# Math 125 A – Fall 2013
## Homework 4: Due Friday, October 11

### Required Problems

**Problem 1:** Decide whether the following statements are **T**rue or **F**alse. Circle the right answer. You don't need to justify your answers.
Use $\mathcal{L} = \{0, 1, +\}$.

**T F** $x \in FreeVar(x + 1 = 1 \wedge \exists x(x + x = x))$.

**T F** If $s(x) = 0$, then $(\mathcal{N}, s) \models \forall x(x + x = x)$.

**T F** $(x + x = x \wedge x + 1 = 0) \in AtomicFrom_{\mathcal{L}}$.

**T F** $Term_{\mathcal{L}} \subseteq AtomicForm_{\mathcal{L}}$.

**T F** If the truth value of $(M, s) \models \varphi$ is independent of $s$, then $\varphi$ has no free variables.

**Problem 2:**
a. Let $\mathcal{L} = \{\mathsf{f}\}$ where $\mathsf{f}$ is a unary function symbol. Show that the class of all $\mathcal{L}$-structures $\mathcal{M}$ such that $\mathsf{f}^{\mathcal{M}}$ is a bijection on $M$ is an elementary class in the language $\mathcal{L}$.
b. A *directed graph* is a nonempty set $V$ of vertices together with a set $E \subseteq V \times V$ where $(v, w) \in E$ intuitively represents an edge originating at $v$ and terminating at $w$. A cycle in a directed graph is a sequence $v_1 v_2 \cdots v_k$ of vertices, such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$. If we let $\mathcal{L} = \{\mathsf{R}\}$, where $\mathsf{R}$ is a binary relation symbol, then directed graphs correspond exactly to $\mathcal{L}$-structures. Show that the class of directed acyclic graphs (that is, directed graphs with no cycles) is a weak elementary class in this language.

**Problem 3:**
a. Let $\mathcal{L} = \{\mathsf{f}\}$ where $\mathsf{f}$ is a binary function symbol. Show that $(\mathbb{N}, +) \not\equiv (\mathbb{Z}, +)$.
b. Let $\mathcal{L} = \{\mathsf{f}\}$ where $\mathsf{f}$ is a binary function symbol. Define $g\colon \{1, 2, 3, 4\}^2 \to \{1, 2, 3, 4\}$ and $h\colon \{a, b, c, d\}^2 \to \{a, b, c, d\}$ by

| g | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 4 | 3 | 1 | 1 |
| 2 | 2 | 2 | 1 | 2 |
| 3 | 1 | 4 | 1 | 4 |
| 4 | 1 | 3 | 2 | 3 |

| h | a | b | c | d |
|---|---|---|---|---|
| a | b | b | c | b |
| b | a | d | d | a |
| c | b | a | c | a |
| d | d | b | c | a |

Interpret the diagrams as follows. If $m, n \in \{1, 2, 3, 4\}$, to calculate the value of $g(m, n)$, go to row $m$ and column $n$. For example, $g(1, 2) = 3$. Similarly for $h$. Show that $(\{1, 2, 3, 4\}, g) \not\equiv (\{a, b, c, d\}, h)$.
c. Let $\mathcal{L} = \{\mathsf{R}\}$ where $\mathsf{R}$ is a 3-ary relation symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-stucture where $M = \mathbb{R}$ and $\mathsf{R}^{\mathcal{M}}$ is the "betweeness relation", i.e. $\mathsf{R}^{\mathcal{M}} = \{(a, b, c) \in \mathbb{R}^3 : \text{Either } a \leq b \leq c \text{ or } c \leq b \leq a\}$. Let $\mathcal{N}$ be the $\mathcal{L}$-stucture where $N = \mathbb{R}^2$ and $\mathsf{R}^{\mathcal{N}}$ is the "collinearity relation", i.e. $\mathsf{R}^{\mathcal{N}} = \{((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in (\mathbb{R}^2)^3 : \text{There exists } a, b, c \in \mathbb{R} \text{ with either } a \neq 0 \text{ or } b \neq 0 \text{ such that } ax_i + by_i = c \text{ for all } i\}$. Show that $\mathcal{M} \not\equiv \mathcal{N}$.