Math 125 A – Fall 2013

Midterm 2: November 4

Name:.........Solutions.................. /30

Problem 1: (10 points) Decide whether the following statements are True or False. Circle the right answer. You don't need to justify your answers.

T **F** Once we prove compactness, the notions of elementary class and of weakly elementary class become equivalent.

T **F** If an embedding between two $\mathcal{L}$-structures is onto, it is an elementary embedding.

T **F** Let $\mathcal{L}$ be countable. Then every infinite $\mathcal{L}$-structure has a proper elementary substructure (proper means not equal).

T **F** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. Then $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \equiv \mathcal{B}$.

T **F** Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent if and only if $Th(\mathcal{N}) \subseteq Th(\mathcal{M})$.

Problem 3: (6 points)
Let $\Sigma = \{\psi_n : n \in \mathbb{N}\} \subseteq Sent_{\mathcal{L}}$ be such that $\varphi_{n+1} \models \varphi_n$ for every n. Prove that if $Mod(\Sigma)$ is elementary, then there exists some n such that $\models \varphi_n \leftrightarrow \varphi_{n+1}$.

Suppose $Mod(\Sigma)$ is elementary and $Mod(\Sigma) = Mod(\tau)$. $\Rightarrow \Sigma \models\dashv \tau$

Then, by thm from class, $\Sigma \models\dashv \Sigma_0$ for some finite $\Sigma_0 \subseteq \Sigma$

Let $n_0 =$ largest n, $\varphi_n \in \Sigma_0$.

~~Th~~ Since $\varphi_{n_0} \models \varphi_n \ \forall n \leq n_0$, $\varphi_{n_0} \models\dashv \Sigma_0 \models\dashv \Sigma$

So $\varphi_{n_0} \models \varphi_{n+1}$

Problem 3: (8 points) Consider $\mathcal{Q} = (\mathbb{Q}, +)$.

(a) Find four subsets of $\mathbb{Q}$ which are definable in $\mathcal{Q}$, and give the formulas that define them.

(b) Prove that there are no other subsets of $\mathbb{Q}$ which are definable in $\mathcal{Q}$.

(a) - $\{0\} \subseteq \mathbb{Q}$ definable by $x + x = x$

- $\mathbb{Q} \setminus \{0\}$ definable by $x + x \neq x$

- $\mathbb{Q}$ definable by $x = x$

- ϕ definable by $x \neq x$

(b) Suppose $D \subseteq \mathbb{Q}$ is definable.

If $D \subseteq \{0\}$ then either $D = \phi$ or $D = \{0\}$

So, suppose $D \nsubseteq \{0\}$ and $d \in D \setminus \{0\}$.

Take any $q \in \mathbb{Q} \setminus \{0\}$. The $h(p) = \frac{qp}{d}$ is an automorphism of $(\mathbb{Q}, +)$

So $q = h(d) \in D$

Thus $\mathbb{Q} \setminus \{0\} \subseteq D$. Hence either $\mathbb{Q} \setminus \{0\} = D$ or $\mathbb{Q} = D$

Problem 4: (6 points) Let $\mathcal{L} = \{\leq\}$, and let $\Sigma \subset Sent_{\mathcal{L}}$ be such that all the models of Σ are linear orderings. Suppose also that for every $n \in \mathbb{N}$, there is an $\mathcal{A} \models \Sigma$ such that $|A| \geq n$.

Prove that for every linear ordering $\mathcal{C} = (C, \leq^{\mathcal{C}})$ there is an $\mathcal{A} \models \Sigma$ for which there exists an $\mathcal{L}$-embedding of $\mathcal{C}$ into $\mathcal{A}$.

Let $\mathcal{L}' = \{\leq\} \cup \{c_b : b \in C\}$

Let $\Gamma = \Sigma \cup \{ \text{"} c_b \leq c_{b'} \text{"} : b, b' \in C, \ b \leq^{\mathcal{C}} b' \}$

Γ is finitely satisfiable, because if $\Gamma_0 \subseteq \Gamma$ is finite it mentions only finitely many constants $c_{b_0}, \ldots, c_{b_K}$.

Then, let $A \models \Sigma$ with $|A| > b_K$

We can interpret $c_{b_0} \ldots c_{b_K}$ in A so that they are in the correct ordering and get a model of Γ_0.

So, by compactness, Γ is satisfiable.

Let $A' \models \Gamma$ and let A be the restriction of A' to L.

The $A \models \Sigma$ and the map $h : \mathcal{C} \to A$, $h(b) = c_b^{A'}$ is an order preserving embedding.

Problem 5: (5 points)

Let $\mathcal{L}$ be a countable language and $\mathcal{M}$ an infinite $\mathcal{L}$-structure. Show that there exists a countable $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$ and $M \neq N$.

Use Up LST to get $\mathcal{N}' \succcurlyeq \mathcal{M}$, $\mathcal{N}'$ uncountable

Let $X = M \cup \{c\}$ for some $c \in N' \setminus M$. (note X is countable)

Use Down LST to get $\mathcal{N}$, $X \subseteq N$, $\mathcal{N}$ countable

$\mathcal{N} \preccurlyeq \mathcal{N}'$

Then $\mathcal{M} \preccurlyeq \mathcal{N}$ and $c \in N \setminus M$.