Math 125 A – Fall 2013

Homework 3: Due Wednesday, September 25

Required Problems

Problem 1: Decide whether the following statements are **True** or **False**. Circle the right answer. You don't need to justify your answers.

T **F** There is $\Gamma \subseteq Sent^P$ which is consistent, but such that for any other sentence $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent.

T **F** If P is finite, there is a finite sequence $\varphi_0, ..., \varphi_n \in Sent^P$ such that any other $\psi \in Sent^P$ is semantically equivalent to one of them.

T **F** If P is finite, then $Sent^P$ is finite.

T **F** If $\Gamma \vdash \varphi \vee \psi$ then either $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$.

T **F** If Γ and Δ are satisfiable, then so is $\Gamma \cup \Delta$

Problem 2: Prove soundness for the rules $(\rightarrow E)$ and $(\rightarrow I)$ from the previous homework.
That is, assume these two rules were actual rules. Then, as in the proof of soundness that we did in class, we should consider the case when the *last rule applied* was one of these. Prove these cases.

Problem 3: Suppose that $\theta \vdash \gamma$ and $\gamma \vdash \theta$. Show that if $\Gamma \vdash \varphi$, then $\Gamma \vdash Subst^{\theta}_{\gamma}(\varphi)$.
Recall the function $Subst^{\theta}_{\gamma}$ from HW1. *Hint:* Staying only on the syntactic side is not advisable.

Problem 4: Suppose that we eliminate the *Contr* rule. Show that the Completeness Theorem no longer holds.
Hint: Notice that this rule is the only one that lets you get something new or smaller in the right hand side of $\Gamma \vdash \varphi$. State precisely what this gives you, and how it establishes the result.

HW3

Problem 2 Suppose the last rule in the derivation of $\Gamma \vdash \varphi$ is $(\to E)$. Therefore there is a $\psi \in \Gamma$ s.t. the last derivation is $\dfrac{\Gamma' \vdash \psi \to \varphi}{\Gamma \vdash \varphi}$ where $\Gamma = \Gamma' \cup \{\psi\}$.

By I.H., $\Gamma' \models \psi \to \varphi$. Take $v : P \to \{T, F\}$ that makes Γ true.

Then v satisfies Γ' and hence $\bar{v}(\psi \to \varphi) = T$.
Also $\bar{v}(\psi) = T$ (because $\psi \in \Gamma$) and hence $\bar{v}(\varphi) = T$ as wanted.

Suppose now the last rule was $(\to I)$
So φ is of the form $\psi_1 \to \psi_2$ and the last rule was

$$\frac{\Gamma \cup \{\psi_1\} \vdash \psi_2}{\Gamma \vdash \psi_1 \to \psi_2}$$

By I.H. $\Gamma \cup \{\psi_1\} \models \psi_2$.
Take v satisfying Γ.
If $\bar{v}(\psi_1) = F$, then $\bar{v}(\varphi) = \bar{v}(\psi_1 \to \psi_2) = T$
If $\bar{v}(\psi_1) = T$, then v satisfies Γ, ψ_1, and since $\Gamma, \psi_1 \models \psi_2$ $\bar{v}(\psi_2) = T$
It follows that, in any case, $\bar{v}(\psi_1 \to \psi_2) = T$

Problem 3 In Prob 3 of H1 we showed that if $\bar{v}(\theta) = \bar{v}(\gamma)$, then $\bar{v}(\varphi) = \bar{v}(\mathrm{Subst}^{\theta}_{\gamma}(\varphi))$

Now we have $\theta \vdash \gamma$ and $\gamma \vdash \theta$, and we want to show that if $\Gamma \vdash \varphi$, then $\Gamma \vdash \mathrm{Subst}^{\theta}_{\gamma}(\varphi)$.

We'll show that $\Gamma \models \mathrm{Subst}^{\theta}_{\gamma}(\varphi)$ which is enough by completeness.

Take v satisfying Γ.

By soundness $\Gamma \models \varphi$, $\theta \models \gamma$ and $\gamma \models \theta$.

Therefore $\bar{v}(\varphi) = T$ and $\bar{v}(\theta) = \bar{v}(\gamma)$.

It follows that $v(\mathrm{Subst}^{\theta}_{\gamma}(\varphi)) = T$ as wanted.

Problem 4

We claim that, without (Contr), $\neg\neg A \nvdash A$, even though we know that $\neg\neg A \vDash A$.

Suppose toward a contradiction that $\neg\neg A \vdash A$.

Let Γ be ~~the~~ a set of sentences such that

- $\Gamma \vdash A$ with the shortest possible deduction
- $A \notin \Gamma$
- no formula of the form $\varphi \vee \psi$ is in Γ.

Some such set exists because $\{\neg\neg A\} = \Gamma$ works.

The last rule in the deduction of $\Gamma \vdash A$

- can't be (BR) because $A \notin \Gamma$
- can't be ($\vee$IL) or ($\vee$IR) because A is not of the form $\varphi \vee \psi$.
- can't be ($\vee$PC) because Γ contains no sentences of thee form $\varphi \vee \psi$
- can't be ($\neg$PC) because the last rule would be
$$\frac{\Gamma, \psi \vdash A \quad \Gamma, \neg\psi \vdash A}{\Gamma \vdash A}$$
and $\Gamma, \neg\psi \vdash A$ would have a shorter deduction contradicting the choice of Γ.