Math 125 A – Fall 2013

Homework 2: Due Wednesday, September 19

Required Problems

Definition: Let $\Gamma_1, \Gamma_2 \subseteq Sent^p$. We say that Γ_1 and Γ_2 are *semantically equivalent* if $\Gamma_1 \vDash \psi$ for all $\psi \in \Gamma_2$ and $\Gamma_2 \vDash \varphi$ for all $\varphi \in \Gamma_1$.

Problem 1:
a. Show that the following are equivalent for $\Gamma_1, \Gamma_2 \subseteq Sent^p$.

1. Γ_1 and Γ_2 are semantically equivalent.
2. For all $\theta \in Sent^p$, we have $\Gamma_1 \vDash \theta$ if and only if $\Gamma_2 \vDash \theta$.

b. Show that if Γ is finite, then Γ has an independent semantically equivalent subset.
c. Show that there exists a set P and an infinite set $\Gamma \subseteq Sent^p$ which has no independent semantically equivalent subset.

Definition: Let $\Gamma \subseteq Sent^p$. We say that Γ is *independent* if there is no $\varphi \in \Gamma$ such that $\Gamma \backslash \{\varphi\} \vDash \varphi$.

Definition: We define $depth \colon Sent^P \to \mathbb{N}$ by recursion:

$$\begin{aligned} depth(A) &= 0, \quad \text{for } A \in P, \\ depth(\neg\varphi) &= depth(\varphi) + 1, \\ depth((\varphi \vee \psi)) &= \max(depth(\varphi), depth(\psi)) + 1. \end{aligned}$$

Problem 2: Let $P = \{\mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3, \mathsf{A}_4, \mathsf{A}_5, \mathsf{A}_6\}$. Show that there exists a Boolean function $f \colon \{T, F\}^7 \to \{T, F\}$ such that $Depth(\varphi) \geq 5$ for all φ with $B_\varphi^7 = f$.
Hint: Do a counting argument. Get an upper bound on the number of formulas φ with $Depth(\varphi) \leq 4$.

Problem 3: Recall that $(\varphi \to \psi)$ is defined as $(\neg\varphi \vee \psi)$. Prove the deduced rules for for $\to$:

$$\frac{\Gamma \vdash \varphi \to \psi}{\Gamma \cup \{\varphi\} \vdash \psi} \ (\to E) \qquad\qquad \frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi} \ (\to I)$$

Problem 4:
a. In class we showed how to obtain one part of the Soundness Theorem easily from the other. Show that this implication can be reversed. That is, give a short proof that "If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$" using the assumption "Every satisfiable set of formulas is consistent".
b. In class we showed (or will show) how to obtain one part of the Completeness Theorem easily from the other. Show that this implication can be reversed. That is, give a short proof that "Every consistent set of formulas is satisfiable" using the assumption "If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$".

Challenge Problems

Problem 1: Show that every countable set is semantically equivalent to an independent set.