Math 125 A – Fall 2013
Homework 5: Due Friday, October 18

Required Problems

Problem 1: Decide whether the following statements are True or False. Circle the right answer. You don’t need to justify your answers.

T  F Every embedding is a homomorphism.

T  F The set of even numbers is definable in \((\mathbb{Z}, S)\), where \(S\) is the successor function \(S(n) = n + 1\).

T  F The set of odd numbers is definable in \((\mathbb{Z}, +)\).

T  F There is an embedding from \((\mathbb{N}, 0, +)\) to \((\mathbb{N}, 0, \times)\).

T  F The structures \((\mathbb{R}, +)\) and \((\mathbb{R}, -)\) are isomorphic.

Problem 2: Let \(\mathcal{L} = \{f\}\) where \(f\) is a binary function symbol. Let \(\mathcal{M}\) be the \(\mathcal{L}\)-structure where \(M = \{0, 1\}^{<\omega}\) (the set of binary strings) and \(f^\mathcal{M} : M^2 \rightarrow M\) is concatenation (i.e. \(f^\mathcal{M}(\sigma, \tau) = \sigma \tau\)).

a. Show that \(\{\lambda\} \subseteq M\) is definable in \(\mathcal{M}\), where \(\lambda\) is the empty string.

b. Show that for each \(n \in \mathbb{N}\), the set \(\{\sigma \in M : |\sigma| = n\}\) is definable in \(\mathcal{M}\).

c. Find all automorphisms of \(\mathcal{M}\).

d. Show that \(\{\sigma \in M : \sigma\text{ contains no }1\text{'s}\} = \{0\}^{<\omega}\) is not definable in \(\mathcal{M}\).

Problem 3: Let \(\mathcal{L} = \{f\}\) where \(f\) is a binary function symbol. Let \(\mathcal{M}\) be the \(\mathcal{L}\)-structure where \(M = \mathbb{N}\) and \(f^\mathcal{M} : M^2 \rightarrow M\) is multiplication (i.e. \(f^\mathcal{M}(m, n) = m \cdot n\)).

a. Show that \(\{0\} \subseteq M\) is definable in \(\mathcal{M}\).

b. Show that \(\{1\} \subseteq M\) is definable in \(\mathcal{M}\).

c. Show that \(\{p \in M : p\text{ is prime}\}\) is definable in \(\mathcal{M}\).

d. Find all automorphisms of \(\mathcal{M}\).

e. Show that \(\{n\} \subseteq M\) is not definable in \(\mathcal{M}\) whenever \(n \geq 2\).

f. Show that \(\{(k, m, n) \in M^3 : k + m = n\}\) is not definable in \(\mathcal{M}\).