HW 2

Problem 1 (a) (1) $\Rightarrow$ (2)

Suppose $\Gamma_1 \models \theta$. We want to show $\Gamma_2 \models \theta$.

Take $v : P \to \{T, F\}$ satisfying Γ_2

By (1) $\Gamma_2 \models \Gamma_1$, so v satisfies Γ_1

Since $\Gamma_1 \models \theta$, $v(\theta) = T$. Thus $\Gamma_2 \models \theta$.

That $\Gamma_2 \models \theta \Rightarrow \Gamma_1 \models \theta$ is analogous.

(2) $\Rightarrow$ (1) Take $\theta \in \Gamma_1$, then $\Gamma_1 \models \theta$, then $\Gamma_2 \models \theta$ (by (2)).

Analogously, if $\theta \in \Gamma_2$, $\Gamma_1 \models \theta$.

(b) Let $\Delta \subseteq \Gamma$ be smallest such that Δ equivalent to Γ

Since Γ is finite, such a smallest set exists.

Claim Δ independent: if not, there is $\varphi \in \Delta$

s.t. $\Delta \setminus \{\varphi\} \models \varphi$. but then

Δ and $\Delta \setminus \{\varphi\}$ are equivalent

contradicting that Δ is smallest

(c) Let $P = \{A_0, A_1, A_2, \ldots\}$

$\Gamma = \{A_0, A_0 \wedge A_1, A_0 \wedge A_1 \wedge A_2, \ldots\}$

HW2

Problem 2

On the one hand $\left|\{f : \{T,F\}^7 \to \{T,F\}\}\right| = 2^{2^7} = 2^{128}$

On the other hand:

Let $D_i = \{\varphi \in Sent^P : depth(\varphi) \leq i\}$.

and $N_i = |D_i|$

So $D_0 = \{A_0, \dots A_6\}$ and $N_0 = 7$

$D_{i+1} = D_i \cup \{\neg\varphi : \varphi \in D_i\} \cup \{\varphi \vee \psi : \varphi, \psi \in D_i\}$

$N_1 = 7 + 7 + 7^2 = 63 < 64 = 2^6$

$N_{i+1} = N_i + N_i + N_i^2 \leq 2N_i^2$

So $N_2 \leq 2N_1^2 < 2 \cdot (2^6)^2 = 2^{13}$

$N_3 \leq 2N_2^2 < 2(2^{13})^2 = 2^{27}$

$N_4 \leq 2N_3^2 < 2 \cdot (2^{27})^2 = 2^{54}$

So $N_4 \leq 2^{2^7}$ (there are many other ways to prove this bound.)

Thus by pigeonhole principle, there is $f : \{T,F\}^7 \to \{T,F\}$ such that for every $\varphi \in D_4$, $f \neq B_\varphi$

So, if $f = B_\varphi$, $depth(\varphi) \geq 5$

Problem 3

$(\to E)$

$$\dfrac{\dfrac{\Gamma \vdash \varphi \to \psi}{\Gamma, \varphi \vdash \neg\varphi \vee \psi}\,(\text{Subset}) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \varphi, \neg\varphi, \neg\psi \vdash \varphi}\,\text{BR} \quad \dfrac{}{\Gamma, \varphi, \neg\varphi, \neg\psi \vdash \neg\varphi}\,\text{BR}}{\Gamma, \varphi, \neg\varphi \vdash \psi}\,(\text{Contr}) \qquad \dfrac{}{\Gamma, \varphi, \psi \vdash \psi}\,\text{BR}}{\Gamma, \varphi, \neg\varphi \vee \psi \vdash \psi}\,\vee\text{PC}}{\Gamma, \varphi \vdash \psi}\,(\text{MP})$$

$(\to I)$

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\varphi \vdash \neg\varphi}\,(\text{BR})}{\Gamma, \neg\varphi \vdash \neg\varphi \vee \psi}\,\vee\text{IL} \qquad \dfrac{\Gamma, \varphi \vdash \psi}{\Gamma, \varphi \vdash \neg\varphi \vee \psi}\,(\vee\text{IR})}{\Gamma \vdash \underbrace{\neg\varphi \vee \psi}_{\varphi \to \psi}}\,\neg\text{PC}$$

Problem 4

Recall that $\Gamma \vdash \varphi \iff \Gamma, \neg\varphi$ inconsistent

$\Gamma \models \varphi \iff \Gamma, \neg\varphi$ not satisfiable.

(a) $\Gamma \vdash \varphi \Rightarrow \Gamma, \neg\varphi$ incon $\overset{\text{by } 2}{\Rightarrow} \Gamma, \neg\varphi$ not satisfiable $\Rightarrow \Gamma \models \varphi$

(b) Suppose Γ not satisfiable

Then for any φ, $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$

The $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ by 1

The Γ inconsistent.