+$ be the set of positive rational numbers.

a. Show that $(\mathbb{Z}^\infty, +) \cong (\mathbb{Q}^+, \times)$. (hint: think of prime numbers)

b. Show that the set $\{(a, b) \in (\mathbb{Q}^+)^2 : a \leq b\}$ is not definable in $(\mathbb{Q}^+, \times)$.

c. Show that the set $\{(a, b) \in \mathbb{Q}^2 : a \leq b\}$ is not definable in $(\mathbb{Q}, \times)$.

Problem 3: Let $\mathcal{L} = \{e, *\}$.

(a) An element g on a group $(G, e, *)$ is said to be *torsion* if there exists an $n \in \mathbb{N}^+$, such that $g^n = e$. Show that there is no $\mathcal{L}$-formula $\varphi(x)$ such that on all groups G, $\varphi(x)$ defines the set of torsion elements of G.

(b) A group G is said to be *torsion* if every element of G is torsion. Show that the class of torsion groups is not weakly elementary.

(c) A group G is *torsion-free* if no element of G is torsion, except for e. Show that the class of torsion-free groups is weakly elementary but not elementary.

Problem 4: Let $\mathcal{L} = \{\leq\}$.

(a) An *anti-chain* on a partial ordering $(P, \leq)$ is a subset $X \subseteq P$ such that no two elements of X are comparable (i.e. $\forall x, y \in X(x \not\leq y \wedge y \not\leq x)$). A partial ordering is said to be *thin* if it has no infinite anti-chains. Show that the class of thin partial orderings is not weakly elementary.

(b) A partial ordering is said to be *short* if it has no infinite chains. Show that the class of short partial orderings is not weakly elementary.

(c) Two elements a, b in a partial ordering are said to be *finitely apart* if $a \leq b$ and there are only finitely many elements in between a and b. Show that there is no formula $\varphi(x, y)$ such that, on any partial ordering P, $\varphi(x, y)$ defines the set of of pairs of elements of P which are finitely apart.

(d) Give an example of a partial ordering where the set $\{(a, b) \in P^2 : a \text{ is finitely apart from } b\}$ is definable.