Math 125 A – Fall 2013

Homework 7: Due Friday, November 1st

Problem 1: Let $\mathcal{L} = \{\leq\}$. Let φ_{DLO} be the axiom for dense linear orderings without endpoints. That is, φ_{DLO} is the conjunction of the axioms for linear orderings, a sentence saying that for any two elements there is one in between them, and the sentence saying that for every element there is another one to the left and another one to the right.

Prove that φ_{DLO} is complete, that is that for every sentence φ, either $\varphi_{DLO} \models \varphi$ or $\varphi_{DLO} \models \neg\varphi$.

Problem 2:

Let $\mathcal{Z} = (\mathbb{Z}, 0, S, P)$, where S is the successor function, $S(n) = n + 1$, and P is the predecessor function, $P(n) = n - 1$. Let $\mathcal{Z}_2 = (\mathbb{Z}^2, \vec{0}, S_0, P_0)$, where $\vec{0} = (0, 0)$, $S_0(n, m) = (n + 1, m)$ and $P_0(n, m) = (n - 1, m)$.

Consider the embedding $h\colon \mathcal{Z} \to \mathcal{Z}_2$ given by $h(n) = (n, 0)$. Prove that h is an elementary embedding (that is, that the image of h is an elementary substructure of $\mathcal{Z}_2$).

Hint: Prove that $\mathcal{Z}_1$ is an elementary substructure of a structure isomorphic to $\mathcal{Z}_2$.

Problem 3:

Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times, \leq)$.

(a) Let $\mathcal{M}$ be such that $\mathcal{N} \preccurlyeq \mathcal{M}$ and $\mathbb{N} \subsetneq M$. Given $a, b \in M$, define the following relation. Let $a << b$ if there are infinitely many elements between a and b in M. Show that if $a << b$ then there exits $c \in M$ such that $a << c$ and $c << b$.

Hint: think of the average.

(b) Prove that there exits $\mathcal{A}$ with $\mathcal{N} \preccurlyeq \mathcal{A}$ for which there exits $a \in A$ such that for every $n \in \mathbb{N}$, $\mathcal{A} \models$" n divides a".

Problem 4:

(a) Suppose that $\mathcal{A} \preccurlyeq \mathcal{B}$ and not equal. Prove that the set A, as a subset of B, is not definable in $\mathcal{B}$.

(b) Let $\mathcal{C}$ be a structure such that for every $a \in C$, there is an $\mathcal{L}$-term t without variables such that $t^{\mathcal{C}} = a$. Prove that if $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{C} \equiv \mathcal{D}$, then $\mathcal{C} \preccurlyeq \mathcal{D}$.

HW 7 Sketches of solutions

(1) If $T_{DLO} \not\models \varphi$ then $T_{DLO} \wedge \neg\varphi$ is satisfiable by some $\mathcal{L}_1$

By Down LST, $\mathcal{L}_1$ can be taken to be countable

If $T_{DLO} \not\models \neg\varphi$, then $T_{DLO} \wedge \varphi$ is satisfiable by some $\mathcal{L}_2$

By Down LST, $\mathcal{L}_2$ can be taken to be countable.

We know that any two countable DLO are isomorphic

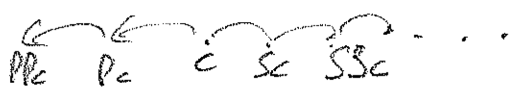
$\mathcal{L}_1 \cong \mathcal{L}_2$ but $\mathcal{L}_1 \models \neg\varphi$ and $\mathcal{L}_2 \models \varphi$, contradiction

So either $T_{DLO} \models \varphi$ or $T_{DLO} \models \neg\varphi$

(2) Let $M \succeq \mathbb{Z}$ be uncountable (it exists by Up LST)

M consists of an uncountable set of Z-chains.

If $c \in M \setminus \mathbb{Z}$, then $\forall n$ $S^n c$ and $P^n c \in M$



We know that $\forall x \; SS\dots S^n(x) = x$ → because it's true in $\mathbb{Z}$

and $\forall x \; SP(x) = x$

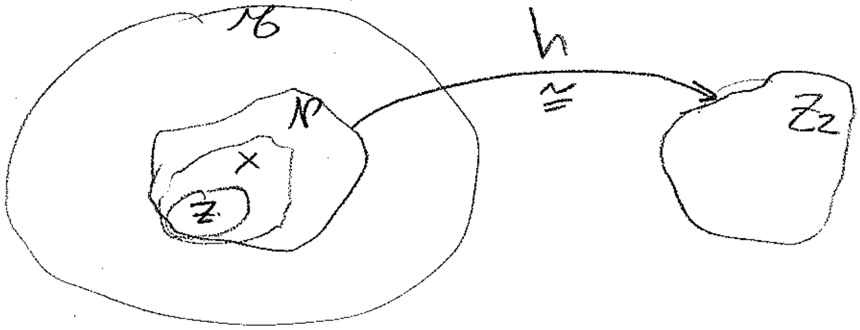
So M has no loops

Let $X \subseteq M$ consists of Z together with countably infinite other chains.

Let $N \preceq M$, $X \subseteq N$ (by Down LST)

The $\mathbb{Z} \subseteq N$ and since $N \preceq M \succeq \mathbb{Z}$, one can prove $\mathbb{Z} \preceq N$

Then one needs to prove that $N' \cong \mathbb{Z}_2$



(3)(a) in $\mathbb{N}$, we know that $\forall a,b \in \mathbb{N}$ the number $\left[\frac{a+b}{2}\right]$ exists

that is $N' \models \forall x,y \, \exists z \, (z+z = x+y \vee z+z+1 = x+y)$
Hence it's true in M.
Take $a, b \in M$, $a \ll b$.
Then exists $c \in M$ such that $c+c = a+b$ or
$c+c = a+b-1$

Then $a < c < b$ and $c-a = b-c$
or $c-a = b-c+1$

by elementary

Then $a \not\ll c \iff c-a \in \mathbb{N} \iff b-c \in \mathbb{N} \iff c \not\ll b$.

Since either $a \ll c$ or $c \ll b$ (because otherwise there are only finitely many elements between a and b)

we have that both $a \ll c$ and $c \ll b$.

3(b) Let $L' = L \cup \{c\}$
Let $\Gamma = \mathrm{Diag}(N) \cup \{\exists x\,(x \times \overbrace{(1+1+\dots+1)}^{n \text{ times}} = c) : n \in \mathbb{N}\}$
Γ is finitely satisfiable $\Rightarrow$ Γ satisfiable $\Rightarrow$

(4) (a) Suppose A is defined by $\varphi(x)$

then $\forall a \in A \quad \mathcal{B} \models \varphi(a)$

$\forall b \in B \setminus A \quad \mathcal{B} \models \neg\varphi(b)$

But since $\mathcal{A} \preceq \mathcal{B}$, $\forall a \in A \quad \mathcal{A} \models \varphi(a)$

$\Rightarrow \mathcal{A} \models \forall x \varphi(x) \Rightarrow \mathcal{B} \models \forall x \varphi(x)$ contradiction

(b) To prove $\mathcal{C} \preceq \mathcal{D}$ take $\varphi(x, x_1 \ldots x_k)$ and $c_1 \ldots c_k \in \mathcal{C}$

s.t. $\mathcal{D} \models \exists x \varphi(x, c_1 \ldots c_k)$

Let $t_1 \ldots t_k$ be s.t. $t_i^{\mathcal{C}} = c_i$. Since $\mathcal{C} \subseteq \mathcal{D}$, $t_i^{\mathcal{D}} = t_i^{\mathcal{C}} = c_i$

Then $\mathcal{D} \models \underbrace{\exists x \varphi(x\, t_1 \ldots t_k)}_{\text{Sentence}}$

Since $\mathcal{C} \equiv \mathcal{D}$, $\quad \mathcal{C} \models \exists x \varphi(x, t_1 \ldots t_k)$

$\Rightarrow \exists c \in \mathcal{C} \quad \mathcal{C} \models \varphi(c\, t_1 \ldots t_k)$

and $t \in Term_L \quad t^{\mathcal{C}} = c \Rightarrow \mathcal{C} \models \varphi(t, t_1 \ldots t_k)$

but then $\mathcal{D} \models \varphi(t, t_1 \ldots t_k)$

$\Rightarrow \mathcal{D} \models \varphi(c, c_1 \ldots c_k)$ as needed for the Tarski criterion