Math 125 A – Fall 2013
Homework 1: Due Wednesday, September 11

Required Problems

Problem 1: Consider the following generating system. Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, and $F = \{f, g\}$, where $g: \text{ } U \rightarrow \text{ } U$ is given by

\[
\begin{align*}
g(1) &= 3 & g(2) &= 1 & g(3) &= 3 & g(4) &= 7 & g(5) &= 5 & g(6) &= 1 & g(7) &= 4 \\
\end{align*}
\]
and $f: U^2 \rightarrow U$ is given by

\[
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 3 & 6 & 6 & 2 & 6 & 1 \\
2 & 7 & 2 & 1 & 1 & 2 & 1 & 3 \\
3 & 6 & 6 & 6 & 1 & 2 & 1 & 3 \\
4 & 1 & 4 & 4 & 4 & 4 & 4 & 7 \\
5 & 4 & 5 & 2 & 5 & 7 & 3 & 4 \\
6 & 1 & 7 & 6 & 1 & 2 & 1 & 2 \\
7 & 7 & 6 & 1 & 5 & 5 & 1 & 5 \\
\end{array}
\]

Interpret the diagram as follows. If $m, n \in U$, to calculate the value of $f(m, n)$, go to row $m$ and column $n$. For example, $f(1, 2) = 6$.

a. Determine the set $C$ generated from $B = \{5\}$ using $F$. Justify your answer.

b. Determine the set $C$ generated from $B = \{1\}$ using $F$. Justify your answer.

c. Are any of these sets freely generated?

Problem 2: Let $U = \mathbb{N}^+, B = \{7, 13\}$, and $F = \{g, f\}$ where $g: U \rightarrow U$ is given by $g(n) = 20 \cdot n + 1$ and $f: U^2 \rightarrow U$ is given by $f(n, m) = 2^n(2m + 1)$. Show that the set generated from $B$ using $F$ is freely generated.

Problem 3: Define, for any $\theta, \gamma \in \text{Sent}^P$, a function $\text{Subst}_\gamma^\theta: \text{Sent}^P \rightarrow \text{Sent}^P$ (intuitively substituting $\theta$ for all occurrences of $\gamma$) recursively as follows.

- $\text{Subst}_\gamma^\theta(A) = \begin{cases} 
\theta & \text{if } \gamma = A \\
A & \text{otherwise}
\end{cases}$

- $\text{Subst}_\gamma^\theta(\neg \phi) = \begin{cases} 
\theta & \text{if } \gamma = \neg \phi \\
\neg \text{Subst}_\gamma^\theta(\phi) & \text{otherwise}
\end{cases}$

- $\text{Subst}_\gamma^\theta(\phi \lor \psi) = \begin{cases} 
\theta & \text{if } \gamma = (\phi \lor \psi) \\
\text{Subst}_\gamma^\theta(\phi) \lor \text{Subst}_\gamma^\theta(\psi) & \text{otherwise}
\end{cases}$

Show by induction that if $v: P \rightarrow \{T, F\}$ is a truth assignment with $v(\theta) = v(\gamma)$, then $v(\phi) = v(\text{Subst}_\gamma^\theta(\phi))$ for every $\phi \in \text{Sent}^P$.

Problem 4: Using the induction principle for generating systems, prove that on every sentence $\phi \in \text{Sent}^P$ the number of left parenthesis is equal to the number of right parenthesis.