Problem 1 \hspace{1cm} (2) \Rightarrow (1)

Suppose \( p \equiv \emptyset \). We want to show \( P_2 \equiv \emptyset \).

Take \( \nu : P \rightarrow \{T,F\} \) satisfying \( P_2 \).

By (1) \( P_2 \subset P_1 \), so \( \nu \) satisfies \( P_1 \).

Since \( P_1 \equiv \emptyset \), \( \nu(\emptyset) = T \). Thus \( P_2 \equiv \emptyset \).

That \( P_2 \equiv \emptyset \Rightarrow P_1 \equiv \emptyset \) is analogous.

By (2)

(2) \Rightarrow (1) Take \( \theta \in P_1 \), then \( P_1 \equiv \emptyset \), then \( P_2 \equiv \emptyset \).

Analogously, if \( \theta \in P_2 \), \( P_1 \equiv \emptyset \).

(2) Let \( \Delta \subset \Gamma \) be smallest such that \( \Delta \equiv \emptyset \).

Since \( \Gamma \) is finite, such a smallest set exists.

Claim \( \Delta \) independent: if not, there is \( \Phi \in \Delta \) such that \( \Delta \cdot \Phi \equiv \emptyset \). But then \( \Delta \cdot \{\emptyset\} \equiv \emptyset \).

\( \Delta \) and \( \Delta \cdot \{\emptyset\} \) are equivalent, contradicting that \( \Delta \) is smallest.

(3) Let \( \Gamma = \{A_0, A_0 \wedge A_1, A_0 \wedge A_1 \wedge A_2, \ldots \} \)

\( \Gamma \)
Problem 2

On the one hand
\[ | \{ f : \{T,F\}^3 \to \{T,F\} \} | = 2^3 = 2^{128} \]

On the other hand:

Let \( D_i = \{ p \in \text{Sent}^P : \text{depth}(p) \leq i \} \).

and \( N_i = |D_i| \)

\[ D_0 = \{ \lambda, A, \ldots, A_6 \} \quad \text{and} \quad N_0 = 7 \]

\[ D_{i+1} = D_i \cup \{ \eta \in \text{Sent}^P : \eta \in D_i 3 \cup \exists \gamma : \gamma \in \text{Sent}^P, \eta \eta \gamma \in D_i \} \]

\[ N_1 = 7 + 7 + 7^2 = 63 < 64 = 2^6 \]

\[ N_{i+1} = N_i + N_i + N_i^2 \leq 2N_i^2 \]

\[ N_2 \leq 2N_1^2 < 2 \cdot (2^6)^2 = 2^{13} \]

\[ N_3 \leq 2N_2^2 < 2 \cdot (2^{13})^2 = 2^{27} \]

\[ N_4 \leq 2N_3^2 < 2 \cdot (2^{27})^2 = 2^{54} \]

\[ N_4 \leq 2^{27} \] (there are many other ways to prove this bound)

Thus, by pigeonhole principle, there is \( f : \{T,F\}^3 \to \{T,F\} \) such that for every \( p \in D_4 \), \( f \neq B_p \).

\[ \therefore \text{if } f = B_p, \text{ depth}(p) \geq 5 \]
Problem 3

\[
\begin{align*}
\Gamma, \phi &\vdash \psi & \text{(Cont)} \quad \frac{\Gamma, \phi, \psi \vdash \Gamma, \phi} \text{BR} \\
\Gamma, \phi &\vdash \psi & \text{(Subset)} \quad \frac{\Gamma, \phi, \psi \vdash \Gamma, \phi} \text{BR} \\
\Gamma, \phi &\vdash \psi & \text{(MP)} \quad \frac{\Gamma, \phi, \psi \vdash \Gamma, \phi} \text{BR} \\
\Gamma, \phi &\vdash \psi & \text{(VI)} \quad \frac{\Gamma, \phi, \psi \vdash \Gamma, \phi} \text{VI} \\
\Gamma, \phi &\vdash \psi & \text{(VIR)} \quad \frac{\Gamma, \phi, \psi \vdash \Gamma, \phi} \text{VIR} \\
\end{align*}
\]
Problem 4

Recall that

\[ \Gamma \vdash \varphi \iff \Gamma, \varphi \text{ inconsistent} \]
\[ \Gamma \vdash \varphi \iff \Gamma, \varphi \text{ not satisfiable.} \]

(a) \[ \Gamma \vdash \varphi \Rightarrow \Gamma, \varphi \text{ inconsistent} \ implies \ \Gamma, \varphi \text{ not satisfiable} \Rightarrow \Gamma \vdash \varphi \]

(b) Suppose \( \Gamma \) not satisfiable.

Then for any \( \varphi, \ \Gamma \vdash \varphi \) and \( \Gamma \vdash \neg \varphi \).

Then \( \Gamma \vdash \varphi \) and \( \Gamma \vdash \neg \varphi \) by (1).

The \( \Gamma \) inconsistent.