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Math 125 A – Fall 2013
Midterm 1: September 30

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Name:..

Problem 1: (10 points) Decide whether the following statements are True or False. Circle the right answer. You don't need to justify your answers.

T F Let C be the subset of $\mathbb{Z}$ generated from $\{2,3,4,5,6\}$ using $f(x) = 2x^2+1$ and $g(x) = 4x$. Then C is freely generated.

T F For $\Gamma, \Delta \subseteq Sent^P$, if Γ is finite and Δ are semantically equivalent to Γ, then Δ is semantically equivalent to a finite subset of itself.

T F If $\Gamma \vdash \varphi \wedge \psi$ then $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$.

T F There exists $v\colon P \to \{T, F\}$ such that the set $\Gamma = \{\varphi \in Sent^P : \bar{v}(\varphi) = F\}$ is consistent and complete.

T F A countable partial ordering Q can be written as a union of k chains if and only if every finite subset of Q can be written as a union of k chains.

The rules:
Basic Proofs: $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$.

Rules for $\vee$: We have two rules for introducing $\vee$.

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee IL) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ (\vee IR)$$

Rules for proofs by cases:

$$\frac{\Gamma \cup \{\varphi\} \vdash \theta \quad \Gamma \cup \{\psi\} \vdash \theta}{\Gamma \cup \{\varphi \vee \psi\} \vdash \theta}\ (\vee PC) \qquad \frac{\Gamma \cup \{\psi\} \vdash \varphi \quad \Gamma \cup \{\neg\psi\} \vdash \varphi}{\Gamma \vdash \varphi}\ (\neg PC)$$

Rule for proof by contradiction:

$$\frac{\Gamma \cup \{\neg\varphi\} \vdash \psi \quad \Gamma \cup \{\neg\varphi\} \vdash \neg\psi}{\Gamma \vdash \varphi}\ (Contr)$$

Problem 2: (10 points) In class use gave an informal proof that if $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$. Write down a proof of this fact using induction on the size of the proof-tree for $\Gamma \vdash \varphi$. Write down the proof of the basic step (where you need to consider (BR)). The induction step is split into five cases depending what was the last rule used to derive $\Gamma \vdash \varphi$. Only consider the rules $(\vee IL)$ and $(\vee PC)$, and omit the other cases.

Basic Step: Suppose $\Gamma \vdash \varphi$ was derived by (BR) $\frac{}{\Gamma \vdash \varphi}$ BR

Then $\varphi \in \Gamma$. Let $\Gamma_0 = \{\varphi\}$. then $\frac{}{\Gamma_0 \vdash \varphi}$ BR

Induction step

$(\vee IL)$ Suppose the last rule was $(\vee IL)$ $\qquad \frac{\Gamma \vdash \psi_1}{\Gamma \vdash \psi_1 \vee \psi_2}(\vee I$

Then φ is of the form $\psi_1 \vee \psi_2$ and

By IH, there is $\Gamma_0 \subseteq \Gamma$, finite st $\Gamma_0 \vdash \psi_1$

But then $\frac{\Gamma_0 \vdash \psi_1}{\Gamma_0 \vdash \psi_1 \vee \psi_2} \vee IL$

$(\vee PC)$ Suppose the last rule was $(\vee PC)$. Then $\Gamma = \Gamma' \cup \{\psi_1 \vee \psi_2\}$

and $\frac{\Gamma', \psi_1 \vdash \varphi \qquad \Gamma', \psi_2 \vdash \varphi}{\Gamma' \cup \{\psi_1 \vee \psi_2\} \vdash \varphi}(\vee PC)$

By IH, there are finite sets $\Gamma_{0,1} \subseteq \Gamma', \psi_1$ and $\Gamma_{0,2} \subseteq \Gamma', \psi_2$

such that $\Gamma_{0,1} \vdash \varphi$ and $\Gamma_{0,2}'' \vdash \varphi$. Let $\Gamma_0' = \frac{(\Gamma_{0,1} \setminus \{\psi_1\}) \cup (\Gamma_{0,2} \setminus \{\psi_2\})}{}$

By the subset rule $\Gamma_0', \psi_1 \vdash \varphi$ and $\Gamma_0', \psi_2 \vdash \varphi$

Then $\Gamma_0', \psi_1 \vee \psi_2 \vdash \varphi$ (by $\vee PC$)

Let $\Gamma_0 = \Gamma_0' \cup \{\psi_1 \vee \psi_2\}$ which is finite and

$\Gamma_0 \vdash \varphi$.

Problem 3: (5 points) Write down a syntactical derivation of $A \vdash \neg\neg A$ using only the rules (BR), $(\neg PC)$, and $(Contr)$.
(Hint: try using $(\neg PC)$ as the last rule.)

$$\dfrac{\dfrac{\dfrac{}{A, \neg A, \neg\neg\neg A \vdash A}\,(BR) \qquad \dfrac{}{A, \neg A, \neg\neg\neg A \vdash \neg A}\,(BR)}{A, \neg A \vdash \neg\neg A}\,(Contr.) \qquad \dfrac{}{A, \neg\neg A \vdash \neg\neg A}\,(BR)}{A \vdash \neg\neg A}\,(\neg PC)$$

Problem 4: (5 points) Consider a new connective $*$ such that for a valuation v,

$$\bar{v}(\varphi * \psi) = T \iff \bar{v}(\varphi) = F \text{ and } \bar{v}(\psi) = F.$$

Show that for every n-place Boolean function f, there is a sentence φ which only uses the connective $*$ and no other connective at all, such that $f = B^n_\varphi$.

semantically equivalent

Observe that $\neg\varphi \models\!\!\!\dashv \varphi * \varphi$ ①

$\varphi \vee \psi \models\!\!\!\dashv \neg(\varphi * \psi)$. ②

We know there is φ using $\neg, \vee$ st $B^n_\varphi = f$

Replace each $\vee$ by the shorthand given in ②

Replace each $\wedge$ " " " " " ①

We end up with an equivalent formula which only uses $*$.