Precalculus - Lecture Notes

March 30, 2026

Contents

These notes heavily follow the textbook: Mathematics for Calculus, 8th Edition, James Stewart; Lothar Redlin; Saleem Watson, ISBN-13: 978-0-357-75363-7.

0 Course Introduction and Syllabus

This is a course in precalculus which essentially covers two topics. 1) Functions and 2) Trigonometry. The goal of this course is to prepare you for success in calculus. Most calculus students do not struggle with calculus; they struggle with the precalculus problem that most calculus problems reduce to.

The course webpage is https://math.berkeley.edu/∼andrewshi/teaching/math9-sp26/math9-sp26.html. This is where all the course material is, including the syllabus and policies, course notes, discussion problems, HW, quizzes, and exams.

Lectures will be given Monday and Wednesdays, and starting next week, pen-and-paper quizzes will be given on Fridays in the first 30 minutes. The remainder of the Friday recitation section will first be dedicated to answering any questions you may have, and with the remaining time I will work out some problems on the board. I will try to give as many examples as possible during the lectures, but it will really be Friday (and really, on your own time) where more complex examples can be worked out. Lecture notes and the recitation problems are posted on the course webpage. My hope is that this frees you up to concentrate and try to follow and participate in the lectures rather than mindlessly scribing everything I write on the board.

My office hours are immediately after class (just hold me after), and my appointment either before class or in the evenings via zoom. This is not a large class, so participation is highly encouraged during lectures and recitations.

The ARC can be a great resource. The ARC Learning Assistant to our class is Bryant Paulino. You can check out the ARC website to see when his drop in hours are what the rules of the ARC are. Experience shows that students who make the time an effort to "get stuck", articulate questions, and try to answer them themselves and then get help from others learn the most. In other words, you should be going to office hours or Bryant with your own agenda of things you encounter difficulty with and want to discuss. Note that you should not be contacting the ARC LA Bryant by email or make appointments with him – these are outside the scope of his duties. I do not know if you can speak with other LAs who are assigned to different courses – it might be possible in theory if they are free but not guaranteed.

I made the conscious decision to not use the software Webassign. Typically in the past, this class has used Webassign for homework and quizzes. However, I believe the problems in Webassign are usually too simple and not representative of the course material. Also, Webassign can be frustrating to use as it is quite finicky in how you must input answers in a particular manner to get credit. In exchange, I had to write a large about of course material from scratch (so there may be typos, please help me find them). Also, this should hopefully save you some money.

Homework is assigned but not turned in or graded. I basically just highlight all of the "groups" of problems you should be able to do on a quiz or an exam. Of course it is not always feasible to do every problem, but do enough so that you feel comfortable with each particular type of problem. The most important part of this class is to practice a lot on your own. It is deceptively easy to watch me work out a problem on the board and fool yourself into thinking that is equivalent to being able to do it yourself. The book is excellent and has lots of great problems. You should read the book and do as many practice problems as possible. Almost all of the problems I use in lecture, on Fridays, on quizzes, and on exams are almost exclusively from the book.

The quiz on Fridays will cover the material from the Monday and Wednesday of the previous week. There will be 10 quizzes, but two will be dropped in the computation of your final grade. There are no makeup or early quizzes given. Quizzes make up 20% of your grade. Closed book, closed notes, closed calculator.

There will be two midterms and one final. The two midterms each make up 20% of your grade and the final makes up 40% of your grade. No calculators, but you can bring one "cheat sheet" of A4 sized paper. There are no makeup or early midterms. However, your final exam grade can override one or both of your midterm grades if it is higher. You cannot miss the final exam and pass the class.

In terms of grading, you will be graded no harsher than the standard cutoffs. However, it is quite possible I will lower the cutoffs or curve the grades upwards. I will give you an idea of where you stand in terms of letter grades after each midterm. Note that on the midterm a maximum score of 110% (11 problems with 10 points each) and on the final a maximum score of 116% (14 problems with 12 points each) will be possible. Letter grades will represent how I feel about your readiness for calculus – a C being the minimum threshold that I think you could move on to calculus and pass. There is no distribution of grades I have in mind. Everyone could get an A or everyone could get an F.

It is tempting to heavily utilize ChatGPT/Gemini or other AI tools to for help or to show you how to get started. Oftentimes this hurts more often than it helps since getting started is typically the hardest part. I do not recommend you use AI tools to generate practice problems. The problems in the book are especially curated so that if you do most or all of them within a single block of problem type, you will encounter all of the nuances or subtleties associated with a particular problem type. AI tools will not do so (yet).

Finally, if you have a disability that you require accommodation for, please let me know as soon as possible so I can make the necessary preparations. At the same time, please get your disability letter from the Moses center as soon as possible. If you have any religious holidays that conflict with a quiz or an exam date, please notify me within the first two weeks of the class so we can make alternative arrangements.

Let me close with this. This is a difficult class. Basically all precalculus students take the course because they "have" to, not because they want to. I don't expect this to be anyone's favorite class, and it is quite likely that many of you have had difficulties with mathematics to some extent in the past. This is also a very small class. Please interrupt me as often as needed to get clarification or to ask me to go over something again or in a different way. This is what the entire point of having a class is (vs say, having you watch Youtube videos). I will do everything I can to make the experience as positive as possible, but ultimately it is up to you to do lots of problems, identify weaknesses, and rectify them.

Questions? If not, let's get started.

FIRST DAY SURVEY

What is your name? (Or preferred name if not your official one). What year are you? Where are you from?

What is your major (or intended major)?

How would you describe your prior math background?

What concerns if any, do you have about taking this course?

This is not a question – rather I want to communicate again there is anything I can do to make the course better, please do not hesitate to let me know. I will solicit feedback from you again around the first midterm.

1 The Fundamentals §1

1.1 Real Numbers, Sets, Absolute Values §1.1

The **natural numbers** are given by

$$1, 2, 3, 4, \ldots$$

The **integers** consist of the natural numbers together with their negatives and 0. Sometimes we refer to these as whole numbers.

$$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

The **rational numbers** refer to the set of numbers that can be expressed as the ratio of integers. So we have

$$r = \frac{m}{n}$$

where any rational number r can be expressed by the ratio of integers m, n where $n \neq 0$ since we cannot divide by 0.

The **irrational numbers** refer to the set of all numbers that are not rational, that is, cannot be expressed as the ratio of two integers. Examples include:

$$\sqrt{2}, \sqrt{3}, \pi, \frac{3}{\pi^2}.$$

In general it is difficult to show a number is irrational and this is outside the scope of this course. Take as a fact that the square root of any positive number that is not a perfect square is an irrational number.

How do we define the **real numbers**? One way is to define them as the set of all rational and irrational numbers. Another way is to contrast them to the set of numbers that is "not real" (otherwise known as imaginary). This is in §1.6 in the section called "Complex numbers" but we will not cover them in this course.

Figure 1.1 below gives a classification of the real numbers.

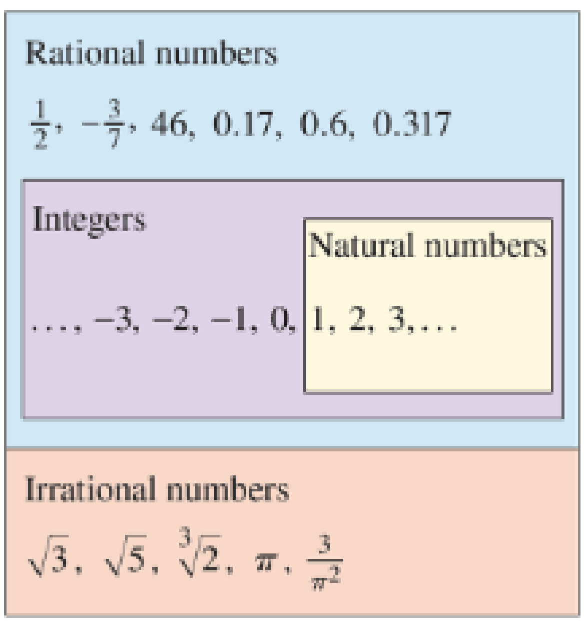


Figure 1: Classification of Real Numbers

Every real number has a decimal representation. If the number is rational, then the corresponding decimal eventually repeats. For example:

1. $\frac{1}{2} = 0.5$
2. $\frac{2}{3} = 0.666666\cdots = 0.\overline{6}$
3. $\frac{157}{495} = 0.317171717\cdots = 0.3\overline{17}$
4. $\frac{9}{7} = 1.285714285714\cdots = 1.\overline{285714}$

where the overbar indicates the sequence of digits repeats forever. If possible, leaving your answer as a fraction is almost always preferable to decimal form. In particular, if an answer is 2/3 and you write 0.666, this is technically not correct.

If the number is irrational, the decimal representation is nonrepeating and seems to have no real order or pattern

$$\begin{aligned}\sqrt{2} &= 1.414213562373095..\\ \pi &= 3.141592653589793..\end{aligned}$$

Truncating the decimal expansion of number creates an approximation of that number. For instance:

$$\pi \approx 3.14159$$

where $\approx$ is read in English as "approximately equal to".

Real numbers follow a set of properties:

- Commutative properties: $a + b = b + a, ab = ba$
- Associative properties: $(a + b) + c = a + (b + c), (ab)c = a(bc)$
- Distributive properties: $a(b + c) = ab + ac, (b + c)a = ab + ac$

Example 1.1: Using the distribute property

Apply the distributive property to the following two quantities:

1. $2(x + 3)$
2. $(a + b)(x + y)$

Solution:

1. $$\begin{aligned}2(x + 3) &= 2 \cdot x + 2 \cdot 3\\ &= 2x + 6\end{aligned}$$

2. Some of you may already know how to do this with "FOIL".

$$\begin{aligned}(a + b)(x + y) &= (a + b)x + (a + b)y\\ &= ax + bx + ay + by\end{aligned}$$

Negative numbers also satisfy a set of properties:

- $(-1)a = -a$

- $-(-a) = a$
- $(-a)b = a(-b) = -(ab)$
- $(-a)(-b) = ab$
- $-(a+b) = -a - b$
- $-(a-b) = -a + b$

Now we move onto multiplication and division of fractions. In a fraction $\dfrac{a}{b}$ which means a divided by b, we have that a is the **numerator** and b is the **denominator**. They follow these properties:

- $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$
- $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
- $\frac{ac}{bc} = \frac{a}{b}$ if $c \neq 0$.
- If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Example 1.2: LCD to add fractions

Evaluate $\frac{5}{36} + \frac{7}{120}$.

Solution: We need to find a common denominator so we can directly add the numerators. An easy way to find a common denominator is to take 36×120 to get

$$\frac{5}{36} + \frac{7}{120} = \frac{5(120)}{36(120)} + \frac{7(36)}{120(36)} = \frac{852}{4320} = \frac{71}{360}$$

It is preferable to find the least common denominator. Factoring each denominator into prime factors gives us

$$36 = 2^2 \cdot 3^2$$
$$120 = 2^3 \cdot 3 \cdot 5$$

so the LCD is formed by the product of the prime factors that occur in these factorizations, using the highest power in each prime factor. Thus the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$.

$$\frac{5}{36} + \frac{7}{120} = \frac{50}{360} + \frac{21}{360} = \frac{71}{360}$$

Now we move onto sets. A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S.

For example, if $\mathbb{Z}$ represents the set of integers, then $-3 \in \mathbb{Z}$ but $\pi \notin \mathbb{Z}$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

A can also be written in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which can be read as "A is the set of all x such that x is an integer and $0 < x < 7$."

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in either in S or T or in both. The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. In other words $S \cap T$ is the common part of S and T.

The **empty set**, denoted by $\emptyset$, is the set that contains no elements.

Example 1.3: Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}, T = \{4, 5, 6, 7\}$ and $V = \{6, 7, 8\}$, find the sets $S \cup T, S \cap T, S \cap V$.

Solutions:

1. $S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$
2. $S \cap T = \{4, 5\}$
3. $S \cap V = \emptyset$

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond to geometrically to line segments.

If $a < b$, then the **open interval** from a to b concists of all numbers between a and b and is denoted by (a, b). The **closed interval** from a and b includes the endpoints and is denoted by $[a, b]$.

Using set-builder notation:

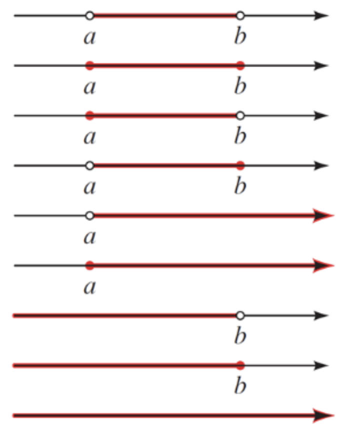
$$(a, b) = \{x \mid a < x < b\} \qquad [a, b] = \{x \mid a \leq x \leq b\}$$

Note that the parenthesis () in the interval notation and open circles on the graph indicate that endpoints are excluded from the interval:

Figure 2: The open interval (a, b)

whereas square brackets and solid circles indicate that the endpoints are included.

Figure 3: The closed interval $[a, b]$

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	
$[a, b)$	$\{x \mid a \leq x < b\}$	
$(a, b]$	$\{x \mid a < x \leq b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \leq x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

The below table shows many different possibilities where intervals can be represented with set builder notation as well as graphically on the number line. You can also have half closed (or half open) set like $(a, b]$. Note that when one side of the interval is ∞ or $-\infty$ it is convention to use open brackets.

Example 1.4: Union and Intersection of Intervals

Graph each set.

1. $(1, 3) \cap [2, 7]$
2. $(1, 3) \cup [2, 7]$

Solutions:

1. The intersection of two intervals consists of the numbers that are in both intervals.

$$\begin{aligned}(1,3) \cap [2,7] &= \{x \mid 1 < x < 3 \text{ and } 2 \leq x \leq 7\} \\ &= \{x \mid 2 \leq x < 3\} = [2,3)\end{aligned}$$

2. The union of two intervals consists of the numbers that are in either one interval or the other.

$$\begin{aligned}(1,3) \cup [2,7] &= \{x \mid 1 < x < 3 \text{ or } 2 \leq x \leq 7\} \\ &= \{x \mid 1 < x \leq 7\} = (1,7]\end{aligned}$$

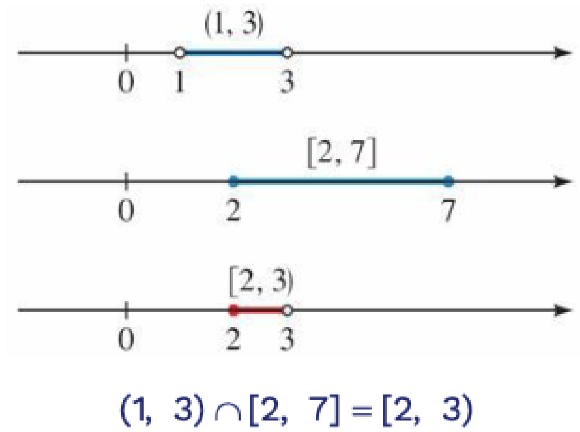


Figure 4: Solution to 1.4

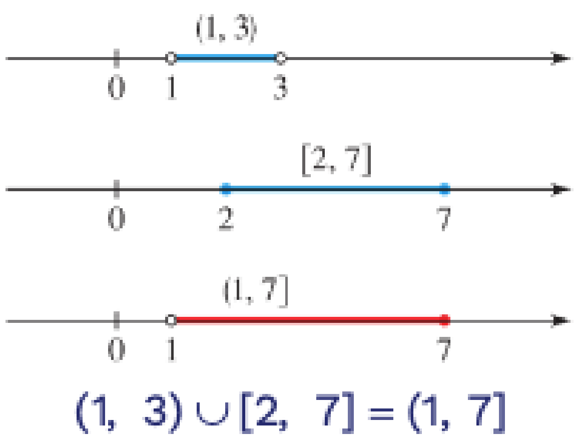


Figure 5: Solution to 1.4

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.

Since distance is always positive or zero, $|a| \geq 0$ for every number a.

If a is a real number, then the absolute value of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example 1.5: Using Absolute Values

Evaluate each quantity.

1. $|3|$
2. $|-3|$
3. $|0|$
4. $|3-\pi|$

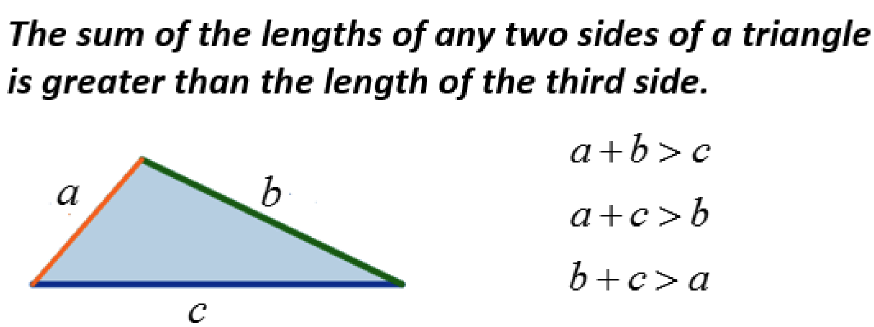
Solutions:

1. $|3| = 3$
2. $|-3| = -(-3) = 3$
3. $|0| = 0$
4. $|3-\pi| = -(3-\pi) = \pi - 3$. This is because $3 < \pi \implies 3 - \pi < 0$

Here are some properties of absolute values:

- $|a| \geq 0$
- $|a| = |-a|$
- $|ab| = |a||b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|a+b| \leq |a| + |b|$

The last property is known as the triangle inequality.



The distance between two numbers on the real line $d(a, b)$ is given by

$$d(a, b) = |b - a|$$

For example, the distance between the numbers -8 and 2 is given by

$$\begin{aligned} d(a, b) &= |2 - (-8)| \\ &= |-10| \\ &= 10 \end{aligned}$$

Example 1.6: Using Absolute Values II

Compute

$$|-5 + |4 - 8||$$

Solutions:

$$|-5 + |4 - 8|| = |-5 + |-4|| = |-5 + 4| = |-1| = 1$$

1.2 Exponents and Radicals §1.2

If a is any real number and n is a positive integer, then the nth power of a is given by

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

We call a the **base** and n the **exponent**.

Example 1.7: Exponents

Simplify the following expressions

1. $\left(\frac{1}{2}\right)^5$
2. $(-3)^4$
3. -3^4

Solution:

1. $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2^5} = \frac{1}{32}$
2. $(-3)^4 = (-3)(-3)(-3)(-3) = 81$
3. $-3^4 = -(3)(3)(3)(3) = -81$

Note that $(-3)^4 \neq -3^4$. This is an order of operations thing.

Now we look at the case of zero and negative exponents. If we have $a \neq 0$ and n a positive integer, then we have

$$a^0 = 1 \text{ and } a^{-n} = \frac{1}{a^n}$$

This may seem a bit unmotivated for now, but we will see shortly why these special cases *have to* take on these values in order to be consistent with the laws of exponents.

Example 1.8: Zero and Negative Exponents

Simplify the following expressions

1. $\left(\frac{4}{7}\right)^0$
2. x^{-1}
3. $(-2)^{-3}$

Solution:

1. $\left(\frac{4}{7}\right)^0 = 1$
2. $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$
3. $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$

Now we present the laws of exponents. Let the bases a, b be real numbers and m, n integers.

1. $a^m a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$

4. $(ab)^n = a^n b^n$

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

6. $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$

7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

Let's discuss these in detail. Seek to memorize as little as possible and understand, digest, and internalize as much as possible. For the first rule, we have

1.
$$\begin{aligned} a^m a^n &= \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ times}} \cdot \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ times}} \\ &= \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m+n \text{ times}} \\ &= a^{m+n} \end{aligned}$$

2.
$$\begin{aligned} \frac{a^m}{a^n} &= \frac{\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ times}}}{\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}} \\ &= \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m-n \text{ times}} \\ &= a^{m-n} \end{aligned}$$

3.
$$\begin{aligned} (a^m)^n &= \underbrace{a^m \cdot a^m \cdot a^m \cdot \cdots \cdot a^m}_{n \text{ times}} \\ &= a^{\overbrace{m + m + m + \cdot + m}^{n \text{ times}}} \text{ (by the first law of exponents)} \\ &= a^{mn} \end{aligned}$$

4.
$$\begin{aligned} (ab)^n &= \underbrace{ab \cdot ab \cdot ab \cdot \cdots \cdot ab}_{n \text{ times}} \\ &= \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n\text{times}} \cdot \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{n\text{times}} \text{ by multiple uses of commutativity} \\ &= a^n b^n \end{aligned}$$

5. By direct expansion.

6. Reduces to the previous once you apply the definition of negative exponents.

7. An application of the definition of negative exponents twice.

Example 1.9: Application of the Laws of Exponents

Using the laws of exponents, simplify the following expressions.

1. $3^2 \cdot 3^5$
2. $\frac{3^5}{3^2}$
3. $(3^2)^5$
4. $(3 \cdot 4)^2$
5. $\left(\frac{3}{4}\right)^2$
6. $\left(\frac{3}{4}\right)^{-2}$
7. $\frac{3^{-2}}{4^{-5}}$

Solutions:

1. $3^2 \cdot 3^5 = 3^{2+5} = 3^7$
2. $\frac{3^5}{3^2} = 3^{5-2} = 3^3$
3. $(3^2)^5 = 3^{2\cdot 5} = 3^{10}$
4. $(3 \cdot 4)^2 = 3^2 \cdot 4^2$
5. $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$
6. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$
7. $\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$ Usually leaving something like this perfectly fine – no need to multiply out.

Example 1.10: Simplfying expressions with exponents

Using the laws of exponents, simplifying the following expressions. Do not leave any negative exponents.

1. $(2a^3b^2)(3ab^4)^3$
2. $\left(\frac{x}{y}\right)^3 \left(\frac{y^2x}{z}\right)^4$
3. $\dfrac{6st^{-4}}{2s^{-2}t^2}$
4. $\left(\dfrac{y}{3z^3}\right)^{-2}$

1.

$$\begin{aligned}(2a^3b^2)(3ab^4)^3 &= (2a^3b^2)(3^3a^3b^{12}) \\ &= 2(3^3)a^3a^3b^2b^{12} = 54a^6b^{14}\end{aligned}$$

2.

$$\begin{aligned}\left(\frac{x}{y}\right)^3 \left(\frac{y^2x}{z}\right)^4 &= \frac{x^3}{y^3}\left(\frac{y^8x^4}{z^4}\right) \\ &= \frac{x^7y^8}{y^3z^4} = \frac{x^7y^5}{z^4}\end{aligned}$$

3.

$$\frac{6st^{-4}}{2s^{-2}t^2} = \frac{6ss^2}{2t^2t^4} = \frac{3s^3}{t^6}$$

4.

$$\left(\frac{y}{3z^3}\right)^{-2} = \left(\frac{3z^3}{y}\right)^2 = \frac{9z^6}{y^2}$$

Now we move to radicals. As an example, we have previously defined the quantity 2^n where n is an integer. Now we look at the case when n is a rational number such as $2^{4/5}$.

We denote the symbol $\sqrt{a}$ as the positive square root of a. Thus we have

$$\sqrt{a} = b \implies b^2 = a \text{ and } b \geq 0$$

The symbol $\sqrt{a}$ only makes sense when $a \geq 0$. For example we have $\sqrt{9} = 3$ because $3^2 = 9$ and $3 \geq 0$. In this class we are not interested in quantities like $\sqrt{-9}$ since there is no real number you can square that will be negative. These are known as imaginary numbers (instead of real numbers) which we will not discuss in this class.

We denote that nth root of a as follows:

$$\sqrt[n]{a} = b \implies b^n = a$$

We can also write the nth root in terms of exponents as follows:

$$\sqrt[n]{a} = a^{1/n}$$

which means that these radicals inherit all the properties of exponents. Frequently it is simpler to work in terms of exponents instead of radicals.

1. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
4. $\sqrt[n]{a^n} = a$ if n is odd.
5. $\sqrt[n]{a^n} = |a|$ if n is even.

A brief justification of these properties:

1. Rewrite radicals in terms of exponents and use rules of exponents. $\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a}\,\sqrt[n]{b}$.
2. Same principle as before.
3. $\sqrt[m]{\sqrt[n]{a}} = (a^{1/n})^{1/m} = a^{1/mn} = \sqrt[mn]{a}$.
4. We previously discussed we can't take square roots of negative numbers. This concept extends to all even roots of negative numbers. However, odd roots of negative numbers are perfectly acceptable and result in odd numbers. For example $\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$.
5. As an example. $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$.

Here is a commonly used "non-property" by students:

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

This is VERY false. Practically any two numbers are a counter-example, e.g. $\sqrt{2} = \sqrt{1+1} \neq \sqrt{1} + \sqrt{1} = 2$

Example 1.11: Application of the Laws of Radicals

Using the laws of exponents and radicals, simplify the following expressions.

1. $\sqrt[3]{-8 \cdot 27}$
2. $\sqrt[4]{\frac{16}{81}}$
3. $\sqrt{\sqrt[3]{729}}$
4. $\sqrt[3]{(-5)^3}$
5. $\sqrt[4]{(-3)^4}$

Solutions:

1. $\sqrt[3]{-8 \cdot 27} = \sqrt[3]{(-8)}\sqrt[3]{27} = -2 \cdot 3 = -6$
2. $\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$
3. $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$ (It's good to learn to recognize common powers of numbers, but I will try not to force you to do so especially on quizzes and exams, especially since no calculators are allowed).
4. $\sqrt[3]{(-5)^3} = -5$
5. $\sqrt[4]{(-3)^4} = |-3| = 3$

Example 1.12: More Application of the Laws of Radicals

Using the laws of exponents, simplify the following expressions.

1. $\sqrt[3]{x^4}$
2. $\sqrt[4]{81x^8y^4}$
3. $\sqrt{32} + \sqrt{200}$
4. $\sqrt{25b} - \sqrt{b^3}$ if $b > 0$
5. $\sqrt{49x^2 + 49}$
6. $a^{1/3}a^{7/3}$
7. $\frac{a^{2/5}a^{7/5}}{a^{3/5}}$
8. $(2a^3b^4)^{3/2}$
9. $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\frac{y^4}{x^{-1/2}}\right)$
10. $\left(\frac{a}{3}\right)^{-3} a^5$
11. $\sqrt[3]{-64x^6y^8}$

Solution:

1. $\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = \sqrt[3]{x^3}\,\sqrt[3]{x} = x\sqrt[3]{x}$ (This came from the book, but I don't really consider it simplifying).
2. $\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\,\sqrt[4]{x^8}\,\sqrt[4]{y^4} = 3x^2|y|$
3. $\sqrt{32} + \sqrt{200} = \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2} = \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2} = 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$
4. $\sqrt{25b} - \sqrt{b^3} = \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b} = 5\sqrt{b} - b\sqrt{b} = (5-b)\sqrt{b}$
5. $\sqrt{49x^2 + 49} = \sqrt{49(1+x^2)} = \sqrt{49}\sqrt{1+x^2} = 7\sqrt{1+x^2}$
6. $a^{1/3}a^{7/3} = a^{1/3+7/3} = a^{8/3}$
7. $\frac{a^{2/5}a^{7/5}}{a^{3/5}} = \frac{a^{9/5}}{a^{3/5}} = a^{6/5}$
8. $(2a^3b^4)^{3/2} = 2^{3/2}(a^3)^{3/2}(b^4)^{3/2} = 2^{3/2}a^{9/2}b^6$
9. $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\frac{y^4}{x^{-1/2}}\right) = \frac{2^3(x^{3/4})^3}{y} \cdot y^4x^{1/2} = \frac{8x^{9/4}}{y} \cdot y^4x^{1/2} = 8x^{11/4}y^3$
10. $\left(\frac{a}{3}\right)^{-3} a^5 = \frac{27}{a^3}a^5 = 27a^2$
11. $\sqrt[3]{-64x^6y^8} = (-64x^6y^8)^{1/3} = (-64)^{1/3}x^2y^{8/3} = -4x^2y^{8/3}$

1.3 Algebraic Expressions and Factoring §1.3

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots a_n$ are real numbers and n is a nonnegative integer. If $a_n \neq 0$ then the polynomial has **degree** n.

For example, the following are not polynomials:

$$x^3 + 2x^2 + x^{1/2} \qquad\qquad 3x^{10} + 2x^3 + x^{-2}$$

the former due to a noninteger exponent, and the latter due to a negative exponent.

When it comes to adding and subtracting polynomials, all we need to do is group like terms.

Example 1.13: Adding and Subtracting Polynomials

1. $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$
2. $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$

Solution:

1. $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x) = (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4 = 2x^3 - x^2 - 5x + 4$
2. $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x) = (x^3 - x^3) + (-6x^2 - 5x^2) + (2x - (-7x)) + 4 = -11x^2 + 9x + 4$

The multiplication of polynomials (or any other algebraic expressions) hinges on the repeated use of the distributive property:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

Example 1.14: Multiplying Polynomials

1. $(2x + 3)(x^2 - 5x + 4)$
2. $(x^2 - 5x + 3)(x + 2)$

Solution:

1.

$$\begin{aligned}(2x + 3)(x^2 - 5x + 4) &= 2x(x^2 - 5x + 4) + 3(x^2 - 5x + 4)\\ &= 2x^3 - 10x^2 + 8x + 3x^2 - 15x + 12\\ &= 2x^3 - 7x^2 - 7x + 12\end{aligned}$$

2.

$$\begin{aligned}(x^2 - 5x + 3)(x + 2) &= (x^2 - 5x + 3)x + (x^2 - 5x + 3)(2)\\ &= x^3 - 5x^2 + 3x + 2x^2 - 10x + 6\\ &= x^3 - 3x^2 - 7x + 6\end{aligned}$$

These next few product formulas (especially the first three) are critical. For now we are just focused on expanding the LHS to the RHS. In a few minutes we will discuss the more difficult direction, which is recognizing something is of the form of the RHS and go to the LHS.

1. $(A+B)(A-B) = A^2 - B^2$
2. $(A+B)^2 = A^2 + 2AB + B^2$
3. $(A-B)^2 + A^2 - 2AB + B^2$
4. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
5. $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

Confirm these yourself.

- For the first, we have $(A+B)(A-B) = A^2 - AB + AB - B^2 = A^2 - B^2$. The cross terms cancel out.
- For the second, we have $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$. The cross terms combine.
- ...

Example 1.15: Expanding out polynomials

Expand the following expressions

1. $(3x+5)^2$
2. $(x^2-2)^3$
3. $(x^2+y^3)^2$

Solution: We could do these by scratch, or utilize the formulas.

1. Let $A = 3x, B = 5$, and utilize the second formula, we have
$$(3x+5)^2 = (3x)^2 + 2(3x)(5) + 5^2 = 9x^2 + 30x + 25$$
2. Let $A = x^2, B = 2$, and utilize the fifth formula, we have
$$(x^2-2)^3 = (x^2)^3 - 3(x^2)^2(2) + 3(x^2)(2)^2 - 2^3 = x^6 - 6x^4 + 12x^2 - 8$$
3. Let $A = x^2, B = y^3$ and use the second formula, we have
$$(x^2)^2 + 2(x^2)(y^3) + (y^3)^2 = x^4 + 2x^2y^3 + y^6$$

Expanding polynomials out is the easy half – the hard part is doing the reverse: factoring. There are some special algebraic expressions for this – the first three of which are simply the distribution formulas in reverse:

1. $A^2 - B^2 = (A-B)(A+B)$
2. $A^2 + 2AB + B^2 = (A+B)^2$
3. $A^2 - 2AB + B^2 = (A-B)^2$
4. $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$
5. $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

Example 1.16: Factoring Polynomials

Factor each expression

1. $3x^2 - 6x$
2. $8x^4y^2 + 6x^3y^3 - 2xy^4$
3. $(2x + 4)(x - 3) - 5(x - 3)$

Solution:

1. $3x^2 - 6x = 3x(x - 2)$
2. $8x^4y^2 + 6x^3y^3 - 2xy^4 = 2xy^2(4x^3 + 3x^2y - y^2)$
3. The two terms have the common factor $x - 3$, so $(2x + 4)(x - 3) - 5(x - 3) = (2x + 4 - 5)(x - 3) = (2x - 1)(x - 3)$

Now we come to the topic of factoring trinomials of the form

$$x^2 + bx + c$$

THIS IS THE MOST IMPORTANT SKILL OF THIS CLASS. Note that $(x + r)(x + s) = x^2 + (r + s)x + rs$, so it is necessary to choose numbers r, s such that $r + s = b$ and $rs = c$. Note that this may not always be possible, and even when it is they might not be nice whole numbers.

Let's first try factoring $x^2 + 7x + 12$ via trial and error. So we must pick two integers such that their sum is 7 and their product is 12. The possible numbers whose product are 12 is 1 and 12, 2 and 6, and 3 and 4. In this case we do not need to consider the pairs in reverse order because r and s are interchangeable. Later we will see when they are not and the reverse pairs must be considered.

Note we do not need to consider the pairs of negative numbers whose product is 12 because their sum is positive 7. Considering all the possibilities makes it clear the numbers that satisfy these two criterion are 3 and 4, so we have

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Example 1.17: More Factoring Examples

Factor each expression completely.

1. $x^2 - 2x - 3$
2. $(5a + 1)^2 - 2(5a + 1) - 3$
3. $x^2 + 6x + 9$
4. $4x^2 - 4xy + y^2$
5. $27x^3 - 1$
6. $x^6 + 8$
7. $2x^4 - 8x^2$
8. $x^5y^2 - xy^6$

Solution:

1. We would like to factor this term as the product of two monomials:

$$x^2 - 2x - 3 = (x + r)(x + s) = x^2 + (r + s)x + rs$$

So we need to find two numbers r, s such that $rs = -3$ and $r + s = -2$. The possible pairs of numbers who have product -3 are $r = -1, s = 3$ and $r = 1, s = -3$. We do not need to consider the pairs in reverse $(3, -1)$ and $(-3, 1)$ because r and s are interchangeable here. We require the sum of those two numbers to be -2 so it must be $r = 1, s = -3$, giving us

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

When you factor polynomials, it is always a good idea to multiply them out to see if you can recover the original expression as a means of checking your work.

2. If you prefer, you can let $y = (5a+1)$ so the original expression then looks like $y^2-2y-3 = (y-3)(y+1)$ which we know how to factor. Resubstituting in $y = 5a + 1$ gives us

$$\begin{aligned}(5a + 1)^2 - 2(5a + 1) - 3 &= (5a + 1 - 3)(5a + 1 + 1)\\ &= (5a - 2)(5a + 2)\end{aligned}$$

3. We can use the same method of trial and error with r, s as before, or recognize this as a perfect square of the form $(A + B)^2 = A^2 + 2AB + B^2$ where $A = x, B = 3$, so we have

$$x^2 + 6x + 9 = (x + 3)^2$$

4. Trial and error. We know it must take the form $(4x+ry)(x+sy)$ or $(2x+ry)(2x+sy)$. We know that $rs = 1$, so we already know $r = s = -1$ since the cross term is negative. So we can try out both:

$$(4x - y)(x - y) = 4x^2 - 5y + y^2 \times$$

$$(2x - y)(2x - y) = 4x^2 - 4y + y^2 \checkmark$$

So actually we could have recognized this as a perfect square.

5. We can recognize this as a perfect cube of the form $A^3 - B^3$ where $A = 3x, B = 1$, giving us

$$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)((3x)^2 + (3x)(1) + 1^2) = (3x - 1)(9x^2 + 3x + 1)$$

For now, it is difficult to factor cubic (and higher) equations except when you can recognize them as a perfect cube or it can be factored by grouping (right after this). We will discuss factoring cubic (and higher) equations in §3.3 − §3.4 in week 6.

6. We can recognize this as the sum of cubes formula with $A = x^2, B = 2$, giving us:

$$x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$$

7. First factor out the power of x with the smallest exponent as well as any constants

$$\begin{aligned}2x^4 - 8x^2 &= 2x^2(x^2 - 4)\\ &= 2x^2(x - 2)(x + 2)\end{aligned}$$

8. First factor out the powers of x and y with the smallest exponents

$$\begin{aligned}x^5y^2 - xy^6 &= xy^2(x^4 - y^4)\\ &= xy^2(x^2 - y^2)(x^2 + y^2)\\ &= xy^2(x^2 + y^2)(x + y)(x - y)\end{aligned}$$

Finally, sometimes the situation is serendipitous enough so that we can factor by grouping.

Example 1.18: Factoring by Grouping

Factor each expression completely.

1. $x^3 + x^2 + 4x + 4$
2. $x^3 - 2x^2 - 9x + 18$

Solutions:

1. $$\begin{aligned} x^3 + x^2 + 4x + 4 &= (x^3 + x^2) + (4x + 4) \\ &= x^2(x + 1) + 4(x + 1) \\ &= (x^2 + 4)(x + 1) \end{aligned}$$

2. $$\begin{aligned} x^3 - 2x^2 - 9x + 18 &= (x^3 - 2x^2) + (-9x + 18) \\ &= x^2(x - 2) - 9(x - 2) \\ &= (x^2 - 9)(x - 2) \\ &= (x - 3)(x + 3)(x - 2) \end{aligned}$$

1.4 Rational Expressions §1.4

We can take the quotient of two algebraic expressions such as the following:

$$\frac{2x}{x-1}, \qquad \frac{y-2}{y^2+4}, \qquad \frac{x^3-x}{x^2-5x+6}, \qquad \frac{x}{\sqrt{x^2+1}}.$$

A **rational expression** is a fraction where both the numerator and the denominator are polynomials. For example, in the above the first three expressions are rational expressions, but the fourth one is not because $\sqrt{x^2+1}$ is not a polynomial.

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of all real numbers that the variable is permitted to have. For example:

- $\frac{1}{x}$ has the domain $\{x \mid x \neq 0\}$
- $\sqrt{x}$ has the domain $\{x \mid x \geq 0\}$
- $\frac{1}{\sqrt{x}}$ has the domain $\{x \mid x > 0\}$

Example 1.19: Finding the Domain

Find the domains of the following expressions

a) $2x^2+3x-1$

b) $\dfrac{x}{x^2-5x+6}$

c) $\dfrac{\sqrt{x}}{x-5}$

Solution:

a) Polynomials are defined for any x. So we say that the domain of a polynomial is all real numbers, which is denoted by the symbol $\mathbb{R}$.

b) We cannot divide by zero. Factoring the denominator gives us $x^2-5x+6=(x-2)(x-3)$. So the domain is $x \neq 2, 3$. In interval notation this is $(-\infty,2)\cup(2,3)\cup(3,\infty)$.

c) We must avoid two things: 1) division by 0 and 2) taking square roots of negative numbers. So we have the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$. In interval notation this is $[0,5)\cup(5,\infty)$.

Now we discuss the issue of multiplying and dividing rational expressions. This follows the rules of multiplying and dividing fractions of real numbers.

To multiply rational expressions, we simply multiply their numerators and multiply their denominators

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}.$$

To divide rational expressions, we simply invert the divisor (or in other words, take its reciprocal) and multiply

$$\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}=\frac{AD}{BC}.$$

Example 1.20: Multiplying and Dividing Rational Functions

Perform each of the indicated operations and simplify your answers.

a) $\dfrac{x^2-1}{x^2+x-2}$

b) $\dfrac{x^2+2x-3}{x^2+8x+16} \cdot \dfrac{3x+12}{x-1}$

c) $\dfrac{x-4}{x^2-4} \div \dfrac{x^2-3x-4}{x^2+5x+6}$

Solution:

a)

$$\frac{x^2-1}{x^2+x-2} = \frac{(x-1)(x+1)}{(x-1)(x+2)} = \frac{x+1}{x+2}$$

b)

$$\begin{aligned}\frac{x^2+2x-3}{x^2+8x+16} \cdot \frac{3x+12}{x-1} &= \frac{(x-1)(x+3)}{(x+4)^2} \cdot \frac{3(x+4)}{(x-1)} \\ &= \frac{3(x-1)(x+3)(x+4)}{(x-1)(x+4)^2} \\ &= \frac{3(x+3)}{x+4}\end{aligned}$$

c)

$$\begin{aligned}\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} &= \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \\ &= \frac{x-4}{(x-2)(x+2)} \cdot \frac{(x+2)(x+3)}{(x-4)(x+1)} \\ &= \frac{(x+3)}{(x-2)(x+1)}\end{aligned}$$

When it comes to adding or subtracting rational expressions, we must find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}.$$

Although any common denominator will work, it is best to use the least common denominator.

Example 1.21: Adding and Subtracting Rational Functions

Perform each of the indicated operations and simplify your answers.

a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$

b) $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

Solution:

a) The least common denominator is given by $(x-1)(x+2)$. So we can proceed as

$$\begin{aligned}\frac{3}{x-1}+\frac{x}{x+2} &= \frac{3(x+2)}{(x-1)(x+2)}+\frac{x(x-1)}{(x-1)(x+2)}\\ &= \frac{3x+6+x^2-x}{(x-1)(x+2)} = \frac{x^2+2x+6}{(x-1)(x+2)}\end{aligned}$$

b) The least common denominator of $x^2-1=(x-1)(x+1)$ and $(x+1)^2$ is given by $(x-1)(x+1)^2$.

$$\begin{aligned}\frac{1}{x^2-1}-\frac{2}{(x+1)^2} &= \frac{x+1}{(x-1)(x+1)^2}-\frac{2(x-1)}{(x-1)(x+1)^1}\\ &= \frac{x+1-2x+2}{(x-1)(x+1)^2} = \frac{-x+3}{(x-1)(x+1)^2}\end{aligned}$$

A **compound fraction** is a fraction in which the numerator, denominator, or both contain a fractional expression. This terminology is not that common and I will probably never use it again.

Example 1.22: Simplifying a compound fraction

Simplify the following:

$$\frac{\frac{x}{y}+1}{1-\frac{y}{x}}$$

Solution:

$$\begin{aligned}\frac{\frac{x}{y}+1}{1-\frac{y}{x}} &= \frac{\frac{x+y}{y}}{\frac{x-y}{x}}\\ &= \frac{x+y}{y}\cdot\frac{x}{x-y}\\ &= \frac{x(x+y)}{y(x-y)}\end{aligned}$$

Last topic of the day - rationalizing the denominator (or numerator). If we have an expression like $\frac{1}{1+\sqrt{2}}$, the denominator is irrational due to the presence of $\sqrt{2}$. We would like to "make the denominator rational" by exploiting the following fact:

If you have an expression of the form $A+B\sqrt{C}$, it can be rationalized by multiplying by the **conjugate radical** $A-B\sqrt{C}$ as follows:

$$(A+B\sqrt{C})(A-B\sqrt{C}) = A^2-B^2C$$

Example 1.23: Rationalizing the numerator or denominator

Rationalize the denominator for the first question. Rationalize the numerator for the second and third questions.

a) $\dfrac{1}{1+\sqrt{2}}$

b) $\dfrac{3-\sqrt{2}}{5}$

c) $\dfrac{1}{\sqrt{x}-\sqrt{y}}$

Solution:

a) $$\frac{1}{1+\sqrt{2}} = \frac{(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{(1-\sqrt{2})}{1-2} = \sqrt{2}-1$$

b) $$\frac{3-\sqrt{2}}{5} = \frac{(3-\sqrt{2})(3+\sqrt{2})}{5(3+\sqrt{2})} = \frac{9-2}{5(3+\sqrt{2})} = \frac{7}{5(3+\sqrt{2})}$$

c) $$\frac{1}{\sqrt{x}-\sqrt{y}} = \frac{\sqrt{x}+\sqrt{y}}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{\sqrt{x}+\sqrt{y}}{x-y}$$

1.5 Solving Equations (Linear and Quadratic) §1.5

An **equation** is simply an expression with an equality where there can be constants and or variables on both sides. The idea of solving an equation is to find a value for the unknown(s) which will make the equation true. For now we will only work with single equations with a single variable.

We have two tools when it comes to solving equations

- $A = B \implies A + C = B + C$
- $A = B \implies CA = CB \quad (C \neq 0)$

Adding the same quantity to both sides of an equation gives an equivalent equation. As does multiplication of both sides by a nonzero constant.

A linear equation in one variable is of the form

$$ax + b = 0$$

where a, b are real numbers and x is the variable.

Example 1.24: Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$

Solution:

$$\begin{aligned} 7x - 4 = 3x + 8 &\implies 4x = 12 \\ &\implies x = 3 \end{aligned}$$

The first implication resulted from adding 4 to both sides and subtracting $3x$ from both sides. It is always a good idea to check your work by plugging in your solution to the original equation and see if the equality holds:

$$7(3) - 4 = 3(3) + 8 = 17 \quad \checkmark$$

Now we move onto quadratic equations. A quadratic equation is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, c are real numbers with $a \neq 0$.

We first note a key property: $AB = 0$ if and only if $A = 0$ or $B = 0$ (or both, but at least 1).

Example 1.25: Solve a quadratic equation by factoring

Find all real solutions of the equation $x^2 + 5x = 24$.

Solution: We first rewrite the equation so the RHS is 0 and factor. We need to find a pair of integers whose product is -24 and whose sum is 5. We settle on the pair $3, -8$.

$$\begin{aligned} x^2 + 5x = 24 &= x^2 + 5x - 24 = 0 \\ &= (x + 8)(x - 3) = 0 \\ &= x = 3, -8 \end{aligned}$$

A simple quadratic equation such as $x^2 = c$ can be solved with the solutions $x = \sqrt{c}, -\sqrt{c}$.

Example 1.26: Solving simple quadratics

Find all real solutions of each equation

a) $x^2 = 5$

b) $(x-4)^2 = 5$

Solution:

a) $x^2 = 5 \implies x = \pm\sqrt{5}$

b) $(x-4)^2 = 5 \implies (x-4) = \pm\sqrt{5} \implies x = 4 \pm \sqrt{5}$

To make $x^2 + bx$ be a perfect square, we need to add $\left(\frac{b}{2}\right)^2$ to get

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 1.27: Solving quadratics by completing the square

Find all real solutions of each equation

a) $x^2 - 8x + 13 = 0$

b) $3x^2 - 12x + 6 = 0$

Solution:

a)

$$\begin{aligned} x^2 - 8x + 13 = 0 &\implies x^2 - 8x + \left(-\frac{8}{2}\right)^2 + 13 - \left(-\frac{8}{2}\right)^2 = 0 \\ &\implies x^2 - 8x + 16 + 13 - 16 = 0 \\ &\implies (x-4)^2 - 3 = 0 \\ &\implies x - 4 = \pm\sqrt{3} \\ &\implies x = 4 \pm \sqrt{3} \end{aligned}$$

b)

$$\begin{aligned} 3x^2 - 12x + 6 = 0 &\implies 3(x^2 - 4x + 2) = 0 \\ &\implies 3(x^2 - 4x + 2 + 2) - 6 = 0 \\ &\implies 3(x-2)^2 - 6 = 0 \\ &\implies (x-2)^2 = 2 \\ &\implies x - 2 = \pm\sqrt{2} \\ &\implies x = 2 \pm \sqrt{2} \end{aligned}$$

We also have the quadratic formula

$$ax^2 + bx + c = 0 \;\; (a \neq 0) \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula comes from completing the square:

$$\begin{aligned} ax^2+bx+c=0 &\implies a\left(x^2+\frac{b}{a}x\right)+c=0 \\ &\implies a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)+c-a\left(\frac{b}{2a}\right)^2=0 \\ &\implies a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c \\ &\implies \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a} \\ &\implies \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \\ &\implies x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \\ &\implies x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \end{aligned}$$

Example 1.28: Solving quadratics the quadratic formula

Find all real solutions of each equation.

a) $3x^2-5x-1=0$

b) $4x^2+12x+9=0$

c) $x^2+2x+2=0$

Solution:

a) $x=\frac{5\pm\sqrt{(-5)^2-4(3)(-1)}}{6}=\frac{5\pm\sqrt{37}}{6}$

b) $x=\frac{-12\pm\sqrt{12^2-4(4)(9)}}{2(4)}=\frac{-12}{8}=-\frac{3}{2}$ In fact we can recognize this polynomial as a perfect square $(2x+3)^2=0$

c) $x=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2}=-1\pm\sqrt{-1}$. $\sqrt{-1}$ is not defined in the real number system, so the equation has no real solutions. Rewrite this in standard form to see

$$x^2+2x+2=0 \implies x^2+2x+1=-1 \implies (x+1)^2+1=0$$

So this parabola opens up and has the minimum value $(-1,1)$, so it does not cross the x-axis and as such does not have any real solutions.

The quantity b^2-4ac is known as the **discriminant** of the quadratic equation $ax^2+bx+c=0, \quad a\neq 0$. This quantity determines the nature of the solutions of the quadratic.

1. $b^2-4ac>0$ implies the quadratic has two distinct real roots.
2. $b^2-4ac=0$ implies the quadratic has one repeated real root.
3. $b^2-4ac=0$ implies the quadratic does not have real roots. It has two imaginary roots - we will not get into that. For us it is just not real.

We have other types of equations to consider:

Example 1.29: Solving an equation with fractional expressions

Solve the equation

$$\frac{3}{x} - \frac{2}{x-3} = \frac{-12}{x^2-9}$$

Solution: Eliminate the denominators by multiplying each side by the lowest common denominator $x(x^2-9)$.

$$\begin{aligned}
\frac{3}{x} - \frac{2}{x-3} &= \frac{-12}{x^2-9} \\
\left[\frac{3}{x} - \frac{2}{x-3}\right] x(x^2-9) &= \left[\frac{-12}{x^2-9}\right] x(x^2-9) \\
3(x^2-9) - 2x(x+3) &= -12x \\
3x^2 - 27 - 2x^2 - 6x &= -12x \\
x^2 - 6x - 27 &= -12x \\
x^2 + 6x - 27 &= 0 \\
(x-3)(x+9) &= 0 \\
x &= 3, -9
\end{aligned}$$

Multiplying by an expression that contains the variable can introduce extraneous solutions, so the answers must be checking. $x = 3$ is not a valid solution since that would result in division by 0 in the original expression, so the only valid solution is $x = -9$.

Example 1.30: Solving an equation with a radical

Solve the equation

$$2x = 1 - \sqrt{2-x}$$

Solution: We see a square root and want to eliminate it. If we try to immediately square we get

$$4x^2 = 1 - 2\sqrt{2-x} + (2-x)$$

which spawns another square root. This is no good.

To eliminate the square root, first isolate it on one side of the equals sign and then square both sides.

$$\begin{aligned}
2x - 1 &= -\sqrt{2-x} \\
(2x-1)^2 &= 2 - x \\
4x^2 - 4x + 1 &= 2 - x \\
4x^2 - 3x - 1 &= 0 \\
(4x+1)(x-1) &= 0 \\
x &= -\frac{1}{4}, 1
\end{aligned}$$

Check both solutions by plugging into the original

$$x = -\frac{1}{4} \implies 2\left(-\frac{1}{4}\right) = 1 - \sqrt{\frac{9}{4}} \implies -\frac{1}{2} = -\frac{1}{2} \quad \checkmark$$

$$x = 1 \implies 2(1) = 1 - \sqrt{1} \implies 2 = 0 \quad \text{(NO GOOD)}$$

So the only solution is $x = -\frac{1}{4}$.

As we have seen in the last two examples, sometimes solving an equation may result in extraneous solutions. Extraneous solutions may be introduced when squaring each side of an equation because the operation of squaring can turn a false equation into a true one. For example, if we square the equation $3 = -3$. That is why it is important to check the answers to make sure that each satisfies the original equation.

Now we discuss solving an equation with an absolute value sign.

Example 1.31: Solving an equation with an absolute value

Solve the equation

$$|3x - 8| = 4$$

Solution: We have that $3x - 8$ can either be equal to 4 or -4. So

$$3x - 8 = 4 \implies 3x = 12 \implies x = 4$$

$$3x - 8 = -4 \implies 3x = 4 \implies x = \frac{4}{3}$$

Finally, we will discuss equations of “quadratic type”.

Example 1.32: Solving a quartic of quadratic type

Solve the equation

$$x^4 - 8x^2 + 8 = 0$$

Solution: This is a quartic equation, but we could use the substitution $w = x^2$ to get

$$\begin{aligned} x^4 - 8x^2 + 8 = 0 &\implies w^2 - 8w + 8 = 0 \\ &\implies w = \frac{8 \pm \sqrt{64 - 4(1)(8)}}{2} = 4 \pm 2\sqrt{2} \end{aligned}$$

So since $x^2 = 4 \pm 2\sqrt{2}$, we have four possible solutions $x = \sqrt{4 \pm 2\sqrt{2}}, -\sqrt{4 \pm 2\sqrt{2}}$. Just to be clear, this means there are four solutions: $\sqrt{4 + 2\sqrt{2}}, \sqrt{4 - 2\sqrt{2}}, -\sqrt{4 + 2\sqrt{2}}, -\sqrt{4 - 2\sqrt{2}}$.

Be careful, do not write $x = \pm\sqrt{4 \pm 2\sqrt{2}}$. This refers to the two numbers $x = +\sqrt{4 + 2\sqrt{2}}, -\sqrt{4 - 2\sqrt{2}}$.

1.6 Inequalities §1.8

Previously we have solved equations. Now we are solving inequalities. So in place of the equal sign we may have $>, \geq, <, \leq$ instead.

To solve an inequality, we have the following tools.

1. $A \leq B \implies A + C \leq B + C$
2. $A \leq B \implies A - C \leq B - C$
3. If $C > 0$, then $A \leq B \implies CA \leq CB$
4. If $C < 0$, then $A \leq B \implies CA \geq CB$ (CAREFUL: the sign flips when you multiply or divide by a negative number).
5. If $A > 0$ and $B > 0$, then $A \leq B \iff \frac{1}{A} \geq \frac{1}{B}$
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$
7. If $A \leq B$ and $B \leq C$, then $A \leq C$ (Inequalities are transitive)

Students tend to have the hardest time with #4. For example, we have

$$x > 1 \implies -x < -1.$$

Example 1.33: Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$.

Solution:

$$\begin{aligned} 3x < 9x + 4 &\implies 3x - 9x < 9x + 4 - 9x \qquad \text{(Subtract } 9x \text{ from both sides)} \\ &\implies -6x < 4 \\ &\implies -\left(\frac{1}{6}\right)(-6x) > 4\left(-\frac{1}{6}\right) \qquad \text{(Multiply both sides by } -\frac{1}{6} \text{ and flip the inequality sign)} \\ &\implies x > -\frac{2}{3} \end{aligned}$$

When it comes to solving nonlinear inequalities, we want to take the following steps.

1. Move everything to one side. (Do NOT clear denominators like with equations – we will see why soon).
2. Factor the expression completely (numerator and denominator)
3. Find the intervals of sign change.
4. Make a sign chart and use test values.

Example 1.34: Solving a Quadratic Inequality

Solve the inequality $x^2 \leq 5x - 6$.

Solution:

$$\begin{aligned} x^2 \leq 5x - 6 &\implies x^2 - 5x + 6 \leq 0 \\ &\implies (x-2)(x-3) \leq 0 \end{aligned}$$

The sign of the LHS can only change at the values $x = 2, 3$. So we divide the number line into three portions, knowing that at each of the three portions that sign is unchanged (Figure 1.6).

So we can test each of the intervals by picking a representative point in the interval and checking its sign.

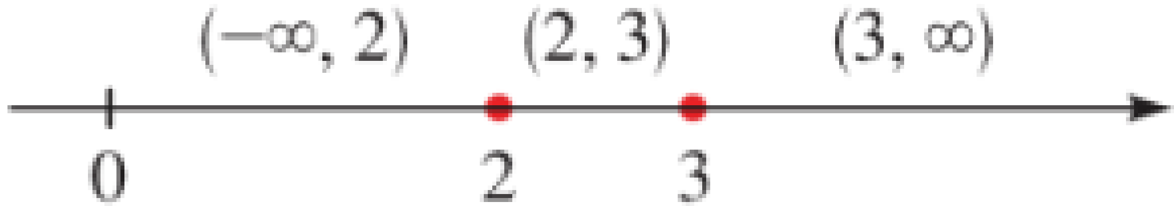


Figure 6: Number line divided up into three intervals.

- For the interval $(-\infty, 2)$, try checking $x = 0$ to deduce $(0-2)(0-3) = 6 > 0$, so we do not include this interval.
- For the interval $(2, 3)$, try checking $x = 2.5$ to deduce $(2.5-2)(2.5-3) < 0$, so we do include this interval.
- For the interval $(3, \infty)$, try checking $x = 4$ to deduce $(4-2)(4-3) = 2 > 0$, so we do not include this interval.

We also confirm the endpoints satisfy the inequality (it makes the expression an equality). From this we conclude the solution of the inequality is the interval $[2, 3]$.

For absolute value inequalities, we have that:

1. $|x| < c \implies -c < x < c$
2. $|x| > c \implies x > c$ or $x < -c$

Example 1.35: Solving an Absolute Value Inequality

Solve the following inequalities:

a) $|x-5| < 2$

b) $|2x+4| > 10$

Solution:

1. $|x-5| < 2 \implies -2 < x-5 < 2 \implies 3 < x < 7$
2. $|2x+4| > 10 \implies 2x+4 > 10$ or $2x+4 < -10 \implies x > 3$ or $x < -7$

Example 1.36: Solving an inequality with fractions

Solve the following inequality.

$$1 + \frac{2}{x+1} \leq \frac{2}{x}$$

Solutions: Unlike with equations, we do not want to multiply to clear out all the denominators. The reason is because we do not know the sign of the denominators, and this can cause the sign to flip. Instead, we combine all on one side leaving the other side zero.

$$\begin{aligned}1+\frac{2}{x+1} \le \frac{2}{x} &\implies \frac{x(x+1)}{x(x+1)}+\frac{2x}{x(x+1)}-\frac{2(x+1)}{x(x+1)} \le 0\\ &\implies \frac{x^2+x+2x-2x-2}{x(x+1)} \le 0\\ &\implies \frac{x^2+x-2}{x(x+1)} \le 0\\ &\implies \frac{(x+2)(x-1)}{x(x+1)} \le 0\end{aligned}$$

So this quantity potentially changes signs at $x = -2, -1, 0, 1$.

So we have to test values within these intervals to conclude that in between -2 and -1 is OK and in between 0 and 1 is OK. We also need to check the endpoints of these intervals. We cannot include 0 or 1 in the solution because that would cause division by 0 in the original expression. So $[-2, -1) \cup (0, 1]$ is the solution to the inequality.

1.7 The coordinate plane; Graphs of equations; circles §1.9

The **coordinate plane** (or Cartesian plane) is given by the intersection of two perpendicular lines which intersect at 0 on each line. The horizontal line with positive direction to the right is called the **x-axis**. The vertical line with positive direction upward is called the **y-axis**.

The point of intersection of the x-axis and the y-axis is denoted as the origin. The two axes divide the plane into four quadrants as follows.

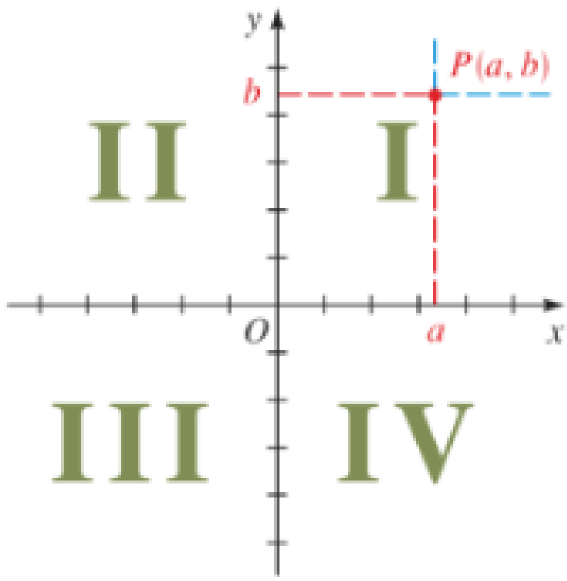


Figure 7: Four quadrants of the coordinate plane

Any point in the coordinate plane can be located by a unique ordered pair of numbers (a, b). The first coordinate is called the x-coordinate of the point, the second coordinate is called the y-coordinate of the point.

Example 1.37: Regions in the Coordinate Plane

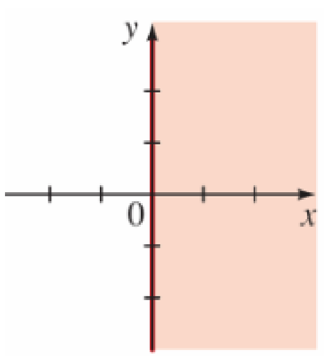
Describe and sketch the regions given by each set

a) $\{(x, y) \mid x \geq 0\}$

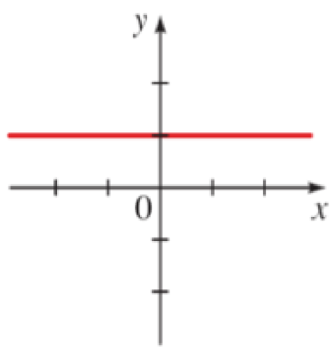
b) $\{(x, y) \mid y = 1\}$

c) $\{(x, y) \mid |y| < 1\}$

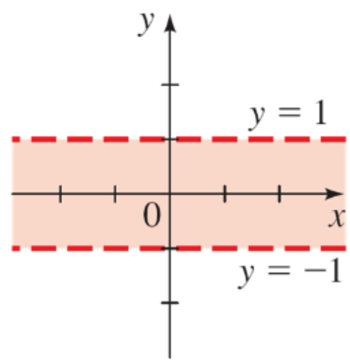
Solution: See plots below. Note in a) since the y-axis included, it is solid. In c) since the lines $y = 1$ and $y = -1$ are not included, they are drawn with dotted lines.



$x \geq 0$



$y = 1$



Recall the pythagorean theorem for right triangles.

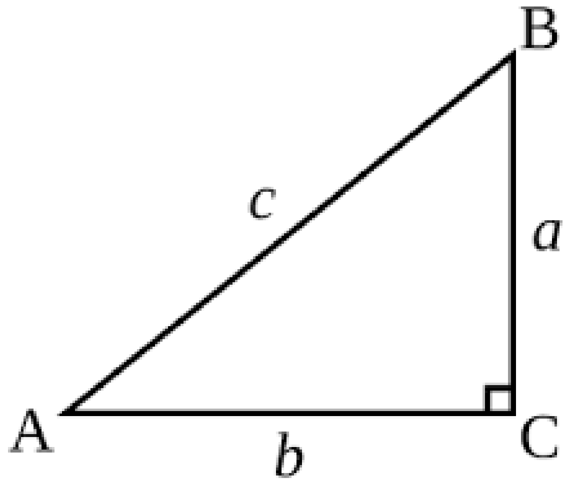


Figure 8: A right triangle with legs of lengths a and b and hypotenuse of length c

We have $a^2 + b^2 = c^2$. Draw two points (x_1, y_1) and (x_2, y_2) in the coordinate plane corresponding to A and B in the above figure, and you can write down the length a is given by $y_2 - y_1$ and the length b as given by $x_2 - x_1$. If you believe in Pythagorus this gives rise to the following.

The **distance** between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is given by

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1.38: Which point is closer?

Which of the points $P = (1, -2)$ or $Q = (8, 9)$ is closer to the point $A = (5, 3)$?

Solution: Computing both distances:

$$d(P, A) = \sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$
$$d(Q, A) = \sqrt{(5-8)^2 + (3-9)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

So P is closer to A.

If we have two points $(x_1, y_1), (x_2, y_2)$, then the **midpoint** of the line connecting these two points is given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Now we move to graphing equations of two variables. The graph of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

Example 1.39: Plot a line

Sketch the graph of the equation $2x - y = 3$.

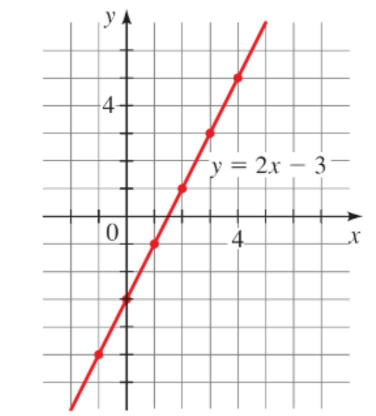


Figure 9: Plot of $y = 2x - 3$

Solution: We can rewrite the equation as $y = 2x - 3$ and plug in various x coordinates to get (x, y) pairs and plot points, and then connect the points.

We also introduce the concepts of **x-intercepts** and **y-intercepts**. The x-intercept (y-intercept) is where the graph crosses the x-axis (y-axis).

We can find the y-intercept by plugging in $x = 0$ and solving for y. Conversely, we can find the x-intercept by plugging in $y = 0$ and solving for x.

Example 1.40: Finding x and y intercepts

Find the x and y intercepts of the graph of the equation

$$y = x^2 - 2$$

Plot the graph.

Solution:. The x-intercepts are found by setting $y = 0$ and solving for x, so

$$x^2 - 2 = 0 \implies x^2 = 2 \implies x = \pm\sqrt{2}$$

The y-intercepts are found by setting $x = 0$ and solving for y.

$$y = (0)^2 - 2 \implies y = -2$$

So the y intercept is -2.

The distance formula gives rise to the equation of a circle. What is a circle? Given a point which we let to be the circle's center (x_0, y_0), it is the set of all points that is of a fixed distance (call it r) away from that center.

So our distance formula tells us that it is the set of all points (x, y) such that

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r \implies (x - x_0)^2 + (y - y_0)^2 = r^2$$

This is the standard form of a circle with center (x_0, y_0) with radius r. Frequently we like to consider the special case of the circle with center $(0, 0)$ and $r = 1$, which we denote to be the **unit circle**.

$$x^2 + y^2 = 1$$

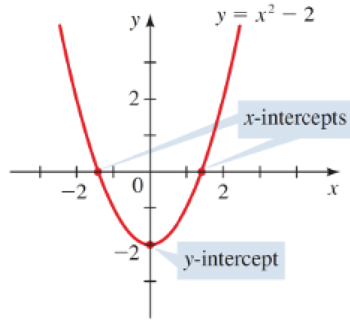


Figure 10: Plot of $y = x^2 - 2$

Example 1.41: Equations of Circles

a) Find an equation of the circle with radius 3 and center $(2, -5)$.

b) Find an equation of the circle that has the points $P = (1, 8)$ and $Q = (5, -6)$ as the endpoints as a diameter.

Solution:

a) By the standard formula of a circle, we have

$$(x-2)^2 + (y+5)^2 = 9$$

b) The center of the circle must be the midpoint of the line PQ, so we have

$$(x_0, y_0) = \left(\frac{1+5}{2}, \frac{8+(-6)}{2}\right) = (3, 1)$$

The radius is the distance from the center to any point on the circle, so we look at the distance between the center and P:

$$r = \sqrt{(1-3)^2 + (8-1)^2} = \sqrt{4+49} = \sqrt{53}$$

which gives us the equation for the circle as

$$(x-3)^2 + (y-1)^2 = 53$$

We will postpone discussion of symmetry until §2.6.

1.8 Lines §1.10

If you draw two distinct points on the $x - y$ coordinate plane and connect them, you have a (straight) line. This will be the object of study in this section.

The **slope** of a nonvertical line that passes through the points $(x_1, y_1), (x_2, y_2)$ is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula is only valid for a nonvertical line because a vertical line such as $x = 3$ has the same x-coordinate which would result in a divide by 0. A vertical line is said to have slope ∞ (some sources call this no slope, but I prefer infinite slope). Refer to the below to see the different possible types of slopes.

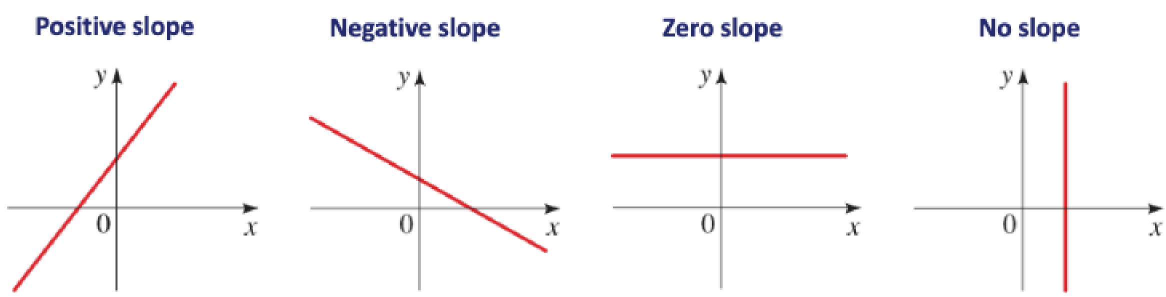


Figure 11: Various slopes of different lines

Example 1.42: Compute slopes of lines

Compute the slopes of the lines determined by the following two points.

a) $(2, 1)$ and $(8, 5)$

b) $(3, 4)$ and $(5, 4)$

c) $(1, 1)$ and $(1, 4)$

Solution:

a) Using the formula, we have:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

It doesn't matter which of the coordinate points you take to be (x_1, y_1) and (y_1, y_2). Suppose we swap the points and you will get the extra negative in the numerator and an extra negative in the denominator cancelling out to get the same results.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 5}{2 - 8} = \frac{-4}{-6} = \frac{2}{3}$$

b) Without using any formulas, we can determine this is a horizontal line, so it must have slope zero. Using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 4}{5 - 3} = 0$$

c) Without using any formulas, we can determine this is a vertical line. Using the formula will result in divide by zero. We say this line has infinite slope. The book refers to this as no slope which is also an acceptable answer.

Consider a point (a, b). An equation of the vertical line through (a, b) is given by $x = a$. An equation of the horizontal line through (a, b) is given by $y = b$.

Lines can be written in two forms:

- Slope-intercept form: $y = mx + b$
- Point-slope form: $(y - y_0) = m(x - x_0)$

In both expressions m represents the slope. b represents the y-intercept. (x_0, y_0) represents any point on the line.

Example 1.43: Slope-intercept form

a) Find an equation of the line with slope 3 and y-intercept -2

b) Find the slope and y-intercept of the line $3y - 2x = 1$

Solution:

a) By definition, the equation has the form $y = 3x - 2$.

b) We put this equation into slope-intercept form with some algebra.

$$3y - 2x = 1 \implies 3y = 2x + 1 \implies y = \frac{2}{3}x + \frac{1}{3}$$

giving us the slope $m = \frac{2}{3}$ and y-intercept $\frac{1}{3}$.

Example 1.44: Point-slope form

a) Find an equation of the line that passes through the point $(1, -3)$ with slope $-\frac{1}{2}$.

b) Sketch the line.

Solution:

a) By point slope form we have the line $(y + 3) = -\frac{1}{2}(x - 1)$ We can rearrange this to slope-intercept form with some algebra:

$$\begin{aligned} y + 3 &= -\frac{1}{2}(x - 1) \\ y + 3 &= -\frac{1}{2}x + \frac{1}{2} \\ y &= -\frac{1}{2}x - \frac{5}{2} \end{aligned}$$

b) See below. Note it has y-intercept $(0, -\frac{5}{2})$.

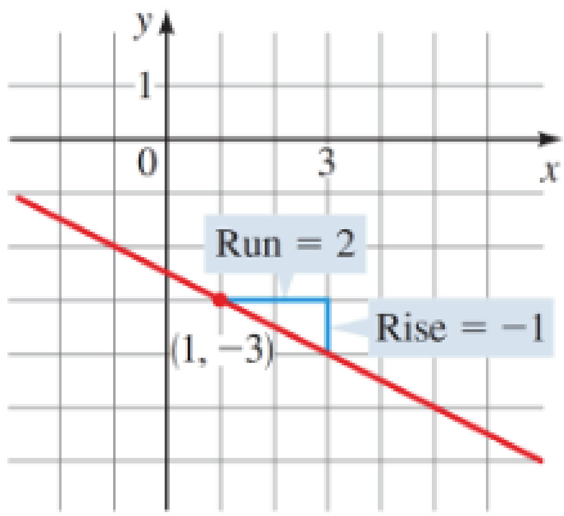


Figure 12: Sketch of the line $(y+3) = -\frac{1}{2}(x-1)$

Finally we close with a discussion of parallel and perpendicular lines.

- Two lines are parallel if and only if they have the same slope. Horizontal lines are parallel to other horizontal lines ,and vertical lines are parallel to other vertical lines.
- Two lines are perpendicular if and only if $m_1m_2 = -1$, that is, the product of their slopes if -1. Or rearranging, $m_1 = -\frac{1}{m_2}$, that their slopes are negative reciprocals. We also consider the special case that a horizontal line is perpendicular to a vertical line.

Example 1.45: Parallel, Perpendicular, or Neither?

Are the lines $3x + 6y = 12, y = 2x - 1$ parallel, perpendicular, or neither?

Solution: We need to find the slopes of the lines in order to tell. The first line has slope:

$$3x + 6y = 12 \implies 6y = -3x + 12 \implies y = -\frac{1}{2}x + 2 \implies m_1 = -\frac{1}{2}$$

The second line has slope $m_2 = 2$. Since the slopes are not the same they cannot be parallel. Since the product of the slopes are $m_1m_2 = -\frac{1}{2}(2) = -1$, the lines are perpendicular. (Sketch the lines on the board time permitting).

2 Functions §2

The term "function" describes the dependence of one quantity on another. For example:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.

For now, forget dependence on more than one factor (for example height is a function of age, genetics, nutrition, etc.). We are interested in one dependent variable that is a function of one independent variable.

2.1 Functions §2.1

We typically use letters like f, g, h to represent functions. A **function** is a rule that assigns to each element in a set A exactly one element, called $f(x)$ in a set B. In this class A and B will be sets of real numbers.

We denote $f(x)$ as "f of x" or "f at x", otherwise known as the value of f at x. The set A is the **domain** of the function. The **range** of the function is the set of all possible values of $f(x)$ as x varies throughout the domain. In set builder notation we have that

$$\text{Range of } f = \{f(x) \mid x \in A\}$$

Think of a function as a machine. If x is in the domain of the function f (it is an acceptable input), then when x enters the machine it is the input and produces as an output $f(x)$.

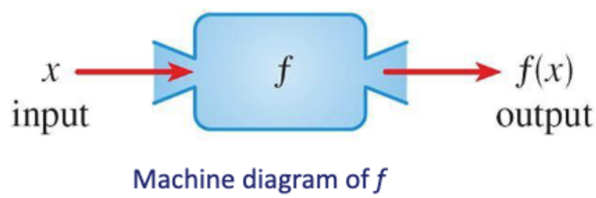


Figure 13: The function as a machine

For the most part we will represent functions mathematically with formulas. For example, consider

$$f(x) = x^2 + 4.$$

What does this function do? It takes an input, squares it, and then adds 4 to the result.

We can evaluate the function at a few values as such:

$$f(3) = 3^2 + 4 = 16 \qquad f(-2) = (-2)^2 + 4 = 8 \qquad f(\sqrt{5}) = (\sqrt{5})^2 + 4 = 9$$

The domain of the function (and of all polynomials) is all real numbers. The range of the function is $[4, \infty)$.

Example 2.1: Evaluating a Function

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

a) $f(-2)$

b) $f(0)$

c) $f(4)$

d) $f\left(\frac{1}{2}\right)$

Solutions:

a) $f(-2) = 3(-2)^2 - 2 - 5 = 12 - 2 - 5 = 5$

b) $f(0) = -5$

c) $f(4) = 3(4)^2 + 4 - 5 = 48 - 1 = 47$

d) $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 + \frac{1}{2} - 5 = \frac{3}{4} + \frac{1}{2} - 5 = \frac{5}{4} - 5 = -\frac{15}{4}$

The domain of a function is given by the set of all real numbers for each the expression is defined as a real number. For example, consider the functions

$$f(x) = \frac{1}{x-4} \qquad g(x) = \sqrt{x}$$

The function f is not defined at $x = 4$, so the domain is $\{x \mid x \neq 4\}$.
The function g is not defined for negative x, so the domain is $\{x \mid x \geq 0\}$.

Example 2.2: Finding domains of functions

Find the domain of each function.

a) $f(x) = \dfrac{1}{x^2 - x}$

b) $g(x) = \sqrt{9 - x^2}$

c) $h(t) = \dfrac{t}{\sqrt{t+1}}$

d) $f(x) = \dfrac{\sqrt{x}}{x^2 - 1}$

Solutions:

1. A rational expression is not defined where the denominator is 0. Since $x^2 - x = x(x-1)$, we have the domain is $x \neq 0, 1$. This can be written in interval notation as $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

2. We cannot take the square root of a negative number, so we require $9 - x^2 \geq 0$. We can solve the inequality to find that $-3 \leq x \leq 3$, or in interval notation the domain is $[-3, 3]$.

3. We cannot take the square root of a negative number, so we require $t + 1 \geq 0 \implies t \geq -1$. We cannot divide by 0, so we require $x \neq -1$. The intersection of these two is $x > -1$. In interval notation this is given by $(-1, \infty)$.

4. We cannot take the square root of a negative number, so we require $x \geq 0$. We also cannot divide by 0, so we must have $x \neq \pm 1$. So the domain is given by $\{x \mid x \geq 0, x \neq 1\}$. In interval notation this is given by $[0, 1) \cup (1, \infty)$.

2.2 Graphs of Functions §2.2

The simplest functions to graph are linear functions. Here are some examples.

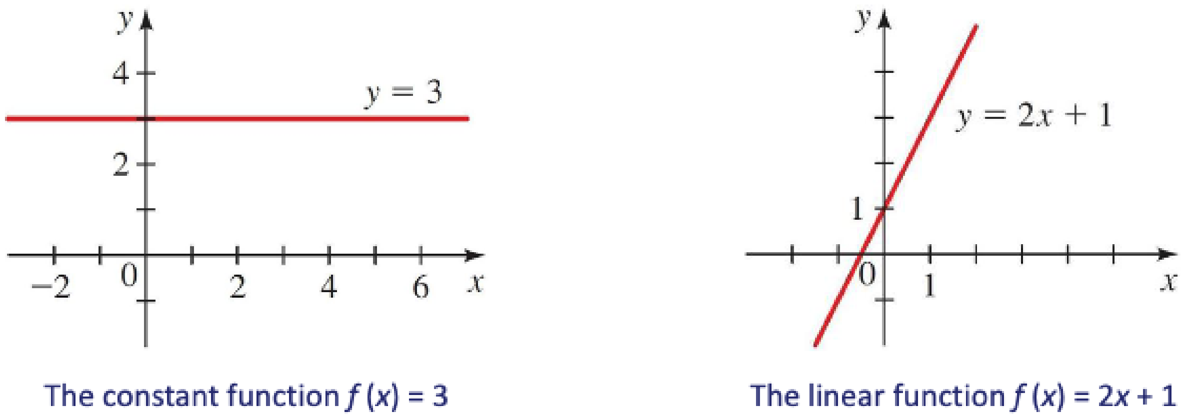


Figure 14: Graphs of linear functions

We can also graph **power functions** such as x^2 and x^3. Note how higher powers essentially look like this as well.

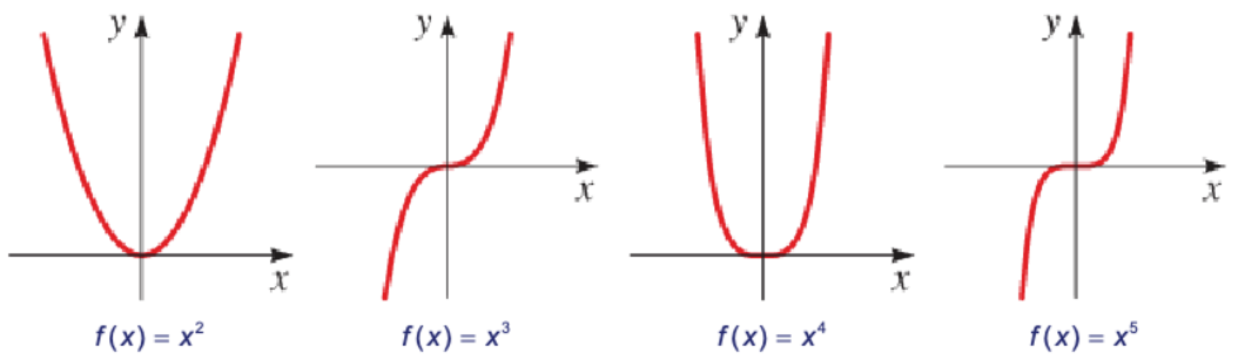


Figure 15: Graphs of power functions

Let's first look at even powers of x. It's clear that all of the plots pass through the points $(-1,1),(0,0),(1,1)$. For $x > 1$ they blow up. How are these graphs related?
We have for $x > 1$:

$$x^6 > x^4 > x^2$$

We have for $x = 1$:

$$x^6 = x^4 = x^2$$

We have for $0 < x < 1$:

$$x^6 < x^4 < x^2$$

You can do something similar for for negative x. We notice that there is a certain symmetry for even powers of x- that if you reflect the function across the y-axis it remains unchanged. Mathematically, this is saying that $f(x) = f(-x)$.

x	x^2	x^4	x^6
-3	9	81	729
-2	4	16	64
-1	1	1	1
$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$
0	0	0	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$
1	1	1	1
2	4	16	64
3	9	81	729

Table 1: Even powers of x

x	x^3	x^5	x^7
-3	-27	-243	-2187
-2	-8	-32	-128
-1	-1	-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{32}$	$-\frac{1}{128}$
0	0	0	0
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{128}$
1	1	1	1
2	8	32	128
3	27	243	2187

Table 2: Odd powers of x

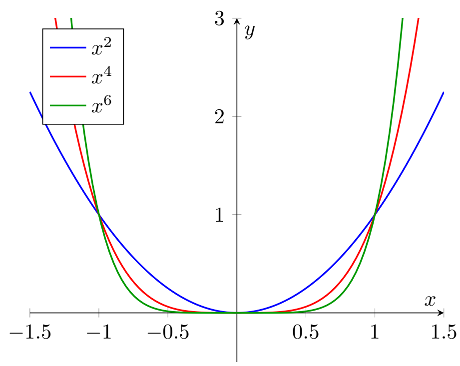


Figure 16: Even Power Functions

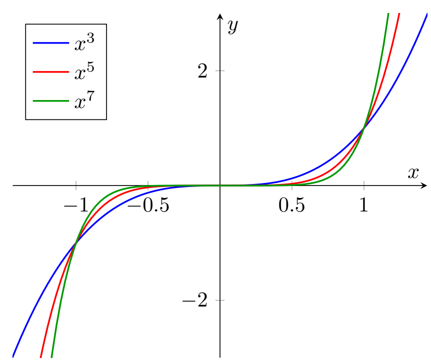


Figure 17: Odd Power Functions

Then we also have the odd powers. The big qualitative difference is that the even power of a negative number is positive, but the odd power of a negative number is negative.
The odd powers of x have the property that if you rotate the function 180 degrees it remains unchanged.

We can also graph **root functions** such as $x^{1/2} = \sqrt{x}$ and $x^{1/3} = \sqrt[3]{x}$.

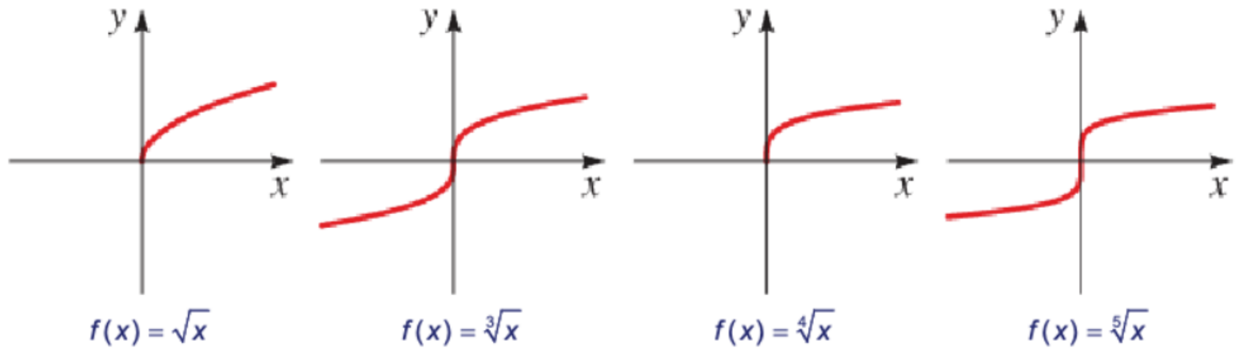


Figure 18: Graphs of root functions

x	$\sqrt{x}$	$\sqrt[4]{x}$	$\sqrt[6]{x}$
0	0	0	0
$\frac{1}{64}$	$\frac{1}{8}$	≈ 0.35	$\frac{1}{2}$
1	1	1	1
64	8	≈ 2.82	2
729	27	≈ 5.20	3

Table 3: Even roots of x ($x \geq 0$)

x	$\sqrt[3]{x}$	$\sqrt[5]{x}$	$\sqrt[7]{x}$
-27	-3	≈ -1.93	≈ -1.60
-1	-1	-1	-1
0	0	0	0
1	1	1	1
27	3	≈ 1.93	≈ 1.60

Table 4: Odd roots of x

The big difference is that you can take odd roots of negative numbers, whereas the even roots of a negative number are not real numbers. Even though these functions look like they are "flattening out" as x grows large in contrast to the power functions that look like they are "blowing up", these functions still approach infinity as x approaches infinity, albeit at a slower rate.

For the even roots, we have for $x > 1$:

$$\sqrt[6]{x} < \sqrt[4]{x} < \sqrt{x}$$

For the even roots, we have for $x = 1$:

$$\sqrt[6]{x} = \sqrt[4]{x} = \sqrt{x}$$

For the even roots, we have for $0 < x < 1$:

$$\sqrt[6]{x} > \sqrt[4]{x} > \sqrt{x}$$

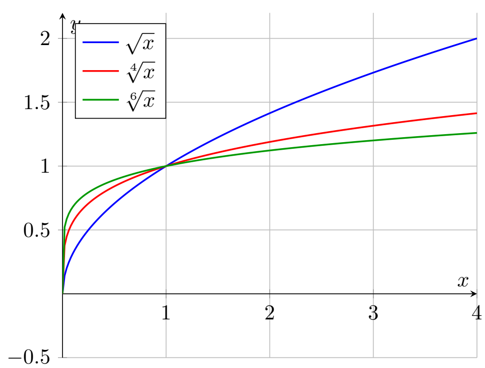


Figure 19: Even Root Functions

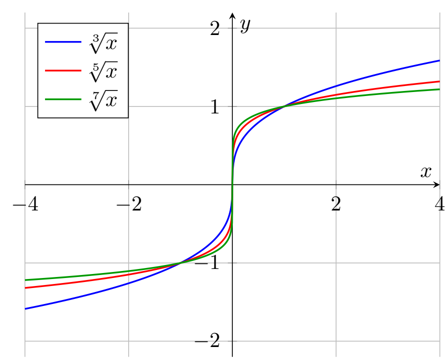


Figure 20: Odd Root Functions

There are also reciprocal functions.

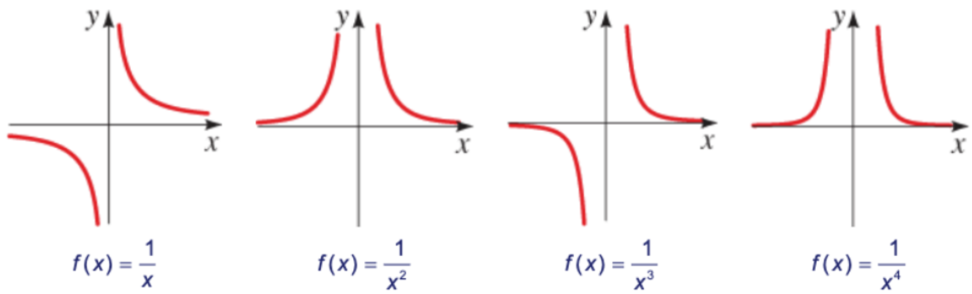


Figure 21: Graphs of reciprocal functions

x	$1/x$	$1/x^3$	$1/x^5$
-2	$-1/2$	$-1/8$	$-1/32$
-1	-1	-1	-1
$-1/2$	-2	-8	-32
$1/2$	2	8	32
1	1	1	1
2	$1/2$	$1/8$	$1/32$

Table 5: Reciprocals: Odd powers

x	$1/x^2$	$1/x^4$	$1/x^6$
-2	$1/4$	$1/16$	$1/64$
-1	1	1	1
$-1/2$	4	16	64
$1/2$	4	16	64
1	1	1	1
2	$1/4$	$1/16$	$1/64$

Table 6: Reciprocals: Even powers

For each one of these graphs, we say that $x = 0, y = 0$ are asymptotes, which are lines that a function approaches but never touches (or crosses). We will come back to this in Chapter 3.

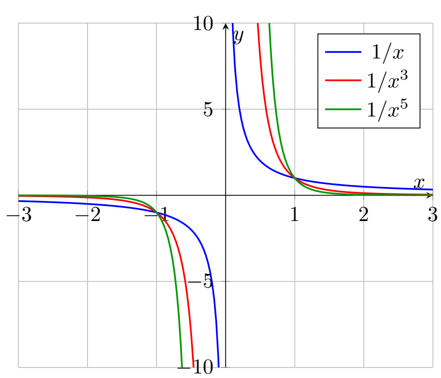


Figure 22: Odd Reciprocal Functions

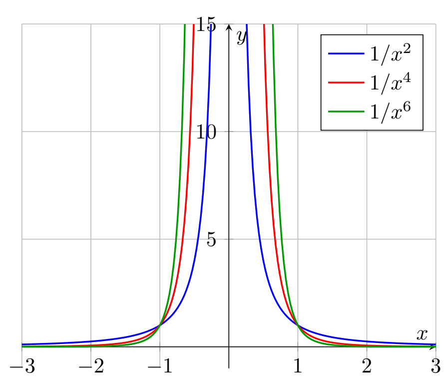


Figure 23: Even Reciprocal Functions

We now introduce the concept of a piecewise function. A piecewise function is defined by different formulas on different parts of its domain. The graph of a piecewise function consists of separate pieces.

Example 2.3: Graph a piecewise function

Sketch the graph of the function

$$f(x) = \begin{cases} x^2 & x \leq 1 \\ 2x+1 & x > 1 \end{cases}$$

Solutions:

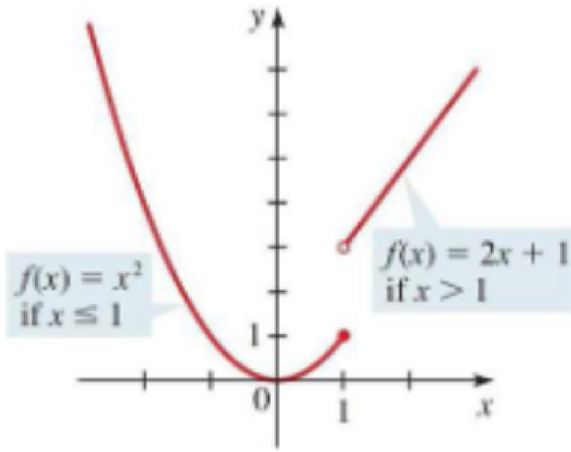


Figure 24: Graphs of the piecewise function

The solid dot at $(1, 1)$ indicates that this point is included in the graph; the open dot at (1, 3) indicates that this point is excluded from the graph.

A function is called **continuous** if it has no "breaks" or "holes". Another way to characterize a continuous function is if you can draw it without picking up your pencil. The linear, power, and root functions we saw are continuous. The piecewise function we just saw is not continuous.

And the absolute value function.

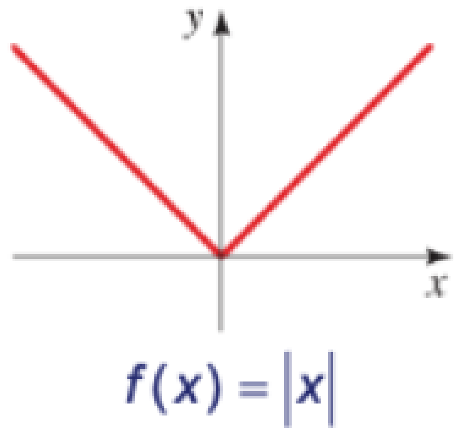


Figure 25: Graphs of absolute value function

x	$y = \|x\|$
-3	3
-2	2
-1	1
$-\frac{1}{2}$	$\frac{1}{2}$
0	0
$\frac{1}{2}$	$\frac{1}{2}$
1	1
2	2
3	3

Table 7: Values for the absolute value function $y = |x|$

If we want to shed the absolute value, we can rewrite it as a piecewise function.

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

What is a function? We earlier characterized it as a machine that takes an input and spits out an output. But a single input cannot map to multiple outputs. If we look at any curve in the xy plane, when is it a function?

We can also represent a function as an arrow diagram. The function maps each input in A to an element in B, the output. The key rule about functions is that it cannot map one input to multiple outputs. A function associates exactly one output to each input.

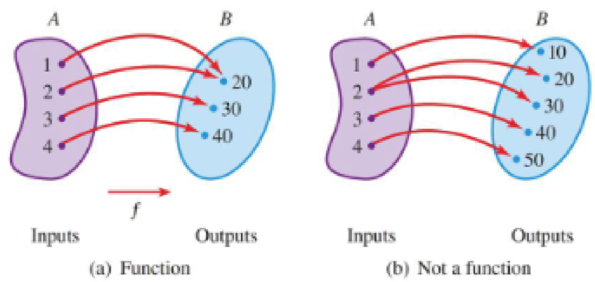


Figure 26: The function as an arrow diagram

Enter the **Vertical Line Test** - a curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

Consider the below. For the graph on the right, two different values are assigned to the input a so it cannot be a function.

Figure 27: Vertical Line test in action

In the below. (b) and (c) represent functions, whereas (a) and (d) are not.

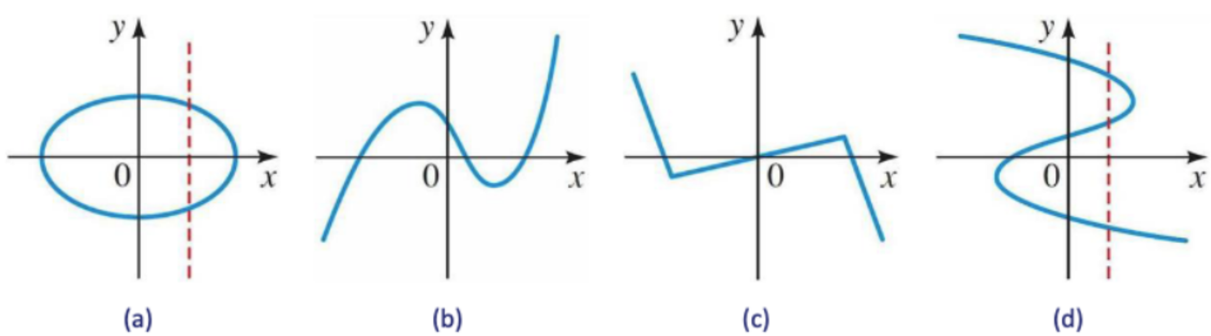


Figure 28: More vertical line test examples

2.3 Transformations of Functions §2.6

FOR THIS LECTURE - USE THE POWERPOINT

First go through the slides on vertical shifting (Slides 5 through 9). Then present the following example.

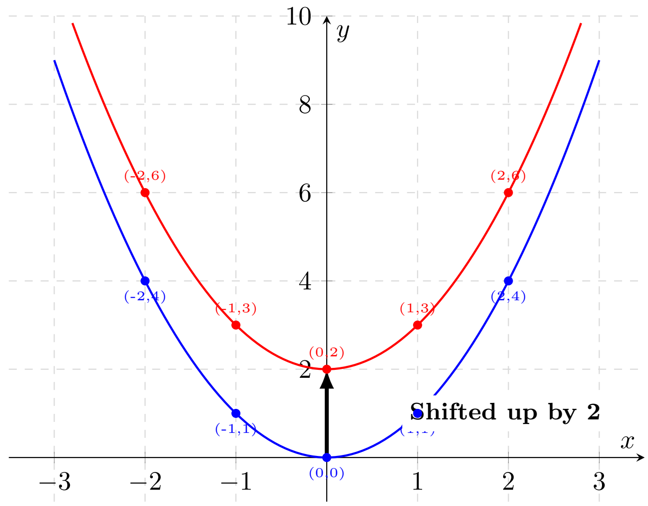
$$y = x^2 \text{ vs. } y = x^2 + 2$$

Table 8: Table of values for $y = x^2$ and $y = x^2 + 2$

x	$y = x^2$	$y = x^2 + 2$
-2	4	6
-1	1	3
0	0	2
1	1	3
2	4	6

It should be pretty clear that the third column is just the second column +2. I omit an example of shifting down.

Vertical Shift Up by 2



Now we go onto horizontal shifting (Slides 10-14). Then present the following example.

$$y = \sqrt{x} \text{ vs. } y = \sqrt{x-4}$$

We first look at $\sqrt{x}$ which is on the left. Then we ask ourselves, in order to get the same y-values, what must we change the x-values by? Since $y = \sqrt{x-4}$ is subtracting 4 from x before taking the square root, we realize we need all of our x values to be increased by 4 to compensate for this in order to get the same y-values.

An example of shifting to the left is omitted.

Table 9: Comparison of $y = \sqrt{x}$ and its horizontal shift $y = \sqrt{x-4}$

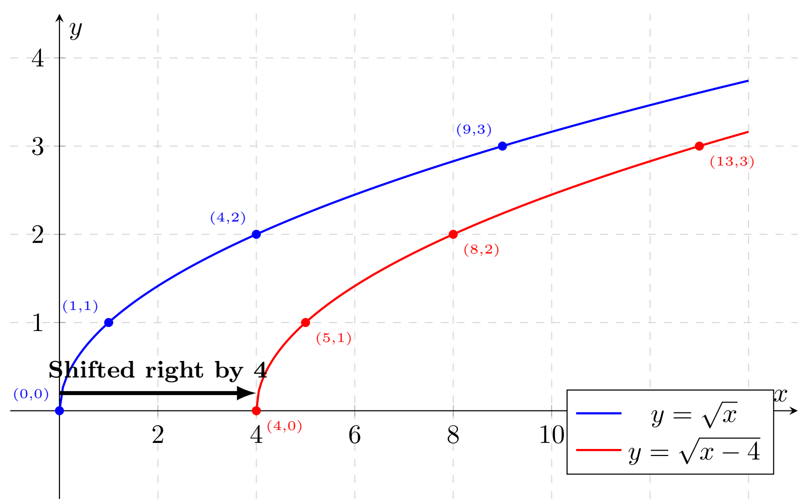
Parent Function

x	$y = \sqrt{x}$
0	0
1	1
4	2
9	3

Horizontal Shift

x	$y = \sqrt{x-4}$
4	0
5	1
8	2
13	3

Horizontal Shift: $y = \sqrt{x}$ vs $y = \sqrt{x-4}$



Now we move onto reflecting graphs (Slides 16-19) $y = -f(x)$ reflects the graph across the x-axis. $y = f(-x)$ reflects the graph across the y-axis.

Table 10: Table of values for $f(x) = x^2$ and $f(x) = -x^2$

x	$f(x) = x^2$	$f(x) = -x^2$
-2	4	-4
-1	1	-1
0	0	0
1	1	-1
2	4	-4

Reflection across the x-axis

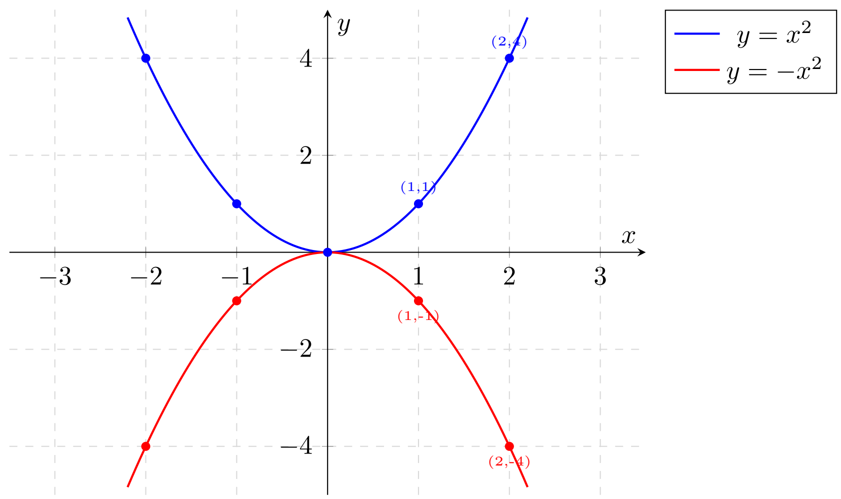


Table 11: Comparison of $y = \sqrt{x}$ and its reflection $y = \sqrt{-x}$

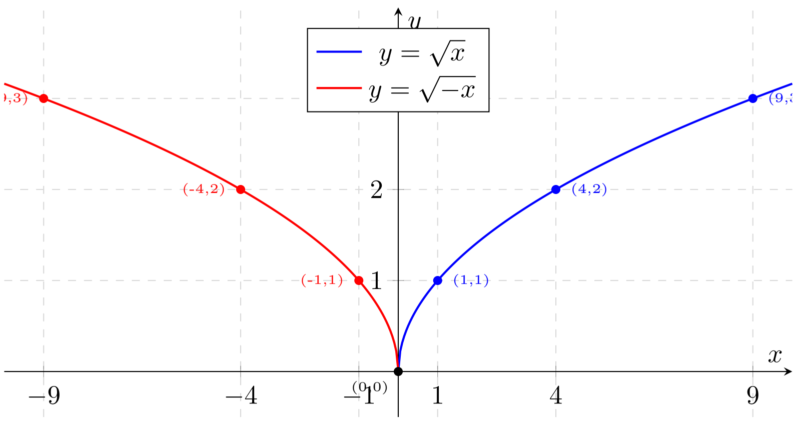
Parent Function

x	$y = \sqrt{x}$
0	0
1	1
4	2
9	3

Reflected Function

x	$y = \sqrt{-x}$
0	0
-1	1
-4	2
-9	3

Horizontal Reflection: $y = \sqrt{x}$ vs $y = \sqrt{-x}$



Now we move onto vertical stretching and shrinking (Slides 20 -24). Consider $f(cx)$. If $c > 1$ we are horizontally shrinking by a factor of c. If $c < 1$ we are horizontally stretching by a factor of c.

Note that in this case $f(x) = (3x)^2 = 9x^2$, so we can interpret horizontally shrinking x^2 by a factor of 3 as vertically stretching by a factor of 9.

Table 12: Comparison of $f(x) = x^2$ and $f(x) = (3x)^2$

Parent Function

x	x^2
-2	4
-1	1
0	0
1	1
2	4

Transformed Function

x	$(3x)^2$
$-2/3$	4
$-1/3$	1
0	0
$1/3$	1
$2/3$	4

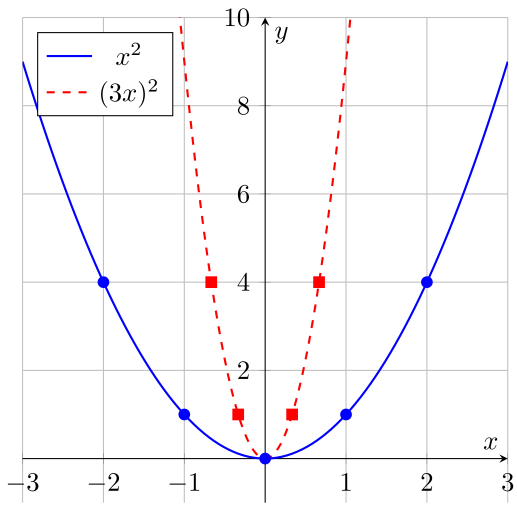


Figure 29: Visualizing the Horizontal Compression

Now we compare $f(x)$ and $f(\frac{1}{2}x)$. Note that in this case $f(\frac{1}{2}x) = \frac{1}{4}x^2$, so we can interpret horizontally stretching x^2 by a factor of 2 as vertically shrinking $f(x)$ by a factor of 4.

Table 13: Comparison of $f(x) = x^2$ and $f(x) = (\frac{1}{2}x)^2$

Parent Function

x	x^2
-2	4
-1	1
0	0
1	1
2	4

Transformed Function

x	$(\frac{1}{2}x)^2$
-8	4
-4	1
0	0
4	1
8	4

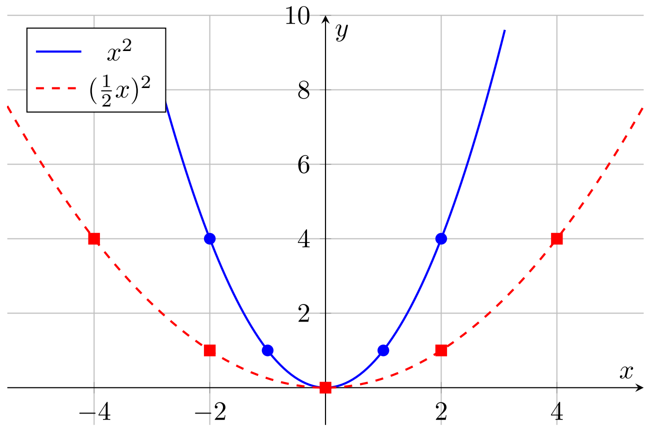


Figure 30: Horizontal Stretch Comparison

Finally, we close with the definition of symmetry and odd or even functions.

If a function f satisfies $f(x) = f(-x)$ for every x in its domain, then f is called an **even function**. The graph of an even function is symmetric with respect to the y-axis.

We can show $f(x) = x^2$ is an even function because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

If a function f satisfies $-f(x) = f(-x)$ for every x in its domain, then f is called an **odd function**. The graph of an odd function is symmetric with respect to the origin. This means we can turn the function around the origin 180 degrees and it will remain unchanged.

We can show $f(x) = x^3$ is an odd function because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

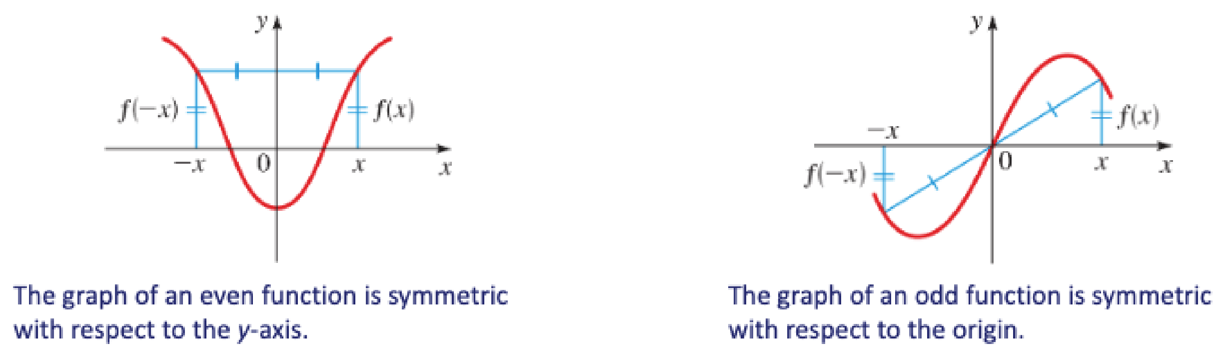


Figure 31: Example of an even and odd function

Example 2.4: Determining Even or Odd Functions

Determine whether the functions are even, odd, or neither even nor odd

a) $f(x) = x^5 + x$

b) $g(x) = 1 - x^4$

c) $h(x) = 2x - x^2$

Solutions:

a)

$$\begin{aligned} f(-x) &= (-x^5) + (-x) \\ &= -x^5 - x \\ &= -(x^5 + x) \\ &= -f(x) \end{aligned}$$

so odd.

b)

$$\begin{aligned} g(-x) &= 1 - (-x)^4 \\ &= 1 - x^4 \\ &= g(x) \end{aligned}$$

so even.

c)

$$\begin{aligned} h(-x) &= 2(-x) - (-x)^2 \\ &= -2x - x^2 \end{aligned}$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, h is neither even nor odd.

Then we did knowledge Check 1 and 2 from the slides.

2.4 Combining Functions §2.7

Two functions f, g can be combined to form new functions $f+g, f-g, fg, \frac{f}{g}$ in a manner similar to the way we add, subtract, multiply, and divide real numbers.

For example, the function $f+g$ is defined as $(f+g)(x) = f(x)+g(x)$. This new function $f+g$ is called the sum of the functions f and g; its value at x is $f(x)+g(x)$.

Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g.

As for the rest of the possibilities: Let f, g be functions with domains A and B respectively. Then the functions $f+g, f-g, fg, \frac{f}{g}$ are defined as follows:

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) \qquad \text{Domain } A\cap B \\
(f-g)(x) &= f(x)-g(x) \qquad \text{Domain } A\cap B \\
(fg)(x) &= f(x)g(x) \qquad \text{Domain } A\cap B \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \qquad \text{Domain } A\cap B, g(x)\neq 0
\end{aligned}$$

Example 2.5: Combining Functions

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$

a) Find the functions $f+g, f-g, fg$ and $\frac{f}{g}$ and their domains.

b) Find $(f+g)(4), (f-g)(4), (fg)(4), \left(\frac{f}{g}\right)(4)$

Solutions: The domain of f is $x \neq 2$. The domain of g is $x \geq 0$. So the intersection of the domains is given by

$$\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0,2) \;\cup\; (2,\infty)$$

So the combinations of the functionsare given by

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) = \frac{1}{x-2}+\sqrt{x} \\
(f-g)(x) &= f(x)-g(x) = \frac{1}{x-2}-\sqrt{x} \\
(fg)(x) &= f(x)g(x) = \frac{\sqrt{x}}{x-2} \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}}
\end{aligned}$$

where the domain of the first three combinations are $\{x \mid x \geq 0, x \neq 2\}$ and the fourth combination $\frac{f}{g}$ has the domain $\{x \mid x > 0, x \neq 2\}$ since it must exclude zero because $g(0) = 0$.

Each of these values exist because $x = 4$ is in the domain of each function

$$(f+g)(4) = f(4) + g(4) = \frac{1}{4-2} + \sqrt{4} = \frac{5}{2}$$
$$(f-g)(4) = f(4) - g(4) = \frac{1}{4-2} - \sqrt{4} = -\frac{3}{2}$$
$$(fg)(4) = f(4)g(4) = \frac{1}{4-2}\sqrt{4} = 1$$
$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4-2)\sqrt{4}} = \frac{1}{4}$$

Now we turn to composition of functions. If we have $f(x) = \sqrt{x}, g(x) = x^2 + 1$, we can define a new function h as

$$h(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}.$$

What's going on here is that the function h is made up from the functions f, g as such: Given a number x, the apply the function g to it, and then apply the function f to the result. We can see the function g as an input to the function f. f only takes in numbers, not functions, but when you apply g to get x and get $g(x)$ it is a number which f can take an input.

Another frequent way to denote function composition $f(g(x))$ is $(f \circ g)(x)$.

Example 2.6: Function Composition

Let $f(x) = x^2, g(x) = x - 3$

a) Find the functions $f(g(x))$ and $g(f(x))$

b) Find $f(g(5))$ and $g(f(7))$.

Solution: We have

$$f(g(x)) = f(x-3) = (x-3)^2$$
$$g(f(x)) = g(x^2) = x^2 - 3$$

The previous example illustrates that $f(g(x)) \neq g(f(x))$. The order in which you apply the functions matters.

To evaluate the composed functions at specific values, we can approach it one or two equivalent ways. If you want to find $f(g(5))$, one way is to look at the composted function $f(g(x)) = (x-3)^2$ and simply plug in 5 to get $f(g(5)) = (5-3)^2 = 4. Analternativeinterpretationistogofrom“insidetooutside”byfirstevaluating$g(5) = 5 - 2$andletthatbetheinputto$f(x)$so$f(2) = 4.

Likewise for $g(f(7)) = 46$.
We can also compose more than two functions such as $f(g(h(x))$.

Example 2.7: Three Function Composition

Let $f(x) = \frac{x}{x+1}, g(x) = x^{10}, h(x) = x + 3$. Find $f(g(h(x)))$

Solution:

$$\begin{aligned} f(g(h(x))) &= f(g(x+3)) \\ &= f((x+3)^{10}) \\ &= \frac{(x+3)^{10}}{(x+3)^{10}+1} \end{aligned}$$

Example 2.8: Domains of composed functions

a) Let $f(x) = \sqrt{x}, g(x) = x^2 + 2x$. What is $g(f(x))$ and its domain?

b) Let $f(x) = \sqrt{x}, g(x) = \frac{2}{x^2}$. What is $g(f(x))$ and its domain?

Solution:

1. $g(f(x)) = g(\sqrt{x}) = x + 2\sqrt{x}$ with the domain $x \geq 0$
2. $g(f(x)) = \frac{2}{x}$ with the domain $x > 0$. Even though the resultant function can take in negative numbers, $f(x)$ cannot, so we need to remember to respect this restriction since this function was the result of function composition.

   To make this concrete, consider $g(f(-1)$. Since $f(-1)$ is not a real number, this quantity cannot be evaluated, even though plugging in $x = -1$ to $\frac{2}{x}$ is perfectly fine.

2.5 One-to-one functions and their inverses §2.8

A function takes an input and maps it to an output. We know by a definition of functions that the same input can never be mapped to multiple outputs. Given a graph we can determine whether it is a function or not by the vertical line test.

Now we are interested in a special class of functions where the same output cannot be mapped from multiple inputs.

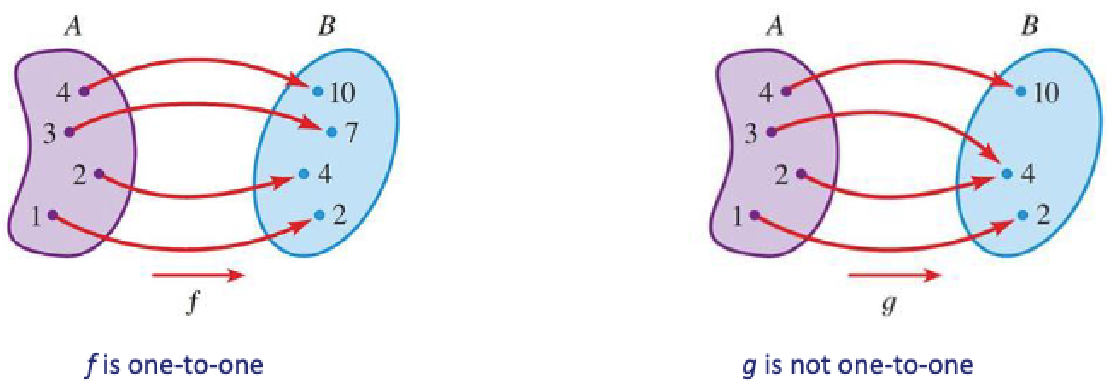


Figure 32: Arrow diagrams of f and g

From the above diagrams, f never takes on the same value twice. g does take on the same value twice because $g(2) = g(3) = 4$. Functions that never take on the same value twice are known as **one-to-one** functions. f is one-to-one, g is not. They are called one-to-one because there is always going to be a "one-to-one" relationship between elements in the domain and the range.

A function is called a **one-to-one** function if no two elements of the domain have the same image. That is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

An equivalent way of writing the condition for a one-to-one function is this:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2$$

So by the above definition, $f(x) = x^2$ is not a one-to-one function, because

$$f(2) = f(-2) \text{ even though } 2 \neq 2$$

or

$$f(2) = f(-2), \text{ but } 2 \neq 2$$

Just like we have the vertical line test to determine whether a graph is a function in order to determine that every input has at most one output, for one-to-one ness we have the horizontal line test to determine that every output has at most one input. The below is a function which fails the horizontal line test and as such is not one-to-one.

Example 2.9: One-to-one test

Is $f(x) = x^3$ one-to-one?

Solution: One solution is by drawing the graph and showing that since the function is strictly increasing, no horizontal line will intersect the graph more than once (in fact, every horizontal line will intersect the graph

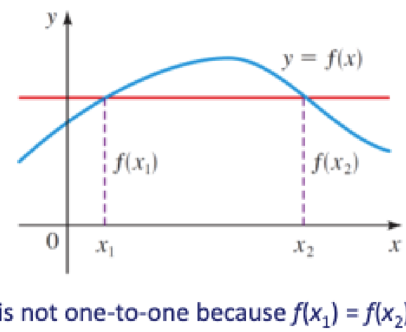


Figure 33: A function which fails the horizontal line test and is not one-to-one

exactly once).

Another way is by the definition. If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two numbers cannot have the same cube).

One-to-one functions are important because they are precisely the functions that possess inverse functions. The **inverse** of a function is a rule that acts on the output of the function and produces the corresponding input. So the inverse "undoes" or reverses what the functions has done.

Not all functions have inverses; those that do are called one-to-one.
Let f be a one-to-one function with domain A and range B. Then its inverse f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any $y \in B$.

So we have that the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Example 2.10: Finding f inverse for specific values

If $f(1) = 5, f(3) = 7, f(8) = -10$, find $f^{-1}(5), f^{-1}(7), f^{-1}(10)$.

Solution: By definition we have

$$f^{-1}(5) = 1 \text{ because } f(1) = 5$$
$$f^{-1}(7) = 3 \text{ because } f(3) = 7$$
$$f^{-1}(-10) = 8 \text{ because } f(8) = -10$$

CAREFUL: The notation can be confusing. Given $f(x)$, the inverse is $f^{-1}(x)$.

$$f^{-1}(x) \text{ DOES NOT MEAN } \frac{1}{f(x)}$$

If f is a one-to-one function with domain A and range B. we have that

$$f^{-1}(f(x)) = x \text{ for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \text{ for every } x \text{ in } B$$

Now we discuss how to find the inverse of a given one-to-one function f.

1. Write $f(x)$
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 2.11: Find inverse

Find the inverse of the function $f(x) = 3x - 2$

Solution:
First let's reason out what it should be. The function $f(x)$ takes an input and 1) multiplies it by 3 and 2) subtracts by 2. So in order to undo this, we would need a function to add by 2 and divide by 3. That would give us the inverse

$$f^{-1} = \frac{x+2}{3}$$

.
Now we follow the algorithm. First write down $y = f(x)$

$$y = 3x - 2$$

Solve this equation for x in terms of y:

$$\begin{aligned} y = 3x - 2 &\implies 3x = y + 2 \\ &\implies x = \frac{1}{3}y + \frac{2}{3} \end{aligned}$$

Interchange x and y to get the inverse function

$$f^{-1}(x) = \frac{x+2}{3}$$

One can easily check that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 2.12: Find inverse

Find the inverse of the function $f(x) = \frac{2x+3}{x-1}$

Solution: First write down $y = f(x)$

$$y = \frac{2x+3}{x-1}$$

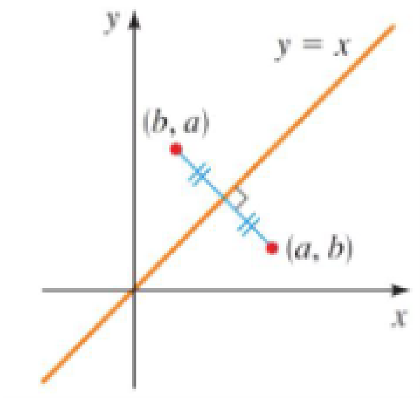
Solve this equation for x in terms of y:

$$\begin{aligned} y = \frac{2x+3}{x-1} &\implies y(x-1) = 2x+3 \\ &\implies yx - y = 2x + 3 \\ &\implies yx - 2x = 3 + y \\ &\implies x(y-2) = 3 + y \\ &\implies x = \frac{y+3}{y-2} \end{aligned}$$

Finally interchange x and y to get the desired inverse:

$$f^{-1}(x) = \frac{x+3}{x-2}.$$

Geometrically, the point (b, a) comes from the point (a, b) by a reflection in the line $y = x$. Therefore, we have that f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



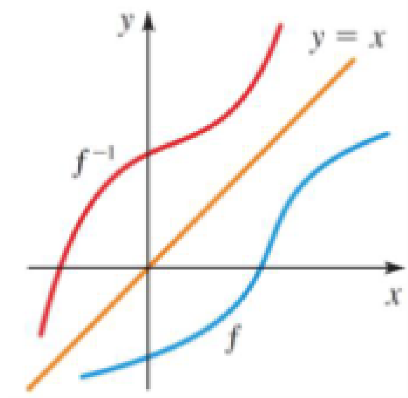


Figure 34: f and f^{-1} related by reflecting across $y = x$

Example 2.13: Graphing the inverse of a function

a) Sketch the graph of $f(x) = \sqrt{x-2}$

b) Use the graph of f to sketch the graph of f^{-1}

c) Find an equation for f^{-1}

Solutions:

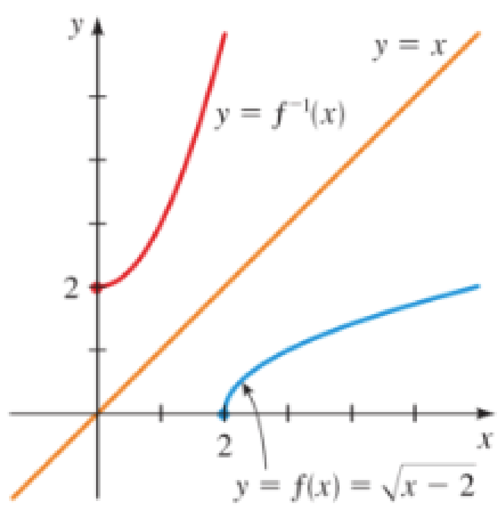


Figure 35: $f(x) = \sqrt{x-2}$ and its inverse

Take $y = \sqrt{x-2}$ and solve for x in terms of y, noting that $y \geq 0$:

$$\begin{aligned} y = \sqrt{x-2} &\implies y^2 = x - 2 \\ &\implies x = y^2 + 2 \end{aligned}$$

Interchange x and y to get that

$$f^{-1}(x) = x^2 + 2, x \geq 0$$

3 Polynomials §3

A polynomial is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a non-negative integer.

The leading term is of degree n (assuming $a_n \neq 0$) so we say this polynomial is of degree n.

We have seen some simpler cases before and given them special names:

$$\begin{aligned}
&P(x) = a_0 \text{ (constant)}\\
&P(x) = a_0 + a_1 x \text{ (linear)}\\
&P(x) = a_0 + a_1 x + a_2 x^2 \text{ (quadratic)}\\
&P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ (cubic)}
\end{aligned}$$

Let's first study the case of the quadratic.

3.1 Quadratic Polynomials §3.1

A quadratic function is a polynomial function of degree 2. So a quadratic function is a function of the form

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

If $a = 1, b = c = 0$, then we have $f(x) = x^2$, whose graph is a parabola. In fact, the graph of any quadratic function is a parabola transformed appropriately.

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form** (or sometimes **vertex form**)

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with vertex (h, k). The parabola opens upwards if $a > 0$ or downwards if $a < 0$.

Let's quickly review completing the square. If we have a quadratic of the form

$$f(x) = ax^2 + bx + c$$

we first factor out a from the first two terms

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

In order to complete the square in the parenthesis, we need to add $\left(\frac{b}{2a}\right)^2$. Accordingly, we also need to subtract the same quantity outside the parenthesis as to not change the value of the expression.

$$\begin{aligned}
f(x) &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2\\
&= a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2
\end{aligned}$$

It will be easier when we apply this to examples with concrete numbers. This idea is not new. We covered this briefly in §1.5 when we were solving equations, and used completion of the square to derive the quadratic formula.

Figure 36: Standard form of a parabola

Example 3.1: Standard form of a Quadratic Function

Let $f(x) = 2x^2 - 12x + 13$

a) Express f in standard form.

b) Find the vertex and x and y intercepts of f.

c) Sketch the graph of f.

d) Find the domain and range of f.

Solution: Since the coefficient of x^2 is not 1, we should first factor it out:

$$\begin{aligned} f(x) &= 2x^2 - 12x + 13 \\ &= 2(x^2 - 6x) + 13 \\ &= 2(x^2 - 6x + 9) + 13 - 18 \\ &= 2(x-3)^2 - 5 \end{aligned}$$

So the standard form of the polynomial is $f(x) = 2(x-3)^2 - 5$.

The vertex is given by $(3, -5)$ and since $a = 2 > 0$, it opens up, so the vertex is a minimum.

To find the y-intercept, simply evaluate $f(0) = 2(-3)^2 - 5 = 13$.

To find the x-intercepts, set $f(x) = 0$ and solve the resulting equation.
Since we have vertex form:

$$\begin{aligned} 2(x-3)^2 - 5 = 0 &\implies (x-3)^2 = \frac{5}{2} \\ &\implies x = 3 \pm \sqrt{\frac{5}{2}} \end{aligned}$$

If you don't want to think about it, just take $f(x) = 2x^2 - 12x + 13$ and apply the quadratic formula. It seems like a waste of a perfectly good vertex form.

$$\begin{aligned} x &= \frac{12 \pm \sqrt{144 - 4 \cdot 2 \cdot 13}}{4} \\ &= \frac{6 \pm \sqrt{10}}{2} \end{aligned}$$

which are the two x-intercepts.

The domain is clearly all real numbers $\mathbb{R}$. The range is given by $[-5, \infty)$.

(Show a graph of this on Wolfram Alpha in class)

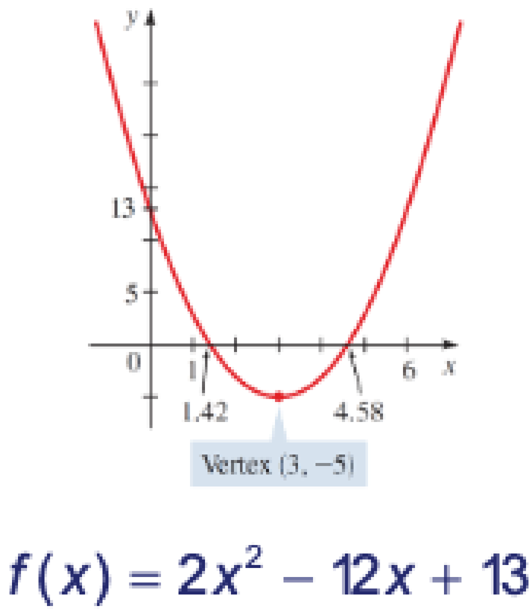


Figure 37: Graph of $f(x) = 2(x-3)^2 - 5$

If $a > 0$ and the parabola opens upwards, then the vertex (h, k) is the minimum. If $a < 0$ and the parabola opens downwards, then the vertex (h, k) is the maximum.

Example 3.2: Standard form of a Quadratic Function II

Let $f(x) = 5x^2 - 30x + 49$

a) Express f in standard form.

b) Sketch the graph of f.

c) Find the minimum value of f.

Solution: First express the parabola in standard form by completing the square.

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 \\ &= 5(x^2 - 6x + 9) + 49 - 45 \\ &= 5(x-3)^2 + 4 \end{aligned}$$

Since $a = 5 > 0$, the parabola opens upwards and the minimum is $(3, 4)$.

Example 3.3: Standard form of a Quadratic Function III

Let $f(x) = -x^2 + x + 2$

a) Express f in standard form.

b) Sketch the graph of f.

c) Find the maximum value of f.

Solution: First express the parabola in standard form by completing the square.

$$\begin{aligned} f(x) &= -x^2 + x + 2 \\ &= -(x^2 - x) + 2 \\ &= -\left(x^2 - x + \frac{1}{4}\right) + 2 + \frac{1}{4} \\ &= -\left(x - \frac{1}{2}\right)^2 + \frac{9}{4} \end{aligned}$$

Since $a = -1 < 0$, the parabola opens downwards and the maximum is $\left(\frac{1}{2}, \frac{9}{4}\right)$.
If we look at the generic form of the parabola $f(x) = ax^2 + bx + c$, we can complete the square as follows:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

So the vertex occurs at $x = -\frac{b}{2a}$, and the vertex point $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ is a minimum if $a > 0$ and a maximum if $a < 0$.

Example 3.4: Minimum and Maximum of Quadratic Functions

Find the minimum or maximum value of each quadratic function

a) $f(x) = x^2 + 4x$

b) $g(x) = -2x^2 + 4x - 5$

Solution:

1. The vertex occurs at $-\frac{b}{2a} = -\frac{4}{2} = -2$. So the vertex is $(-2, -4)$ and since $a > 0$ it is a minimum.

2. The vertex occurs at $-\frac{b}{2a} = -\frac{4}{-4} = 1$. So the vertex is $(1, -3)$ and since $a < 0$ it is a maximum.

The upshot is that we can find the vertex of a polynomial and classify it as a maximum or a minimum without putting into standard form (vertex form). We can find it simply from the form $P(x) = ax^2 + bx + c$.

3.2 Polynomial Functions and Their Graphs §3.2

We have already discussed how to graph the parent functions $f(x) = x, x^2, x^3, x^4, \ldots$

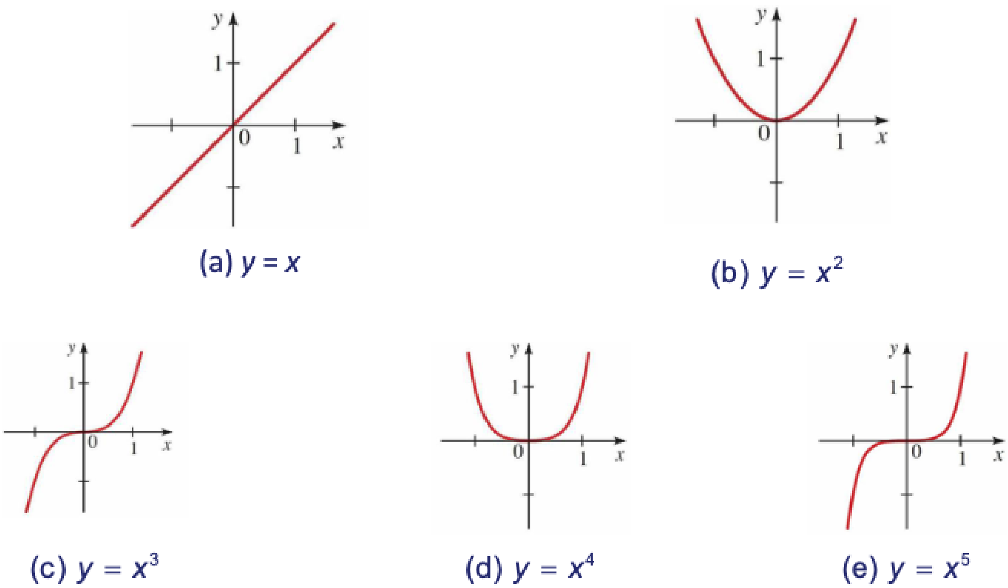


Figure 38: Graphs of monomials $f(x) = x, x^2, x^3, x^4, x^5$

We also discussed how to sketch transformations of these monomials in §2.6, but let's review a little bit.

Example 3.5: Transformations on Monomials

Sketch the following functions.

a) $P(x) = -x^3$

b) $Q(x) = (x - 2)^4$

c) $R(x) = -2x^5 + 4$

Solution: Done in class. Check with Wolfram Alpha or Desmos.

Now we discuss the end behavior of polynomials. This is referring to the behavior of the polynomial $P(x)$ as $x \to \infty$ or $x \to -\infty$.

As an example, the function $y = x^2$ has end behavior $y \to \infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$.

The function $y = x^3$ has the end behavior $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

For general polynomials, the end behavior is determined by the term that contains the highest power of x, because when x is large, the other terms are relatively insignificant in size.

The below describes the end behavior of the four cases:

1. P has odd degree and leading coefficient positive

2. P has odd degree and leading coefficient negative.

3. P has even degree and leading coefficient positive
4. P has even degree and leading coefficient negative.

.

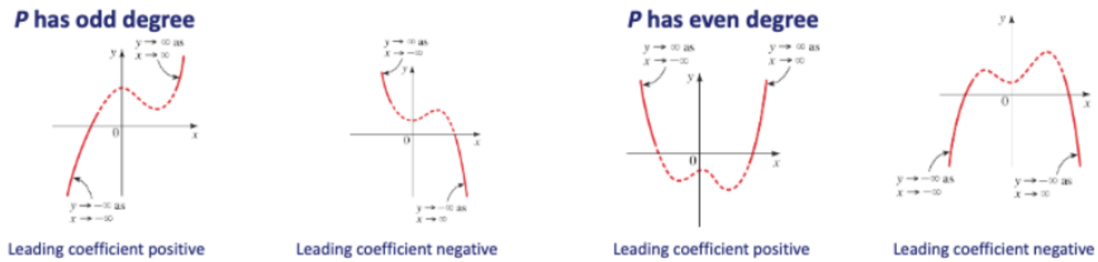


Figure 39: End behavior of polynomials

Example 3.6: End Behavior of a Polynomial

Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

Solution: It has even degree and leading coefficient negative. So it has the following end behavior.

$$y \to -\infty \text{ as } x \to -\infty \text{ and } y \to -\infty \text{ as } x \to \infty$$

Now we discuss how to graph polynomial functions.

1. Find the zeros (the x-intercepts) of the graph.
2. Determine the end behavior of the polynomial.
3. Plot the graph to go through zeros. The graph goes through zeros of odd multiplicity and "bounces off" zeros of even multiplicity.

For example, consider

$$P(x) = (x-1)(x-3)^3(x-2)^4$$

$x = 1$ has a multiplicity at 1, $x = 3$ has a multiplicity of 3, and $x = 4$ has a multiplicity of 4.

Example 3.7: Sketch Polynomial

Let $P(x) = x^3 - 2x^2 - 3x$

a) Find the zeros of $P(x)$

b) Sketch the graph of $P(x)$

Solution:

a) To find the zeros, factor completely.

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 3x \\ &= x(x^2 - 2x - 3) \\ &= x(x-3)(x+1) \end{aligned}$$

Thus the zeros are $x = 0, 3, -1$.

b) This polynomial has the end behavior

$$y \to -\infty \text{ as } x \to -\infty \text{ and } y \to \infty \text{ as } x \to \infty$$

.

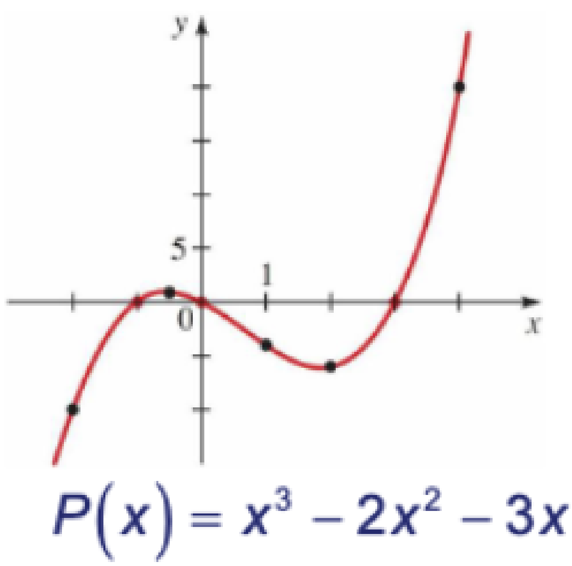


Figure 40: Plot of $P(x) = x^3 - 2x^2 - 3x$

Example 3.8: Sketch Polynomial II

Graph the polynomial

$$P(x) = x^4(x-2)^3(x+1)^2$$

This polynomial has the end behavior

$$y \to -\infty \text{ as } x \to -\infty \text{ and } y \to \infty \text{ as } x \to \infty$$

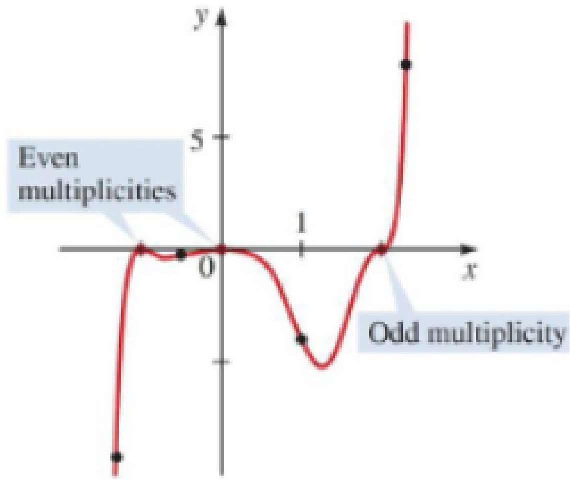


Figure 41: Plot of $P(x) = (x^4)(x-2)^3(x+1)^2$

3.3 Dividing Polynomials §3.3

Sometimes in order to factor polynomials (degree 3 or larger) we typically have to divide polynomials in order to factor them. This is very similar to long division of real numbers.

As an example, let's consider the division of $6x^2 - 26x + 12$ with the divisor $x - 4$.

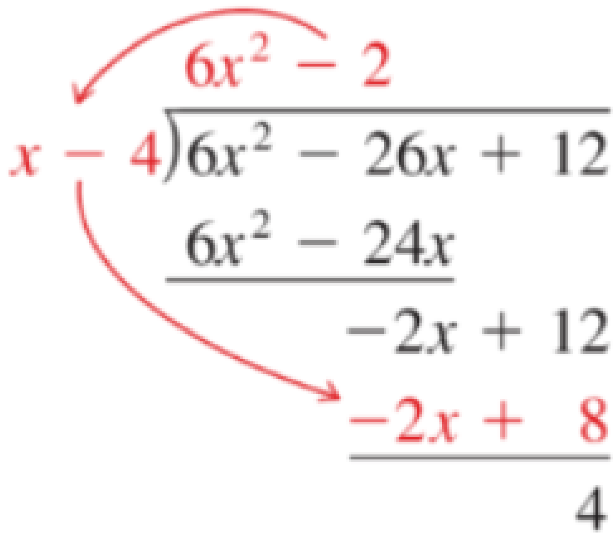


Figure 42: Long division of $6x^2 - 26x + 12$ by $x - 4$.

So we have

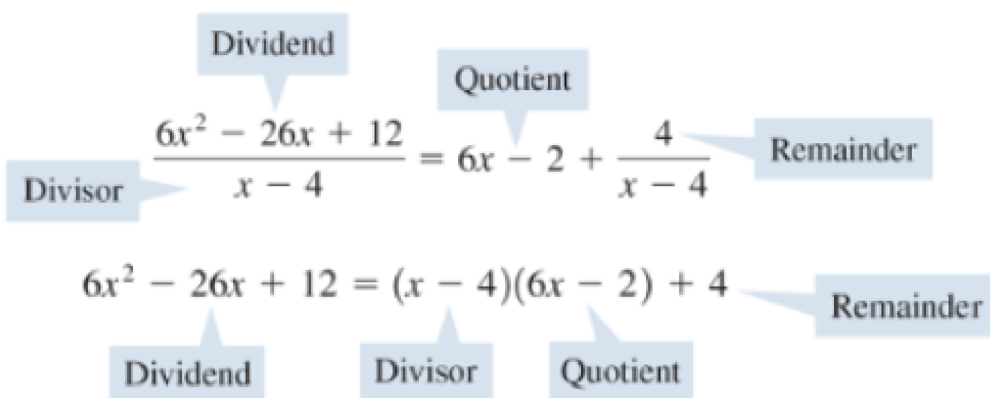


Figure 43: Long division of $6x^2 - 26x + 12$ by $x - 4$.

Here's a fact: c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

Example 3.9: Factor cubic given a zero

Let

$$P(x) = x^3 - 7x + 6$$

Show that $P(1) = 0$ and use this fact to factor $P(x)$ completely.

Solution: Check that $x = 1$ is indeed a zero.

$$P(1) = 1^3 - 7(1) + 6 = 0$$

So $x - 1$ is a factor of $P(x)$. Now use polynomial long division.

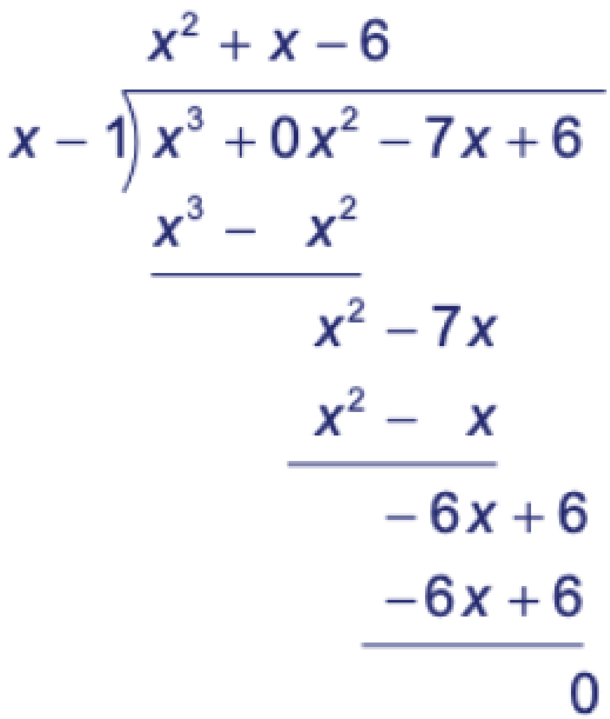


Figure 44: Long division of $x^3 - 7x + 6$ by $x - 1$.

We can factor the resulting quadratic as

$$x^2 + x - 6 = (x + 3)(x - 2)$$

So the polynomial $P(x)$ is factored as

$$P(x) = (x - 1)(x + 3)(x - 2).$$

Now we discuss an alternative but equivalent mechanism to do long division called synthetic division. If you don't want to learn this it's completely fine – but most students usually find it quicker and simpler.

I will not motivate this very well – but just to show you that it works. For example, compare the following example with long division and a completed synthetic division (I have not told you how to do the synthetic division yet).
Here's how to do it.

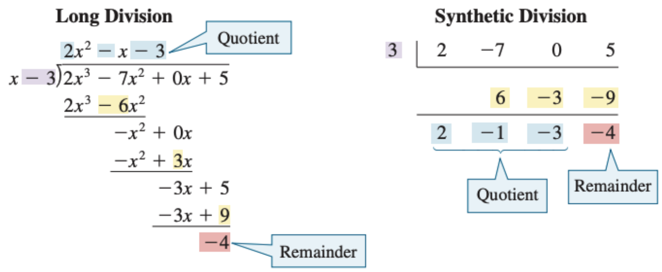


Figure 45: Long division vs Synthetic Division

Here's how to do it.

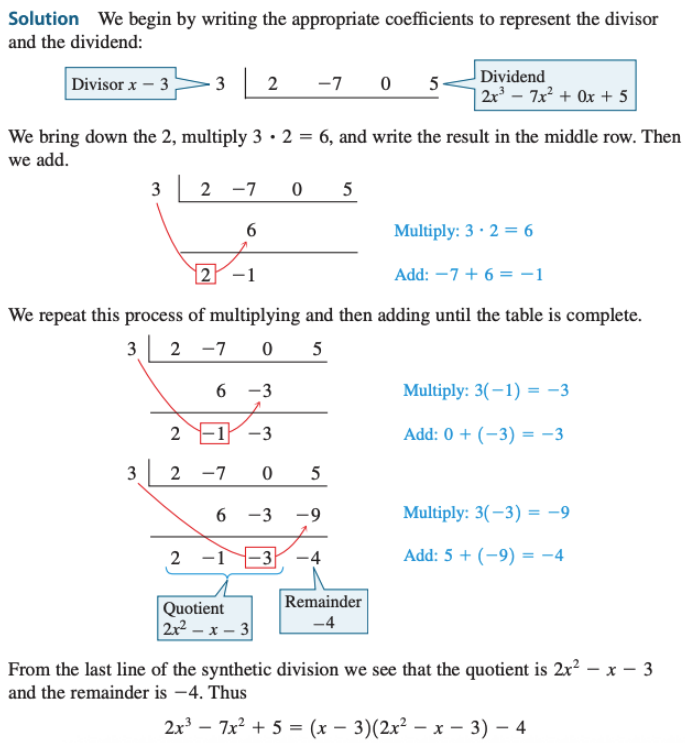


Figure 46: Steps of Synthetic Division

Example 3.10: Synthetic Division

Let

$$P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$$

Use synthetic division to divide by $x + 2$.

Solution:

$$\begin{array}{r|rrrrrr} & 3 & 5 & -4 & 0 & 7 & 3 \\ -2 & & -6 & 2 & 4 & -8 & 2 \\ \hline & 3 & -1 & -2 & 4 & -1 & 5 \end{array}$$

So the quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$ with remainder 5.

Example 3.11: Synthetic/Long Division

Let

$$P(x) = x^3 - 5x^2 - 2x + 10$$

Show $c = 5$ is a zero of this polynomial and find all other zeros.

Solution: It turns out we can factor this polynomial by grouping:

$$\begin{aligned} x^3 - 5x^2 - 2x + 10 &= x^2(x-5) - 2(x-5) \\ &= (x^2-2)(x-5) \end{aligned}$$

but let's pretend I don't know that for now.

Synthetic division gives us

$$\begin{array}{r|rrrr} & 1 & -5 & -2 & 10 \\ 5 & & 5 & 0 & -10 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

So we have $P(x) = (x^2 - 2)(x - 5) = (x - \sqrt{2})(x + \sqrt{2})(x - 5)$.

Polynomial long division gives us:

$$\begin{array}{r} X^2 \qquad\qquad -2 \\ X-5\overline{\smash{)}\; X^3 - 5X^2 - 2X + 10} \\ \underline{-X^3 + 5X^2} \qquad\qquad\quad \\ -2X + 10 \\ \underline{2X - 10} \\ 0 \end{array}$$

3.4 Real Zeros of Polynomials §3.4

We first start off with an observation - zeros of polynomials are factors of the constant term.

For example, consider the polynomial

$$\begin{aligned} P(x) &= (x-2)(x-3)(x+4) \\ &= x^3 - x^2 - 14x + 24 \end{aligned}$$

We can see the zeros of the polynomial are all factors of the constant term.

This brings up to the **Rational Zeros Theorem**, sometimes known as the **Rational Root Theorem**, stated below.

If the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0 \quad a_0 \neq 0$$

has integer coefficients, then every rational zero is of the form $\frac{p}{q}$ where p and q are integers where p is a factor of the constant coefficient a_0 and q is a factor of the leading coefficient a_n.

Example 3.12: Using the Rational Roots Theorem

Find the rational zeros of

$$P(x) = x^3 - 3x + 2$$

Solution: By the theorem, the possible rational zeros are $\pm 1, \pm 2$. Test each of the possibilities (usually starting from the smallest and simplest)

$$\begin{aligned} P(1) &= 1^3 - 3(1) + 2 = 0 \\ P(-1) &= (-1)^3 - 3(-1) + 2 = 4 \\ P(2) &= 2^3 - 3(2) + 2 = 4 \\ P(-2) &= (-2)^3 - 3(-2) + 2 = 0 \end{aligned}$$

So the rational zeros of $P(x)$ are 1 and -2.

This list was short enough that we could go through it. Usually the list is quite long, and we start with the smallest and simplest possibilities. Once we find it, we typically do long division to continue factoring it.

For example, in this case, we would have discovered that $x = 1$ was a root. So we divide $P(x)$ by $x - 1$ by long division

$$\begin{array}{r|l} & X^2 + X - 2 \\ \hline X - 1 & X^3 \qquad\quad - 3X + 2 \\ & -X^3 + X^2 \\ \hline & \qquad X^2 - 3X \\ & \qquad -X^2 + X \\ \hline & \qquad\qquad -2X + 2 \\ & \qquad\qquad 2X - 2 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

or by synthetic division

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

and factor the resulting quadratic was $x^2 + x - 2 = (x + 2)(x - 1)$.

Note one thing – the rational root theorem does not tell us whether a root is a double root like $x = 1$ was in this case. So when you are trying all rational roots and "one of them works", you need to try it again – you can't just discard it because the case of roots of higher than 1 multiplicity are possible.

Example 3.13: Rational Roots Theorem on Quintic

Completely Factor

$$P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$$

The possible rational zeros are $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm 9$.

We always start small. I could start with the smallest, but let me start with ± 1 first.
Attempting to divide out by $x = 1$ gives us

$$\begin{array}{r|rrrrrr} & 2 & 5 & -8 & -14 & 6 & 9 \\ 1 & & 2 & 7 & -1 & -15 & -9 \\ \hline & 2 & 7 & -1 & -15 & -9 & 0 \end{array}$$

Trying $x = 1$ again:

$$\begin{array}{r|rrrrr} & 2 & 7 & -1 & -15 & -9 \\ 1 & & 2 & 9 & 8 & -7 \\ \hline & 2 & 9 & 8 & -7 & -16 \end{array}$$

So 1 is a root of multiplicity 1. Trying -1 gives us

$$\begin{array}{r|rrrrr} & 2 & 7 & -1 & -15 & -9 \\ -1 & & -2 & -5 & 6 & 9 \\ \hline & 2 & 5 & -6 & -9 & 0 \end{array}$$

Trying $x = -1$ again doesn't works.

$$\begin{array}{r|rrrr} & 2 & 5 & -6 & -9 \\ -1 & & -2 & -3 & 9 \\ \hline & 2 & 3 & -9 & 0 \end{array}$$

Trying $x = -1$ yet again does not work. So $x = -1$ is a zero of multiplicity 2.

$$\begin{array}{r|rrr} & 2 & 3 & -9 \\ -1 & & -2 & -1 \\ \hline & 2 & 1 & -10 \end{array}$$

At this point we could just completely factor $2x^2 + 3x - 9 = (2x - 3)(x + 3)$, but for the sake of this exercise pretend we can't. Typically though when you get to a quadratic you can just factor directly and/or use the quadratic formula. Trying $x = \frac{1}{2}$ doesn't work.

$$\begin{array}{r|rrr} & 2 & 3 & -9 \\ \frac{1}{2} & & 1 & 2 \\ \hline & 2 & 4 & -7 \end{array}$$

Trying $x = -\frac{1}{2}$ doesn't work.

$$\begin{array}{r|rrr} & 2 & 3 & -9 \\ -\frac{1}{2} & & -1 & -1 \\ \hline & 2 & 2 & -10 \end{array}$$

Trying $x = \frac{3}{2}$ does work.

$$\begin{array}{r|rrr} & 2 & 3 & -9 \\ \frac{3}{2} & & 3 & 9 \\ \hline & 2 & 6 & 0 \end{array}$$

And what's left is $2x + 6$. So we have $(x-1)(x+1)^2(x-\frac{3}{2})(2x+6) = (x-1)(x+1)^2(2x-3)(x+3)$.

3.5 Rational Functions §3.6

A **Rational Function** is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. Even though $r(x)$ is constructed from polynomials, the graphs of rational functions look quite different from the graphs of polynomial functions.

Let's look at the simplest rational function

$$f(x) = \frac{1}{x}.$$

Let's plot some points like $(1, 1)$ and $(-1, -1)$ and look at the end behavior

$$f(x) \to 0 \text{ as } x \to \pm\infty.$$

Let's look at what happens as x approaches 0 from both sides

$$f(x) \to \infty \text{ as } x \to 0^+$$

$$f(x) \to -\infty \text{ as } x \to 0^-$$

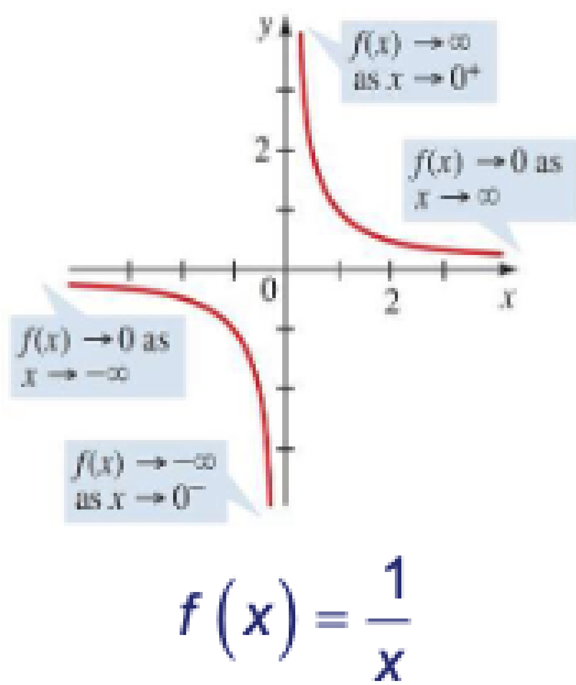


Figure 47: Graph of $f(x) = \frac{1}{x}$

Here we have that $x = 0$ is a **vertical asymptote** and $y = 0$ is a **horizontal asymptotes**. Informally speaking, an asymptote of a function is a line to which the graph of the function gets closer and closer as one travels along that line but never touches it.

For a more formal definition, the line $x = a$ is a vertical asymptote of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or the left.

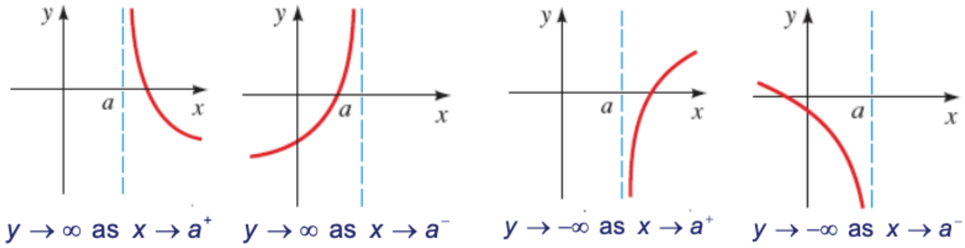


Figure 48: Vertical Asymptotes

Figure 49: Horizontal Asymptotes

The line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.

Example 3.14: Plotting Rational Functions

Graph each rational function, and state the domain and range.

a) $r(x) = \dfrac{2}{x-3}$

b) $s(x) = \dfrac{3x+5}{x+2}$

Solution:

a) This can be found by simply transforming the graph of $\frac{1}{x}$ by the techniques in §2.6.

$$\frac{1}{x} \to \frac{1}{x-3} \to \frac{2}{x-3}$$

So we start with $\frac{1}{x}$, shift it to the right by 3 units, and vertically stretch it by a factor of 2. This function has the domain $x \neq 3$ or in interval notation $(-\infty, 3) \cup (3, \infty)$. There is a vertical asymptote at $x = 3$. The function has range $y \neq 0$ or in interval notation $(-\infty, 0) \cup (0, \infty)$. There is a horizontal asymptote at $y = 0$.

b) For this one, we need to do polynomial long division.

$$\begin{array}{r} 3 \\ X+2\overline{\big)\;\; 3X+5} \\ -\,3X-6 \\ \hline -\,1 \end{array}$$

So the result is

$$\frac{3x+5}{x+2} = 3 - \frac{1}{x+2}$$

Now we can interpret this as a transformation of the parent function $\frac{1}{x}$.

$$\frac{1}{x} \to \frac{1}{x+2} \to -\frac{1}{x+2} \to 3 - \frac{1}{x+2}$$

So we take $\frac{1}{x}$, shift it to the left by 2 units, reflect it across the x axis, and then shift it up by 3. There is a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 3$.

For the graphs – done in class. See Wolfram Alpha or Desmos.

In general, for rational functions

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. If $x = a$ is a root of the denominator but not of the numerator, then $x = a$ is a vertical asymptote.
2. If $n < m$, then r has horizontal asymptote $y = 0$.
3. If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.
4. If $n > m$, then r has no horizontal asymptote.

So far, it has been assumed that the numerator and denominator of a rational function have no factor in common. However, if $s(x) = \frac{p(x)}{q(x)}$ and if p and q do have a factor in common, then that factor cancels, but only for those values of x for which that factor is not zero (because division by zero is not defined).

Since s is not defined at those values of x, its graph has a "**hole**" at those points.

Example 3.15: Rational functions with common factors in the numerator and denominator

Graph the following functions.

a) $s(x) = \frac{x-3}{x^2-3x}$

b) $t(x) = \frac{x^3-2x^2}{x-2}$

Solution:

a) Factor

$$\frac{x-3}{x^2-3x} = \frac{x-3}{x(x-3)} = \frac{1}{x}$$

So this is just the graph of $\frac{1}{x}$ but with a "hole" at $x = 3$.

b) Factor

$$\frac{x^3-2x^2}{x-2} = \frac{x^2(x-2)}{x-2} = x^2$$

Hole at $x = 2$.

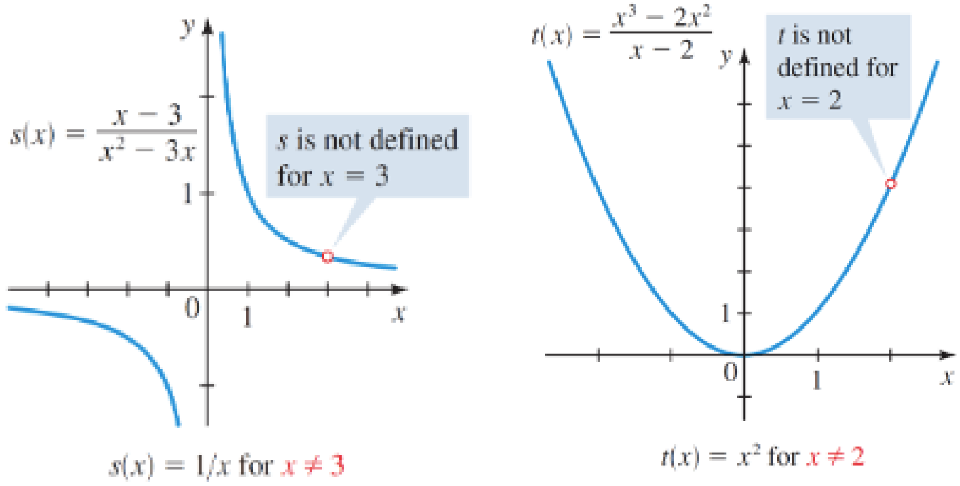


Figure 50: Graphs with holes

Example 3.16: Rational functions plot

Find the vertical and horizontal asymptotes of

$$r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$$

Solution: Factor the denominator:

$$\frac{3x^2 - 2x - 1}{2x^2 + 3x - 2} = \frac{3x^2 - 2x - 1}{(2x - 1)(x + 2)}$$

We can check that neither $x = \frac{1}{2}, x = -2$ are not factors of the numerator, so they are vertical asymptotes. Since the numerator and the denominator have the same degree, the horizontal asymptote is the ratio of their coefficients $y = \frac{3}{2}$.

Example 3.17: Rational functions graph

Graph

$$r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$$

Solution: Factor the numerator and the denominator:

$$\frac{2x^2 + 7x - 4}{x^2 + x - 2} = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$$

The vertical asymptotes are $x = 1, -2$ and the horizontal asymptote is $y = 2$. We can tell $x = \frac{1}{2}, -4$ are x-intercepts and the y-intercepts is $y = 2$.

So now we sketch the asymptotes and the intercepts. Then we test for the behavior near asymptotes to determine whether $y \to \infty$ or $y \to -\infty$ by using test values near the asymptotes. For example, if we want to know what's going on from the LHS of the vertical asymptote $x = 1$, try plugging in a close value like $x = 0.9$. We want to do the same thing for both sides of the both vertical asymptotes. This gives us a rough plot as below.

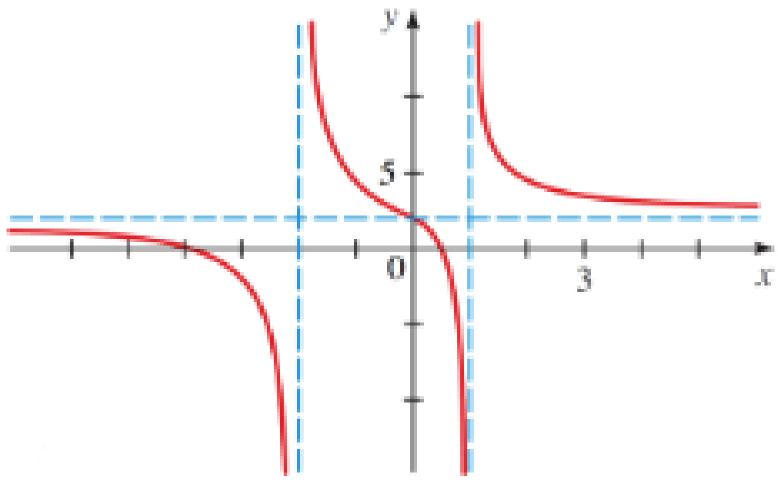


Figure 51: Sketch of $\dfrac{2x^2+7x-4}{x^2+x-2}$

Finally, we consider the special case of

$$r(x) = \frac{P(x)}{Q(x)}$$

where the degree of the numerator $P(x)$ is one more than the degree of the denominator $Q(x)$.

We can divide to express the quotient as

$$f(x) = ax + b + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $a \neq 0$. This means that as $x \to \pm\infty \implies \frac{R(x)}{Q(x)} \to 0$, so for large values of $|x|$ the graph of $y = r(x)$ approaches the graph of the line $y = ax + b$. In this situation, $y = ax + b$ is known as a **slant asymptote**.

Example 3.18: Rational functions with a slant asymptote

Graph the rational function

$$r(x) = \frac{x^2 - 4x - 5}{x - 3}$$

Solution: First factor:

$$r(x) = \frac{x^2 - 4x - 5}{x - 3} = \frac{(x-1)(x-5)}{x-3}$$

The x intercepts are $x = -1, 5$. The y intercept is $y = \frac{5}{3}$. The vertical asymptote is $x = 3$. By polynomial long division, we have

$$\begin{array}{r|l} & X - 1 \\ \hline X - 3) & X^2 - 4X - 5 \\ & -X^2 + 3X \\ \hline & -X - 5 \\ & X - 3 \\ \hline & -8 \end{array}$$

So we have

$$r(x) = \frac{x^2 - 4x - 5}{x - 3} = x - 1 - \frac{8}{x - 3}$$

so the slant asymptote is $x - 1$. There is no horizontal asymptote as the degree of the numerator is larger than the degree of the denominator.

We test the behavior of the function on either side of the vertical asymptote to seee

$$y \to \infty \text{ as } x \to 3^- \text{ and } y \to -\infty \text{ as } x \to 3^+$$

Using the information so far and plugging in a few additional values, we see that

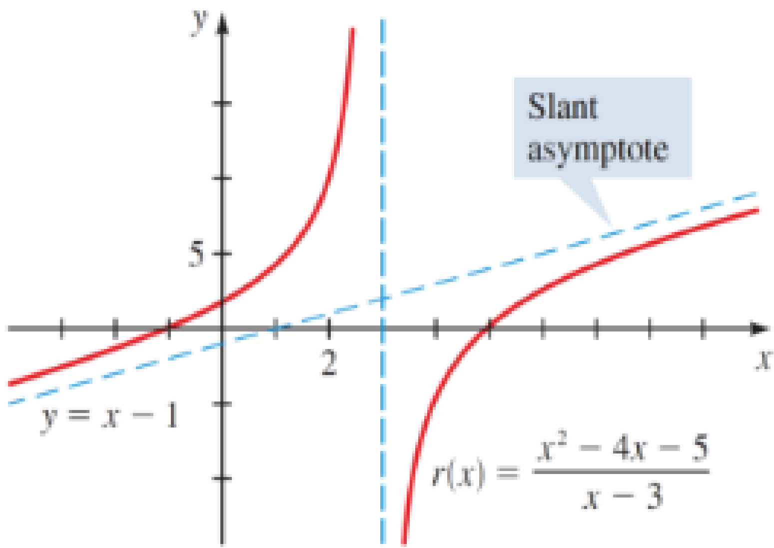


Figure 52: Sketch of $\dfrac{x^2 - 4x - 5}{x - 3}$

3.6 Polynomial and Rational Inequalities §3.7

In order to solve polynomial inequalities, we move all the terms to one side, factor the polynomial, and create a sign chart at the zeros. Conceptually there is nothing new here from when we first introduced this topic in Chapter 1. The main difference is that now we have discussed how to factor polynomials of larger degree using the rational roots theorem and have given a name to rational functions. You may want to review what we did in §1.8.

Example 3.19: Polynomial Inequality I

Solve the inequality

$$2x^3 + x^2 + 6 \geq 13x$$

Solution: Move everything to the left hand side to get

$$2x^3 + x^2 - 13x + 6 \geq 0.$$

Since all the coefficients of the polynomial are integers, we can apply the Rational Root Theorem.
The factors of the leading term are $\pm 1, \pm 2$. The factors of the constant term are $\pm 1, \pm 2, \pm 3, \pm 6$. So all possible candidates are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Try $x = 1$ using long division (or synthetic division).

$$\begin{array}{rrrrr}
 & & 2X^2 & +3X & -10 \\ \cline{2-5}
X-1) & 2X^3 & +X^2 & -13X & +6 \\
 & -2X^3 & +2X^2 & & \\ \cline{2-3}
 & & 3X^2 & -13X & \\
 & & -3X^2 & +3X & \\ \cline{3-4}
 & & & -10X & +6 \\
 & & & 10X & -10 \\ \cline{4-5}
 & & & & -4
\end{array}$$

or by synthetic division

$$\begin{array}{r|rrrr}
 & 2 & 1 & -13 & 6 \\
1 & & 2 & 3 & -10 \\ \hline
 & 2 & 3 & -10 & -4
\end{array}$$

no good. Try $x = -1$ with long division

$$\begin{array}{rrrrr}
 & & 2X^2 & -X & -12 \\ \cline{2-5}
X+1) & 2X^3 & +X^2 & -13X & +6 \\
 & -2X^3 & -2X^2 & & \\ \cline{2-3}
 & & -X^2 & -13X & \\
 & & X^2 & +X & \\ \cline{3-4}
 & & & -12X & +6 \\
 & & & 12X & +12 \\ \cline{4-5}
 & & & & 18
\end{array}$$

or by synthetic division

$$\begin{array}{r|rrrr}
 & 2 & 1 & -13 & 6 \\
-1 & & -2 & 1 & 12 \\ \hline
 & 2 & -1 & -12 & 18
\end{array}$$

also no good. Try $x = 2$ with long division

$$\begin{array}{r} 2X^2 \quad +5X - 3 \\ X-2 \overline{\smash{)}\; 2X^3 \quad + X^2 - 13X + 6} \\ \underline{-\,2X^3 + 4X^2} \qquad\qquad \\ 5X^2 - 13X \quad \\ \underline{-\,5X^2 + 10X} \quad \\ -\,3X + 6 \\ \underline{3X - 6} \\ 0 \end{array}$$

or by synthetic division

	2	1	−13	6
2		4	10	−6
	2	5	−3	0

success! So we have

$$2x^3 + x^2 - 13x + 6 = (x-2)(2x^2 + 5x - 3) = (x-2)(2x-1)(x+3)$$

Notice that we would have eventually found the other two rational roots (or if we tried them in a different order).

For the inequality

$$(x-2)(2x-1)(x+3) \geq 0$$

We create a sign chart with $x = -3, \frac{1}{2}, 2$ to conclude the solution is $\left[-3, \frac{1}{2}\right] \cup [2, \infty)$

Recall what the rational roots theorem says. It says that **IF** a polynomial with integer coefficients has rational roots, then they must be in that list. Consider the polynomial

$$P(x) = x^2 + 2x + 2$$

We can rewrite this in vertex (or standard form) as follows by completing the square:

$$P(x) = x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x+1)^2 + 1$$

So we can sketch this as a parabola with vertex $(-1, 1)$ which opens upwards. So this never crosses the x axis and as such has no real zeros. This does not contradict the rational roots theorem, which says that **IF** this function has rational zeros, it must be in the list $\pm 1, \pm 2$. But since this function does not have any real (much less rational) zeros, there is no contradiction.

We also consider

$$P(x) = x^2 - 2$$

The rational roots theorem again gives us the candidates $\pm 1, \pm 2$. But this polynomial does not have any rational roots. It has two real roots at $x = \pm\sqrt{2}$, which are irrational.

Example 3.20: Polynomial Inequality II

Solve the inequality

$$3x^4 - x^2 - 4 < 2x^3 + 12x$$

Solution: The mechanics of this are identical to that of the previous problem. Move everything to the left hand side to get

$$3x^4 - 2x^3 - x^2 - 12x - 4 < 0$$

Since all the coefficients of the polynomial are integers, we can apply the Rational Root Theorem.
The factors of the leading term are $\pm 1, \pm 3$. The factors of the constant term are $\pm 1, \pm 2, \pm 4$. So all possible candidates are $\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$.

Try $x = 1$ and see we fail.

$$
\begin{array}{r|rrrrr}
 & 3X^3 & +X^2 & & & -12 \\
\hline
X-1 & 3X^4 & -2X^3 & -X^2 & -12X & -4 \\
 & -3X^4 & +3X^3 & & & \\
\hline
 & & X^3 & -X^2 & & \\
 & & -X^3 & +X^2 & & \\
\hline
 & & & & -12X & -4 \\
 & & & & 12X & -12 \\
\hline
 & & & & & -16
\end{array}
$$

$$
\begin{array}{r|rrrrr}
 & 3 & -2 & -1 & -12 & -4 \\
1 & & 3 & 1 & 0 & -12 \\
\hline
 & 3 & 1 & 0 & -12 & -16
\end{array}
$$

Try $x = -1$ and see we fail.

$$
\begin{array}{r|rrrrr}
 & 3X^3 & -5X^2 & +4X & -16 & \\
\hline
X+1 & 3X^4 & -2X^3 & -X^2 & -12X & -4 \\
 & -3X^4 & -3X^3 & & & \\
\hline
 & & -5X^3 & -X^2 & & \\
 & & 5X^3 & +5X^2 & & \\
\hline
 & & & 4X^2 & -12X & \\
 & & & -4X^2 & -4X & \\
\hline
 & & & & -16X & -4 \\
 & & & & 16X & +16 \\
\hline
 & & & & & 12
\end{array}
$$

$$
\begin{array}{r|rrrrr}
 & 3 & -2 & -1 & -12 & -4 \\
-1 & & -3 & 5 & -4 & 16 \\
\hline
 & 3 & -5 & 4 & -16 & 12
\end{array}
$$

Try $x = 2$ and see we **succeed**.

$$
\begin{array}{r|rrrrr}
 & 3X^3 & +X^2 & & & -12 \\
\hline
X-1 & 3X^4 & -2X^3 & -X^2 & -12X & -4 \\
 & -3X^4 & +3X^3 & & & \\
\hline
 & & X^3 & -X^2 & & \\
 & & -X^3 & +X^2 & & \\
\hline
 & & & & -12X & -4 \\
 & & & & 12X & -12 \\
\hline
 & & & & & -16
\end{array}
$$

$$
\begin{array}{r|rrrrr}
 & 3 & -2 & -1 & -12 & -4 \\
2 & & 6 & 8 & 14 & 4 \\
\hline
 & 3 & 4 & 7 & 2 & 0
\end{array}
$$

So we have

$$3x^4 - 2x^3 - x^2 - 12x - 4 = (x-2)(3x^3 + 4x^2 + 7x + 2)$$

We keep going down the list, but now with the smaller polynomial. We try $x = 2$ again (since it could be a root of multiplicity greater than 1 and fail.

$$\begin{array}{r|rrrr} \multicolumn{5}{r}{3X^2 + 10X + 27} \\ \cline{2-5} X-2 & 3X^3 & +4X^2 & +7X & +2 \\ & -3X^3 & +6X^2 & & \\ \cline{2-3} & & 10X^2 & +7X & \\ & & -10X^2 & +20X & \\ \cline{3-4} & & & 27X & +2 \\ & & & -27X & +54 \\ \cline{4-5} & & & & 56 \end{array}$$

$$\begin{array}{r|rrrr} & 3 & 4 & 7 & 2 \\ 2 & & 6 & 20 & 54 \\ \hline & 3 & 10 & 27 & 56 \end{array}$$

Try $x = 4$ and see we fail.

$$\begin{array}{r|rrrr} \multicolumn{5}{r}{3X^2 + 16X + 71} \\ \cline{2-5} X-4 & 3X^3 & +4X^2 & +7X & +2 \\ & -3X^3 & +12X^2 & & \\ \cline{2-3} & & 16X^2 & +7X & \\ & & -16X^2 & +64X & \\ \cline{3-4} & & & 71X & +2 \\ & & & -71X & +284 \\ \cline{4-5} & & & & 286 \end{array}$$

$$\begin{array}{r|rrrr} & 3 & 4 & 7 & 2 \\ 4 & & 12 & 64 & 284 \\ \hline & 3 & 16 & 71 & 286 \end{array}$$

Try $x = -4$ and see we fail.

$$\begin{array}{r|rrrr} \multicolumn{5}{r}{3X^2 - 8X + 39} \\ \cline{2-5} X+4 & 3X^3 & +4X^2 & +7X & +2 \\ & -3X^3 & -12X^2 & & \\ \cline{2-3} & & -8X^2 & +7X & \\ & & 8X^2 & +32X & \\ \cline{3-4} & & & 39X & +2 \\ & & & -39X & -156 \\ \cline{4-5} & & & & -154 \end{array}$$

$$\begin{array}{r|rrrr} & 3 & 4 & 7 & 2 \\ -4 & & -12 & 32 & -156 \\ \hline & 3 & -8 & 39 & -154 \end{array}$$

Try $x = \frac{1}{3}$ and see we fail.

$$
\begin{array}{r}
3X^2 \quad + 5X + \frac{26}{3} \\
X - \frac{1}{3} \overline{\big) \; 3X^3 + 4X^2 \quad + 7X \quad + 2} \\
-\,3X^3 \quad + X^2 \qquad\qquad\quad \\
\hline
5X^2 \quad + 7X \qquad\; \\
-\,5X^2 \quad + \frac{5}{3}X \qquad\; \\
\hline
\frac{26}{3}X \quad + 2 \\
-\,\frac{26}{3}X + \frac{26}{9} \\
\hline
\frac{44}{9}
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 3 & 4 & 7 & 2 \\
\frac{1}{3} & & 1 & \frac{5}{3} & \frac{26}{9} \\
\hline
 & 3 & 5 & \frac{26}{3} & \frac{44}{9}
\end{array}
$$

Try $x = \frac{-1}{3}$ and see we **succeed**.

$$
\begin{array}{r}
3X^2 + 3X + 6 \\
X + \frac{1}{3} \overline{\big) \; 3X^3 + 4X^2 + 7X + 2} \\
-\,3X^3 \quad - X^2 \qquad\qquad\quad \\
\hline
3X^2 + 7X \qquad \\
-\,3X^2 \quad - X \qquad \\
\hline
6X + 2 \\
-\,6X - 2 \\
\hline
0
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 3 & 4 & 7 & 2 \\
-\frac{1}{3} & & -1 & -1 & -2 \\
\hline
 & 3 & 3 & 6 & 0
\end{array}
$$

So we have

$$3x^4-2x^3-x^2-12x-4 = (x-2)(3x^3+4x^2+7x+2) = (x-2)(x-\frac{1}{3})(3x^2+3x+6) = (x-2)(3x-1)(x^2+x+2)$$

This polynomial can be factored as

$$(x-2)(3x+1)(x^2+x+2) < 0$$

We can see the third term has no real roots and in fact is always positive. This can be done by applying the quadratic formula and seeing the discriminant $b^2 - 4ac = 1 - 4(1)(2) = -7 < 0$, or by realizing this is a parabola with the vertex $\left(-\frac{1}{2}, \frac{7}{4}\right)$ that opens upward, so the vertex is a minimum and this has range $\left[\frac{7}{4}, \infty\right)$. So we can essentially ignore it.

Create a sign chart with $x = -\frac{1}{3}, 2$ to conclude the solution is $\left(-\frac{1}{3}, 2\right)$.

In order to solve rational inequalities, we move all the terms to one side, factor the numerator and the denominator, and create a sign chart at where both the numerator and the denominator are zero.

Example 3.21: Rational Inequality I

Solve the inequality

$$\frac{1-2x}{x^2-2x-3} \geq 1$$

Solution: Move everything to the left hand side to get

$$\begin{aligned}\frac{1-2x}{x^2-2x-3} \geq 1 &\implies \frac{1-2x}{x^2-2x-3} - \frac{x^2-2x-3}{x^2-2x-3} \geq 0 \\ &\implies \frac{(1-2x)-(x^2-2x-3)}{x^2-2x-3} \geq 0 \\ &\implies \frac{4-x^2}{x^2-2x-3} \geq 0 \\ &\implies \frac{-(x-2)(x+2)}{(x-3)(x+1)} \geq 0\end{aligned}$$

So we create a sign chart at $x = -2, -1, 2, 3$ and get the interval between -2 and -1 as well as the interval between 2 and 3. We must exclude $x = -1, 3$ since they would result in division by 0. to conclude that $[-2, -1) \cup [2, 3)$.

Example 3.22: Rational Inequality II

Solve the inequality

$$\frac{(x-1)^2}{(x+1)(x+2)} > 0$$

Solution: We need to create a sign chart at $x = -2, -1, 2$ to conclude that $(-\infty, -2) \cup (-1, 1) \cup (1, \infty)$.

Example 3.23: Polynomial Inequality WITHOUT rational roots theorem

Solve the inequality

$$2x^4 + 5x^2 - 3 \geq 0$$

Solution: After this class you are probably tempted to take the Rational Roots Theorem and try to find zeros of this. But actually, we can factor this using techniques for factoring binomials. If it helps, define a new variable $y = x^2$ so we have

$$2y^2 + 5y - 3 = (2y-1)(y+3) = (2x^2-1)(x^2+3) \geq 0$$

Create a sign chart with $x = 0, \pm\sqrt{\frac{1}{2}}$ to conclude $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$.

4 Exponential and Logarithmic Functions §4

4.1 Exponential functions §4.1

Here we discuss a new class of functions called exponential functions. For example

$$f(x) = 2^x$$

is an exponential function (with base 2).

Notice how quickly the values of this function increase

$$\begin{aligned} f(3) &= 2^3 = 8 \\ f(10) &= 2^{10} = 1024 \\ f(30) &= 2^{30} = 1,073,741,824 \end{aligned}$$

hence the term "exponential growth".

Compare this with the function $f(x) = x^2$ (polynomial growth):

$$\begin{aligned} f(3) &= 3^2 = 9 \\ f(10) &= 10^2 = 100 \\ f(30) &= 30^2 = 900 \end{aligned}$$

The exponential function $f(x) = a^x$ is defined for all numbers where $a > 0$ and $a \neq 1$. $a = 1$ is not interesting because then it is just $f(x) = 1$.

$a > 1$ corresponds to exponential growth. $0 < a < 1$ corresponds to exponential decay.

Example 4.1: Graph Exponential Functions

Graph each function.

a) $f(x) = 3^x$

b) $g(x) = \left(\frac{1}{3}\right)^x$

Solution:

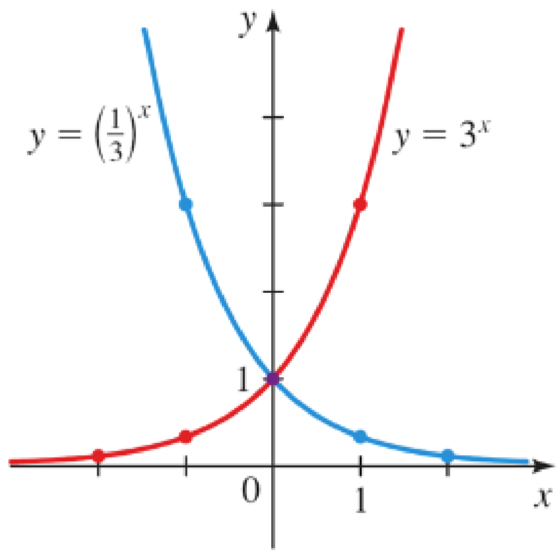


Figure 53: Graphs of 3^x and $\left(\frac{1}{3}\right)^x$

Actually, $g(x) = \left(\frac{1}{3}\right)^x = 3^{-x} = f(-x)$. So $f(x)$ and $g(x)$ are the same graphs but reflected across the y-axis.

We can see various exponential functions for different bases a. All these graphs pass through the point $(0, 1)$ because $a^0 = 1$ for $a \neq 0$. If $0 < a < 1$ the exponential function decreases rapidly. If $a > 1$ the function increases rapidly. For these exponential functions, they have a horizontal asymptote $y = 0$.

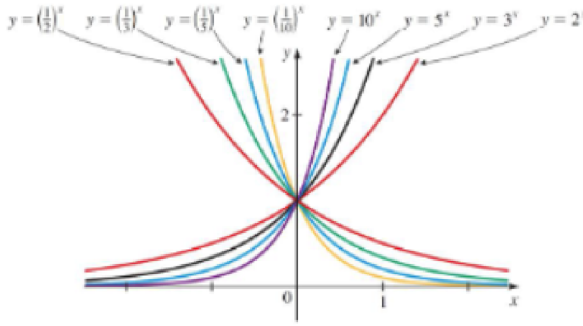
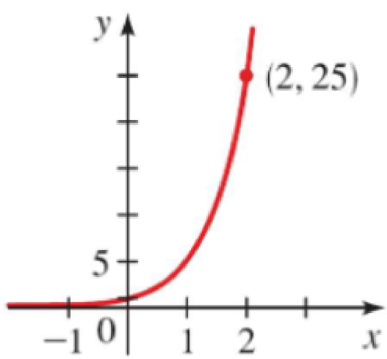
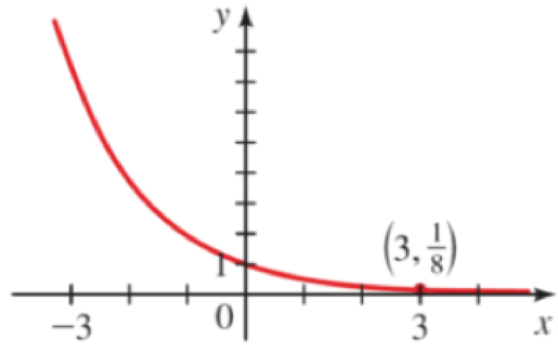


Figure 54: A family of exponential graphs with different bases x.

Example 4.2: Find equations of exponential functions

Find the exponential $f(x) = a^x$ whose graph is given.





Solution: If $y = a^x$, plugging in the point $(2, 25)$ gives us $25 = a^2 \implies a = 5$. Plugging in the point $\left(3, \frac{1}{8}\right)$ gives us $\frac{1}{8} = a^3 \implies a = \frac{1}{2}$.

Now we have the exponential function as another one of our "parent" functions, so we can apply transformations to them as such.

Example 4.3: Transformations of exponential functions

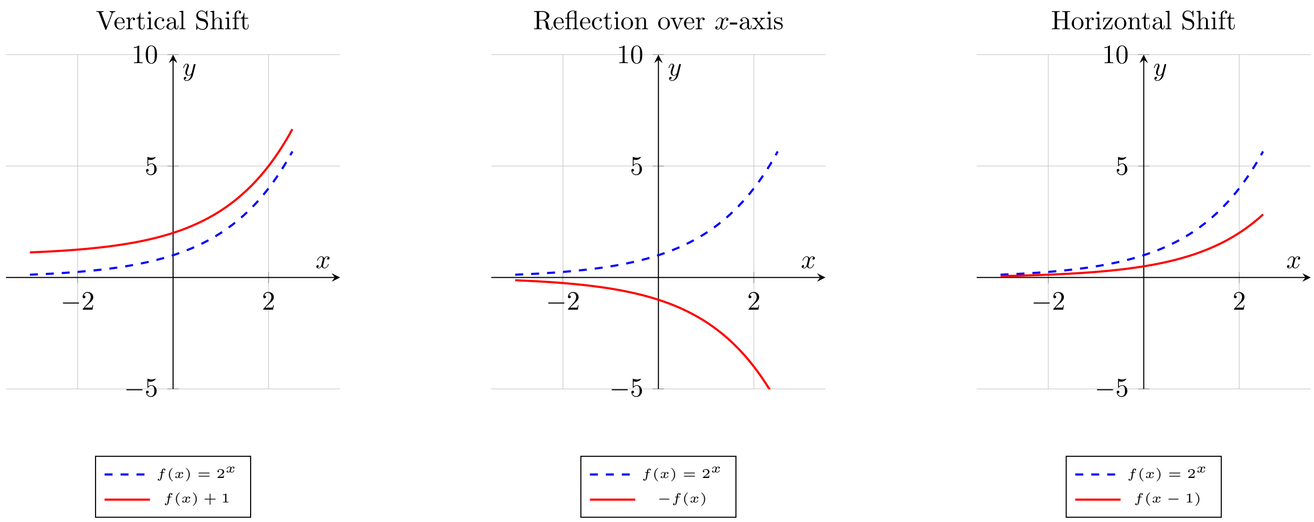
Given the parent function $f(x) = 2^x$, sketch the following by interpreting them as transformations of the parent function $f(x)$.

a) $g(x) = 1 + 2^x$

b) $h(x) = -2^x$

c) $k(x) = 2^{x-1}$

Solution:
a) is Shifting up by 1 unit $f(x)+1$. b) is reflection over the x-axis $-f(x)$. c) is a bit tricky. Recall laws of exponents and rewrite $2^{x-1} = 2^x 2^{-1} = \frac{1}{2}2^x$. So you can interpret this as a horizontal shrink. Or you can think of this as $f(x-1)$ a horizontal shift. These two things do not conflict!



Let's close with a quick discussion on a type of "exponential function" we don't really care about. From the outset we have been considering the cases $0 < a < 1$ for exponential decay and $a > 1$ for exponential growth. But what about a negative a?

Suppose we have $(-5)^x$. What does this look like? Even just looking at a few points $(-5)^1 = -5, (-5)^2 = 25, (-5)^3 = -125$, etc. it should be clear this oscillates in sign! There is nothing wrong with this function mathematically. It's just that we don't really particularly care about this case. It just is not that useful – exponential growth and decay both model realistic phenomena in our universe. This is nothing more than a purely mathematical construct.

4.2 The natural exponential function §4.2

Any positive number can be used as a base for an exponential function. This short section concerns the special base e.

What is e? It is a real, irrational number which has the value $e \approx 2.71828$. In calculus speak, it is the limit of the expression $\left(1 + \frac{1}{n}\right)^n$ as n goes to infinity. It is the number that expression approaches as n grows large.

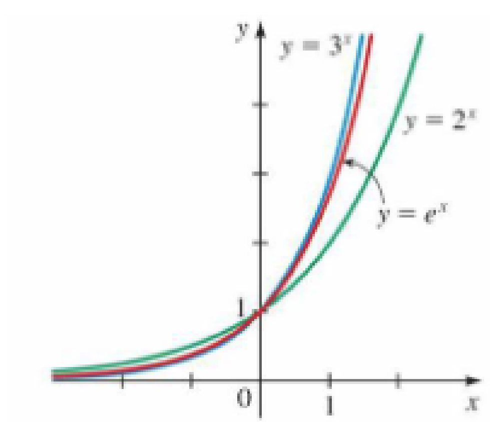
n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,00,0000	2.71828

One way to interpret this is as follows. Suppose you have 1 dollar. and the bank promises you 100% interest after a year. So you would end up with 2. Now suppose you get 100% compounded twice a year. So you would get $(1 + 0.5)(1 + 0.5) = 1.5^2 = 2.25$.

Now suppose it is compounded five times a year. You would get $(1.2)^5 = 2.48832$. In the as n goes to infinity, this is known as continuously compounded interest and would approach the value e.

This value is not well motivated in this class but will be in calculus - just accept it for now as a very common and important exponential function base.

Since $2 < e < 3$, we have that it lies in between the graphs of 2^x and 3^x.



4.3 Logarithmic functions §4.3

Exponential functions are one-to-one by the horizontal line test and as such have inverse functions. These are known as logarithmic functions.

Let a be a positive number with $a \neq 1$. Then the **logarithmic function with base** a, denoted by $\log_a(x)$ is defined by

$$\log_a x \iff a^y = x$$

So $\log_a x$ is the exponent to which the base a must be raised to give x.

The logarithmic and exponential forms are equivalent. It is useful to switch between one and the other. Here are some examples:

$$\begin{aligned}
\log_{10} 100,000 = 5 &\iff 10^5 = 100,000 \\
\log_2 8 = 3 &\iff 2^3 = 8 \\
\log_2 \left(\frac{1}{8}\right) &\iff 2^{-3} = \frac{1}{8} \\
\log_5 s = r &\iff 5^r = s
\end{aligned}$$

Example 4.4: Evaluating logarithmic functions

a) $\log_{10} 1000$

b) $\log_2 32$

c) $\log_{10} 0.1$

d) $\log_{16} 4$

e) $\log_{1/2} 16$

Solution:

a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

b) $\log_2 32 = 5$ because $2^5 = 32$

c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$

e) $\log_{1/2} 16 = -4$ because $\left(\frac{1}{2}\right)^{-4} = 16$.

Here are some important properties of logarithms which arise from the laws of exponents.

a) $\log_a 1 = 0$ because $a^0 = 1$

b) $\log_a a = 1$ because $a^1 = a$

c) $\log_a a^x = x$ because $a^x = a^x$

d) $a^{\log_a x} = x$ because $\log_a x$ is the power to which a must be raised to get x.

The third property is useful because if we have a variable in the exponent, we can take logarithms to bring it down.

The fourth property is useful because if we have a log, we can get rid of it by using it as an exponent to raise the base by. It might seem a little "circular" at first – but this can be verified by taking log base a of both sides.

The following are concrete examples of these properties.

a) $\log_5 1 = 0$

b) $\log_5 5 = 1$

c) $\log_5 5^8 = 8$

d) $5^{\log_5 12} = 12$

Now we interpret graphically the logarithm as an inverse of the exponential function. If a one-to-one function f has domain A and range B, then the inverse function f^{-1} has domain B and range A.

Since the exponential function has domain $\mathbb{R}$ and range $(0, \infty)$, then the logarithmic function has domain $(0, \infty)$ and range $\mathbb{R}$.

We obtain the graph of $f^{-1}(x) = \log_a x$ by reflecting the graph of $f(x) = a^x$ across the line $y = x$.

The below show the case where $a > 1$.

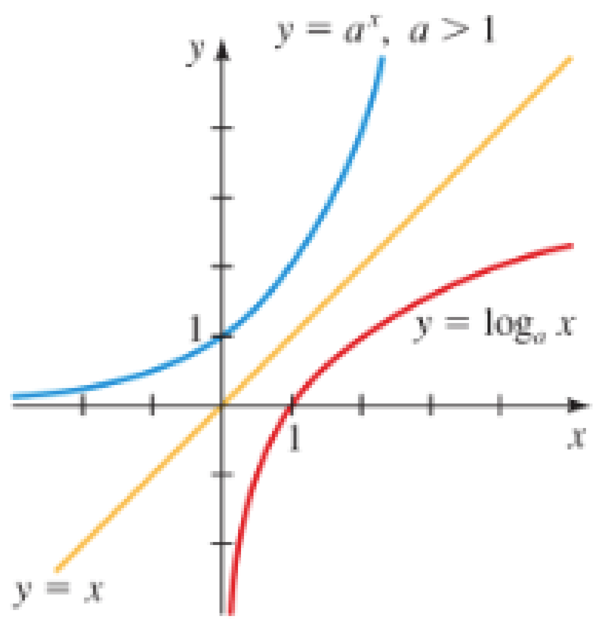


Figure 55: Graphs of a^x and $\log_a x$

Since a^x is a very rapidly increasing function, then $\log_a x$ is a very slowly increasing function.

Since a^x has a horizontal asymptote at $y = 0$, then $\log_a x$ has a vertical asymptote at $x = 0$.

Example 4.5: Sketch of $\log_2 x$

Sketch the graph of $\log_2 x$ by plotting points.

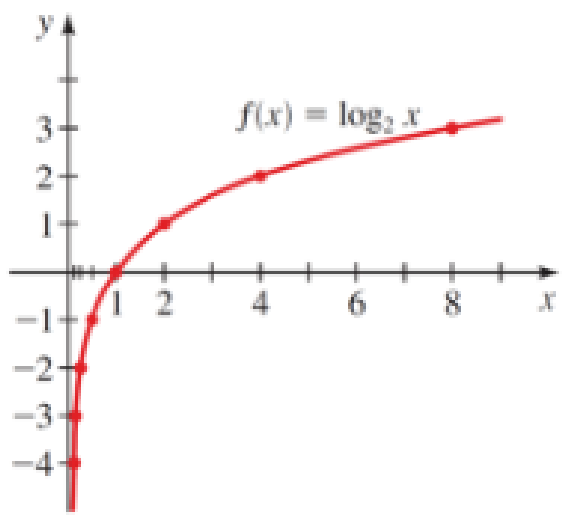


Figure 56: Graphs of $\log_2 x$

The below is a family of logarithmic functions for various a.

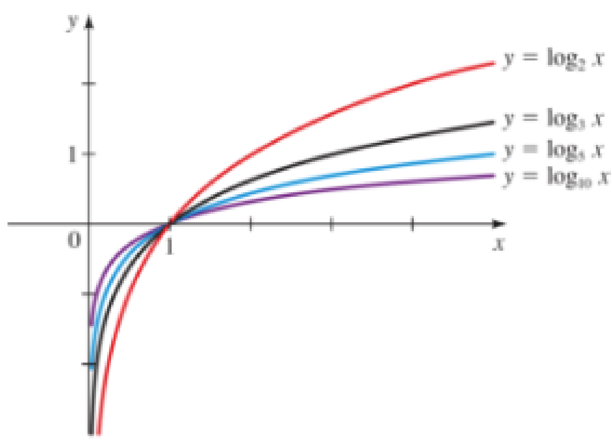


Figure 57: A family of logarithmic functions

Example 4.6: Sketch of transformed logs

Sketch the graph of each function. State the domain, range and asymptote.

a) $g(x) = -\log_2 x$

b) $h(x) = \log_2(-x)$

c) $g(x) = 2 + \log_5 x$

d) $h(x) = \log_{10}(x - 3)$

Solution: For a), we first start with the graph $f(x) = \log_2 x$ and reflect it across the x-axis. For b), we reflect it across the y-axis.

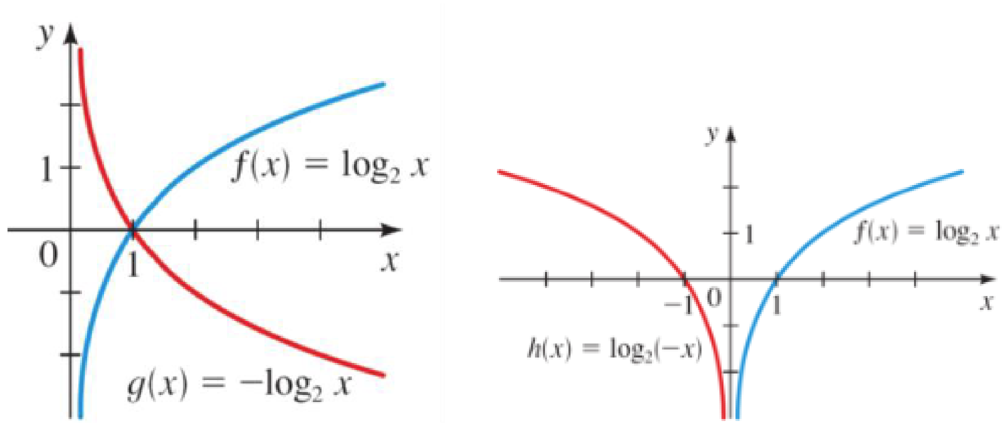


Figure 58: Two reflected logs

For c) we shift it up by 5 units. For d) we shift it to the right by 3 units.

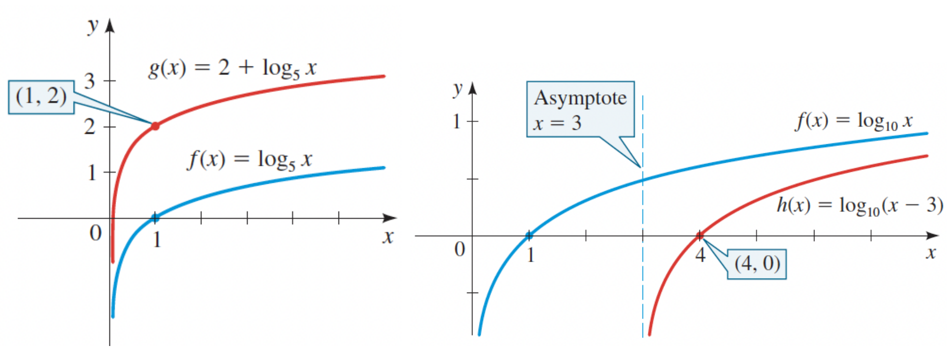


Figure 59: Two shifted logs

When you just write log, it is frequently understood to be base 10.

$$\log x = \log_{10} x$$

We write ln to represent $\log_e$

$$\ln x = \log_e x$$

So $\ln x$ is the inverse function of e^x.

Example 4.7: Domains of logs

Find the domains of the following functions.

a) $g(x) = -\ln(x-2)+3$

b) $f(x) = \ln(4-x^2)$

Solution: We require the argument of the logarithmic to be strictly positive

a) We require $x - 2 > 0 \implies x > 2$.

b) This boils down to solving the inequality $4 - x^2 > 0$. So we factor and create the sign chart at $x = \pm 2$

$$(2-x)(2+x) > 0$$

to conclude that the domain is $(-2, 2)$.

Example 4.8: Evaluate logs

Evaluate the expression.

a) $\log_2 2$

b) $\log_5 1$

c) $\log_{1/2} 2$

d) $\log_3 3^7$

e) $\log_4 64$

f) $\log_{1/2} 0.25$

g) $\log_3 \frac{1}{27}$

h) $\log_{1/3} 27$

i) $\log_7 \sqrt{7}$

j) $3^{\log_3 5}$

k) $5^{\log_5 27}$

l) $e^{\ln 10}$

m) $e^{\ln \sqrt{3}}$

n) $e^{\ln\left(\frac{1}{\pi}\right)}$

o) $10^{\log_{10} 13}$

p) $\log_8 0.25$

q) $\ln e^4$

r) $\ln\left(\frac{1}{e}\right)$

s) $\log_4 \sqrt{2}$

t) $\log_4\left(\frac{1}{2}\right)$

u) $\log_4 8$

Solution:

a) $\log_2 2 = 1$

b) $\log_5 1 = 0$

c) $\log_{1/2} 2 = -1$

d) $\log_3 3^7 = 7$

e) $\log_4 64 = 3$

f) $\log_{1/2} 0.25 = 2$

g) $\log_3 \frac{1}{27} = -3$

h) $\log_{1/3} 27 = -3$

i) $\log_7 \sqrt{7} = \frac{1}{2}$

j) $3^{\log_3 5} = 5$

k) $5^{\log_5 27} = 27$

l) $e^{\ln 10} = 10$

m) $e^{\ln \sqrt{3}} = \sqrt{3}$

n) $e^{\ln\left(\frac{1}{\pi}\right)} = \frac{1}{\pi}$

o) $10^{\log_{10} 13} = 13$

p) $\log_8 0.25 = -\frac{2}{3}$

q) $\ln e^4 = 4$

r) $\ln\left(\frac{1}{e}\right) = \frac{1}{3}$

s) $\log_4 \sqrt{2} = \frac{1}{4}$

t) $\log_4 \left(\frac{1}{2}\right) = -\frac{1}{2}$

u) $\log_4 8 = \frac{3}{2}$

Example 4.9: Sole simple logarithmic equations.

Evaluate the expression.

a) $\log_6 x = 2$

b) $\log_{10} 0.001 = x$

c) $\log_{1/3} x = 0$

d) $\log_4 1 = x$

e) $\ln x = 3$

f) $\ln e^2 = x$

g) $\ln x = -1$

h) $\ln\left(\frac{1}{e}\right) = x$

i) $\log_4 \frac{1}{64} = x$

j) $\log_{\frac{1}{2}} x = 3$

k) $\log_9 \frac{1}{3} = x$

l) $\log_9 x = 0.5$

m) $\log_x 1000 = 3$

n) $\log_x 25 = 2$

Solution:

a) $\log_6 x = 2 \implies x = 36$

b) $\log_{10} 0.001 = x \implies x = -\frac{1}{3}$

c) $\log_{1/3} x = 0 \implies x = 1$

d) $\log_4 1 = x \implies x = 0$

e) $\ln x = 3 \implies x = e^3$

f) $\ln e^2 = x \implies x = 2$

g) $\ln x = -1 \implies x = e^{-1}$

h) $\ln\left(\frac{1}{e}\right) = x \implies x = -1$

i) $\log_4 \frac{1}{64} = x \implies x = -3$

j) $\log_{\frac{1}{2}} x = 3 \implies x = \frac{1}{8}$

k) $\log_9 \frac{1}{3} = x \implies x = -\frac{1}{2}$

l) $\log_9 x = 0.5 \implies x = 3$

m) $\log_x 1000 = 3 \implies x = 10$

n) $\log_x 25 = 2 \implies x = 5$

4.4 Laws of Logarithms §4.4

The laws of exponents give rise to the laws of logarithms:

1. $\log_a(AB) = \log_a A + \log_a B$
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$
3. $\log_a(A^c) = C\log_a A$

Let me explain that statement a bit more in detail by proving each one of these properties of logarithms.

Proof 1: Let $x = b^p$ and $y = b^q$, which imply $xy = b^{p+q}$. Then we have

$$\begin{aligned}\log_b(xy) &= p + q \\ &= \log_b x + \log_b y\end{aligned}$$

Proof 2: Let $x = b^p$ and $y = b^q$, which imply $\frac{x}{y} = b^{p-q}$. Then we have

$$\begin{aligned}\log_b(xy) &= p - q \\ &= \log_b x - \log_b y\end{aligned}$$

Proof 3: Let $x = b^p$which implies $x^n = (b^p)^n = b^{pn}$. Then we have

$$\begin{aligned}\log_b(x^n) &= pn \\ &= n\log_x b\end{aligned}$$

Example 4.10: Applying Laws of Logarithms

a) $\log_4 2 + \log_4 32$

b) $\log_2 80 - \log_2 5$

c) $-\frac{1}{3}\log 8$

Solution:

a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) = \log_4 64 = 3$

b) $\log_2 80 - \log_2 5 = \log_2 16 = 4$

c) $-\frac{1}{3}\log 8 = \log 8^{-1/3} = \log\frac{1}{2}$

Example 4.11: Expanding expressions with laws of logarithms

a) $\log_2(6x)$

b) $\log_5(x^3y^6)$

c) $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$

d) $\log_5(xy)^6$

e) $\ln(y^2\sqrt{x})$

f) $\ln\frac{x^2}{\sqrt{x+1}}$

g) $\log\sqrt{x\sqrt{y\sqrt{z}}}$

Solution:

a) $\log_2(6x) = \log_2 6 + \log_2 x$

b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$

c) $\ln\left(\frac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c} = \ln a + \ln b - \frac{1}{3}\ln c$

d) $\log_5(xy)^6 = 6\log_5(xy) = 6(\log_5 x + \log_5 y)$

e) $\ln(y^2\sqrt{x}) = \ln(y^2) + \ln(\sqrt{x}) = 2\ln y + \frac{1}{2}\ln x$

f) $\ln\frac{x^2}{\sqrt{x+1}} = \ln(x^2) - \ln\sqrt{x+1} = 2\ln x - \frac{1}{2}\ln(x+1)$

g) $\log\sqrt{x\sqrt{y\sqrt{z}}} = \frac{1}{2}\log(x\sqrt{y\sqrt{z}}) = \frac{1}{2}(\log x + \log\sqrt{y\sqrt{z}}) = \frac{1}{2}(\log x + \frac{1}{2}\log(y\sqrt{z})) = \frac{1}{2}(\log x + \frac{1}{2}(\log y + \frac{1}{2}\log z)) = \frac{1}{2}\log x + \frac{1}{4}\log y + \frac{1}{8}\log z$

Example 4.12: Combining logarithmic expressions

a) $3\log x + \frac{1}{2}\log(x+1)$

b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$

c) $\ln(x+3) - \ln(x^2-9)$

d) $4(\log_3 a - 3\log_3 b + 2\log_3 c)$

Solution:

a) $3\log x + \frac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{1/2} = \log(x^3(x+1)^{1/2})$

b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4 = \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right)$

c) $\ln(x+3) - \ln(x^2-9) = \ln\left(\frac{x+3}{x^2-9}\right) = \ln\left(\frac{1}{x-3}\right) = -\ln(x-3)$

d) $4(\log_3 a - 3\log_3 b + 2\log_3 c) = \log_3 a^4 - \log_3 b^{12} + \log_3 c^8 = \log_3\left(\frac{a^4c^8}{b^{12}}\right)$

4.5 Solving Equations with Exponential and Logarithmic Functions §4.5

We know that since exponential functions and logarithmic functions are one-to-one, that

$$a^x = a^y \implies x = y$$
$$\log_a x = \log_a y \implies x = y$$

Example 4.13: Solving an exponential equation I

Solve each exponential equation:

a) $5^x = 125$

b) $5^{2x} = 5^{x+1}$

Solution: These are relatively straightforward in the sense that we don't need to do the typical "do some algebra and solve for x". For a) it is quite clear $x = 3$. However, if we wanted to solve for x we could take $\log_3$ on both sides to conclude

$$5^x = 125 \implies \log_5(5^x) = \log_5(125) \implies x = 3$$

We will see that we have to do this going forward for more general, difficult examples. As for b), we need to ensure the exponents are the same, so we have

$$2x = x + 1 \implies x = 1$$

Example 4.14: Solving an exponential equation II

Solve the equation $3^{x+2} = 7$

Solution: Take the logarithm (any base) of both sides to bring the unknown down, and then do algebra to isolate for x.

$$\begin{aligned} 3^{x+2} = 7 &\implies (x+2)\log 3 = \log 7 \\ &\implies x = \frac{\log 7}{\log 3} - 2 \end{aligned}$$

We can also take $\log_3$ of both sides.

$$\begin{aligned} 3^{x+2} = 7 &\implies x + 2 = \log_3 7 \\ &\implies x = \log_3 7 - 2 \end{aligned}$$

Either is fine. Let me justify their equivalence by the so called "change of basis formula".

Suppose we know $\log_a x$, but we need to know $\log_b x$. Start with $y = \log_b x$.

$$\begin{aligned} b^y &= x \\ \log_a(b^y) &= \log_a x \\ y \log_a b &= \log_a x \\ y &= \frac{\log_a x}{\log_a b} \end{aligned}$$

This gives us the change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Recall that by the change of base formula we have

$$\log_3 7 = \frac{\log 7}{\log 3}$$

Example 4.15: Solving an exponential equation III

Solve the equation $8e^{2x} = 20$

Solution: Divide both sides by 8:

$$8e^{2x} = 20 \implies e^{2x} = \frac{5}{2}.$$

Take natural logs of both sides:

$$2x = \ln\left(\frac{5}{2}\right)$$

Divide both sides by 2

$$x = \frac{1}{2}\ln\left(\frac{5}{2}\right)$$

Example 4.16: Solving an exponential equation IV

Solve the equation $e^{2x} - e^x - 6 = 0$

Solution: x appears twice here. So it's not as if I can easily just isolate x with the operations of $+, -, \times, \div$ Suppose I tried this. I would get

$$\begin{aligned} e^{2x} - e^x - 6 = 0 &\implies e^{2x} = e^x + 6 \\ &\implies 2x = \ln(e^x + 6) \\ &\implies x = \frac{1}{2}\ln(e^x + 6) \end{aligned}$$

While this is true, x is on both sides and we have not solved the equation.

We can interpret this as a quadratic by rewriting the original equation as so:

$$(e^x)^2 - e^x - 6 = 0$$

So factoring gives us

$$(e^x)^2 - e^x - 6 = 0 \implies (e^x - 3)(e^x + 2) = 0$$

So we must have either $e^x = 3$ or $e^x = -2$. The latter is not possible for any value of x since the exponential function only takes on positive values. So the only solution to this equation is

$$e^x = 3 \implies x = \ln(3)$$

Example 4.17: Solving an exponential equation V

Solve $10^{x^2 - 2x} = 1000$

Solution: We know the exponent must be 3 in order for this to be true. So we have

$$\begin{aligned} x^2 - 2x = 3 &\implies x^2 - 2x - 3 = 0 \\ &\implies (x - 3)(x + 1) = 0 \\ &\implies x = 3, -1. \end{aligned}$$

Example 4.18: Solving an exponential equation VI

Solve $3xe^x + x^2e^x = 0$

Solution: Factor the left side of the equation

$$\begin{aligned} 3xe^x + x^2e^x = 0 &\implies x(3+x)e^x = 0 \\ &\implies x(3+x) = 0 \implies x = 0, -3 \end{aligned}$$

The division of e^x is justified because it is never 0,

Example 4.19: Solving an exponential equation VII

Solve $10^{x+3} = 6^{2x}$

Solution: Unlike before, it's not clear what base of logarithmic you should use to make things "cleaner looking". The truth is it doesn't matter what base you use. Suppose I use $\log_{10}$:

$$\begin{aligned} 10^{x+3} = 6^{2x} &\implies x + 3 = (2x)\log_{10} 6 \\ &\implies x - (2x)\log_{10} 6 = -3 \\ &\implies x = \frac{-3}{1 - 2\log_{10} 6} \end{aligned}$$

Suppose I use $\log_6$:

$$\begin{aligned} 10^{x+3} = 6^{2x} &\implies (x+3)\log_6 10 = 2x \\ &\implies x\log_6 10 - 2x = -3\log_6 10 \\ &\implies x = \frac{-3\log_6 10}{\log_6 10 - 2} \end{aligned}$$

Suppose I use a log without specifying the base yet.

$$\begin{aligned} 10^{x+3} = 6^{2x} &\implies (x+3)\log 10 = 2x\log 6 \\ &\implies x\log 10 - 2x\log 6 = -3\log 10 \\ &\implies x = \frac{-3\log 10}{\log 10 - 2\log 6} \end{aligned}$$

All of these are acceptable. I can show the third reduces to the first or the second. Let the base be 10, then

$$x = \frac{-3\log 10}{\log 10 - 2\log 6} = \frac{-3}{1 - 2\log_{10} 6}$$

which agrees exactly with our previous result. Let be base be 6, then

$$x = \frac{-3\log 10}{\log 10 - 2\log 6} = \frac{-3\log_6 10}{\log_6 10 - 2}$$

which agrees exactly with out previous result.

Example 4.20: Solving an logarithmic equation I

Solve $\log_5(x^2+1) = \log_5(x-2) + \log_5(x+3)$

Solution: By properties of logarithms, combine the RHS and then equate the arguments.

$$\begin{aligned} \log_5(x^2+1) = \log_5(x-2) + \log_5(x+3) &\implies \log_5(x^2+1) = \log_5((x-2)(x+3)) \\ &\implies x^2 + 1 = x^2 + x - 6 \\ &\implies x = 7 \end{aligned}$$

Example 4.21: Solving an logarithmic equation II

Solve $\log_2(25 - x) = 3$

Solution:

$$\begin{aligned}\log_2(25 - x) = 3 &\implies 25 - x = 2^3 \\ &\implies x = 17\end{aligned}$$

Example 4.22: Solving an logarithmic equation III

Solve $4 + 3\log_{10}(2x) = 16$

Solution:

$$\begin{aligned}4 + 3\log_{10}(2x) = 16 &\implies 3\log_{10}(2x) = 12 \\ &\implies \log_{10}(2x) = 4 \\ &\implies 2x = 10^4 \\ &\implies x = 5000\end{aligned}$$

Example 4.23: Solving an logarithmic equation IV

Solve $\log_{10}(x + 2) + \log_{10}(x - 1) = 1$

Solution:

$$\begin{aligned}\log_{10}(x + 2) + \log_{10}(x - 1) = 1 &\implies \log_{10}((x + 2)(x - 1)) = 1 \\ &\implies (x + 2)(x - 1) = 10 \\ &\implies x^2 + x - 12 = 0 \\ &\implies (x + 4)(x - 3) = 0 \\ &\implies x = 3, -4\end{aligned}$$

However, notice that we solved this equation without any consideration of the restriction that the argument of a logarithm must be strictly positive. So plugging back in gives us that $x = -4$ is an extraneous solution.

5 Trigonometric Functions: Unit Circle Approach §5

Now we begin the longest and core unit of the course – trigonometry. This topically is objectively more difficult because it builds up so much new machinery. I will try to take it relatively slow but literally every day depends on the previous day.

5.1 The Unit Circle §5.1

Recall that the equation for a circle is given by

$$(x - x_0)^2 + (y - y_0)^2 = r^2,$$

which is the circle with center (x_0, y_0) and radius r.

So the unit circle, with center $(0, 0)$ and radius 1, has the equation

$$x^2 + y^2 = 1$$

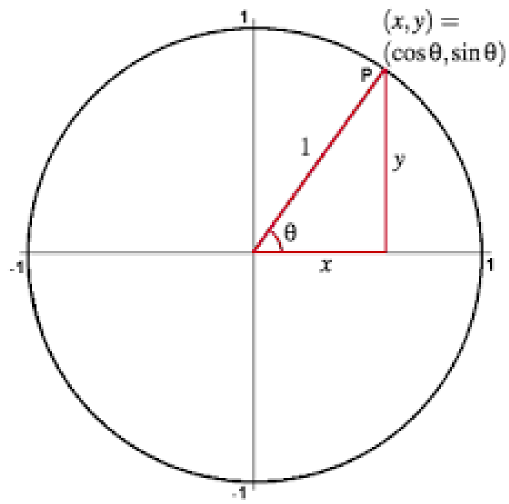


Figure 60: Unit circle

Example 5.1: Show a point is on the unit circle

Show the point $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.

Solution:

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

Example 5.2: Find point on unit circle with specified quadrant

Suppose (x, y) is a point on the unit circle in quadrant IV. If $x = \frac{\sqrt{2}}{7}$, then what is y?

Solution: We recall quadrants from Section 1.10. Quadrant IV corresponds to points with positive x coordinate and negative y coordinate. So if (x, y) is on the unit circle we have

$$\begin{aligned} x^2 + y^2 = 1 &\implies y^2 = 1 - x^2 \\ &\implies y^2 = 1 - \left(\frac{\sqrt{2}}{7}\right)^2 \\ &\implies y^2 = \frac{45}{49} \\ &\implies y = -\frac{\sqrt{47}}{7} \end{aligned}$$

In the last equality, we take the negative square root instead of the positive square root since we are expecting y to be negative for a point in quadrant IV.

Now we introduce the concept of **terminal points**. Suppose t is a real number. If $t \geq 0$, we take the convention that we start at $(1, 0)$ and travel the length of t in a counterclockwise direction. If $t < 0$, then we go t units in a clockwise direction.

In this way we arrive at a point on the unit circle. This point is called the terminal point determined by the real number t.

The circumference of a circle with radius r is given by $2\pi r$. Since the unit circle has radius 1, it has circumference 2π. So if a point starts at $(1, 0)$ and moves counterclockwise all the way around the unit circle and returns to $(1, 0)$, it travels a distance of 2π.

To move halfway around the circle, we travel a distance of $\frac{1}{2}(2\pi) = \pi$. To move a quarter circle, it travels a distance of $\frac{1}{4}(2\pi) = \frac{\pi}{2}$.

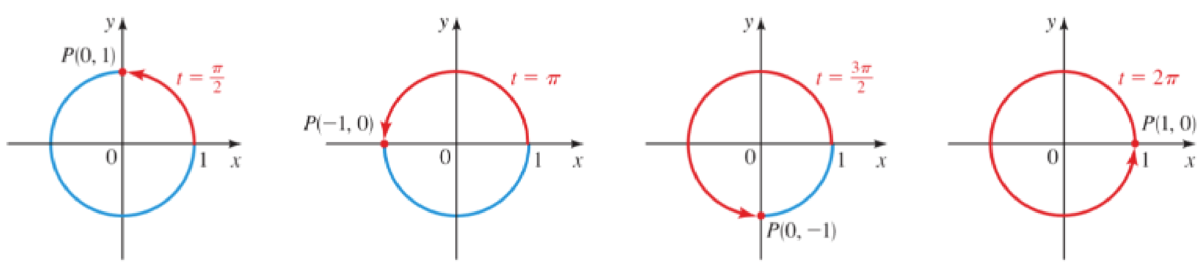


Figure 61: Quarter circle terminal points

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Example 5.3: Terminal Points

Find the terminal point on the unit circle determined by each real number t.

a) $t = 3\pi$

b) $t = -\pi$

c) $= -\frac{\pi}{2}$

Solution:

a) $(-1, 0)$

b) $(-1, 0)$

c) $(0, -1)$

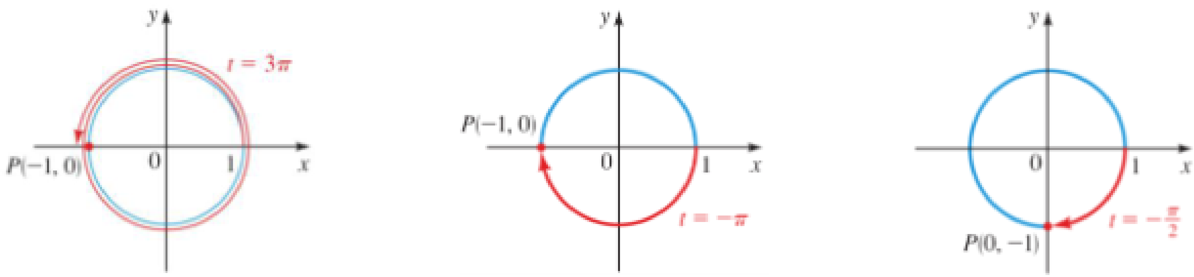

Figure 62: Terminal points

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Notice that different values of t can determine the same terminal point. They differ by 2π.

We have examined terminal values at multiples of $\frac{\pi}{2}$. Now we look at the terminal point corresponding to $t = \frac{\pi}{4}$.

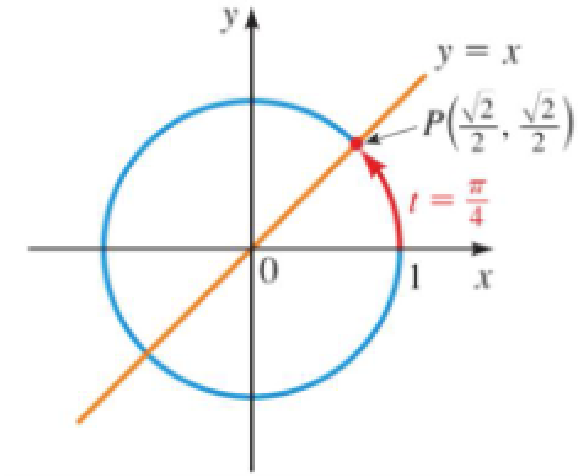

Figure 63: Terminal point at $t = \frac{\pi}{4}$

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This point is the same distance from $(1, 0)$ and $(0, 1)$. It lives on the line $y = x$, so we can plug this into the equation of the unit circle

$$x^2 + y^2 = 1 \implies 2x^2 = 1 \implies x^2 = \frac{1}{2} \implies x = \frac{\sqrt{2}}{2}$$

where we take the positive square root since the point lives in the first quadrant. Likewise $y = \frac{\sqrt{2}}{2}$.

So the terminal point corresponding to $t = \frac{\pi}{4}$ is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

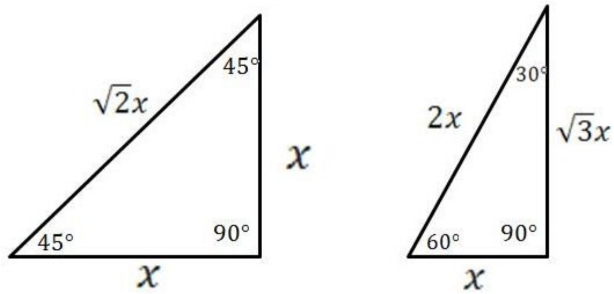

Figure 64: Two special right triangles: 45-45-90 and 30-60-90

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Similar methods can be used to find the terminal points determined by $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$. This is via the use of the 30-60-90 triangle. So in the first quadrant we have the points as follows.

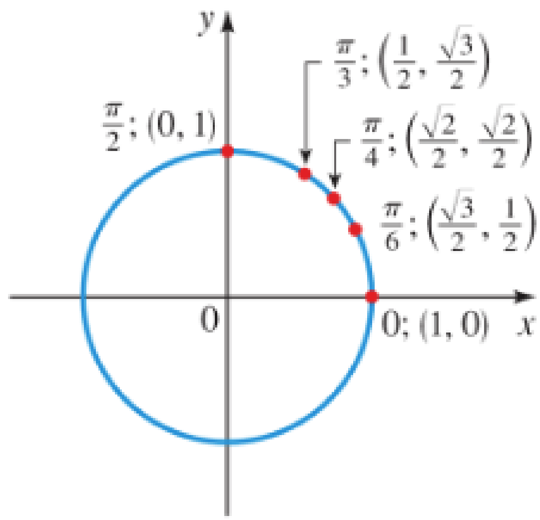

Figure 65: Terminal points in Quadrant I

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Example 5.4: Terminal Points

Find the terminal point on the unit circle determined by each real number t.

a) $t = -\frac{\pi}{4}$

b) $t = \frac{3\pi}{4}$

c) $= -\frac{5\pi}{6}$

Solution: You can do these in an ad-hoc manner, or do them systematically by adding or subtracting multiples of 2π to the terminal point t until it lives in between $[0, 2\pi]$.

a) Add 2π to get $t = \frac{7\pi}{4}$ and get $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

b) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

c) Add 2π to get $t = \frac{7\pi}{6}$ and get $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

I am going to skip the concept of **reference number**. Too much terminology that doesn't really add much.

As mentioned earlier, we can add or subtract multiples of 2π to t without changing the corresponding reference point.

Example 5.5: Large Terminal Point

Find the terminal point determined by $t = \frac{29\pi}{6}$

Solution: We can subtract off $2\pi\left(\frac{12\pi}{6}\right)$ to get $t = \frac{17\pi}{6}$. Do it again to get $t = \frac{5\pi}{6}$ and conclude it is the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

If more time – do more practice.

$$t = \frac{3\pi}{4}, -\frac{5\pi}{4}, -\frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, -\frac{\pi}{6}, \frac{13\pi}{4}, \frac{13\pi}{6}, \frac{41\pi}{6}, \frac{17\pi}{4}, \frac{-11\pi}{3}, \frac{31\pi}{6}, \frac{16\pi}{3}, -\frac{41\pi}{4}$$

5.2 Trigonometric Functions of Real Numbers §5.2

Last time we talked about the unit circle and terminal points associated with traveling a distance t starting at $(1, 0)$ going counterclockwise for positive t.

The sin function assigns to each real number the y-coordinate of the terminal point determined by t. The cos function is assigned to the x-coordinate.

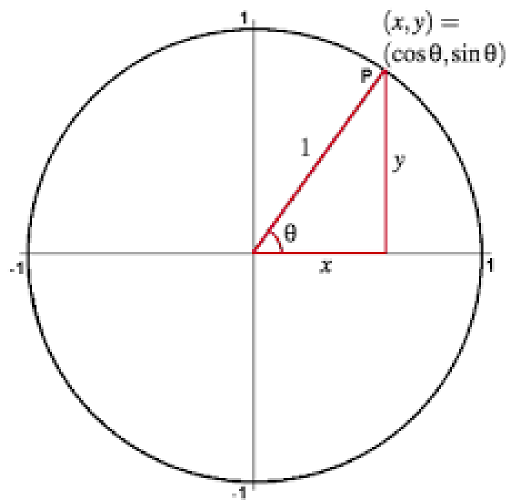


Figure 66: Unit circle

Note that we measure the angle θ typically in radius rather than degrees. For example 2π radians is equivalent to 360 degrees.

The rest of the functions are defined in terms of sin and cos for a total of six trig functions.

$$\tan x = \frac{\sin x}{\cos x}$$
$$\csc x = \frac{1}{\sin x}(\sin x \neq 0)$$
$$\sec x = \frac{1}{\cos x}(\cos x \neq 0)$$
$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}(\sin x \neq 0)$$

Example 5.6: Evaluate trig functions

Find the six trigonometric functions at each given real number t.

a) $t = \frac{\pi}{3}$

b) $t = \frac{\pi}{2}$

Solution: We have the terminal point determined by $t = \frac{\pi}{3}$ is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$\begin{aligned}
\sin\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\
\cos\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\
\tan\left(\frac{\pi}{3}\right) &= \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \\
\csc\left(\frac{\pi}{3}\right) &= \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}} \\
\sec\left(\frac{\pi}{3}\right) &= \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2 \\
\cot\left(\frac{\pi}{3}\right) &= \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}}
\end{aligned}$$

We have the terminal point determined by $t = \frac{\pi}{2}$ is $(0, 1)$.

$$\begin{aligned}
\sin\left(\frac{\pi}{2}\right) &= 1 \\
\cos\left(\frac{\pi}{2}\right) &= 0 \\
\tan\left(\frac{\pi}{2}\right) &= \frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)} = \frac{1}{0} \text{ so undefined.} \\
\csc\left(\frac{\pi}{2}\right) &= \frac{1}{\sin\left(\frac{\pi}{2}\right)} = \frac{1}{1} = 1 \\
\sec\left(\frac{\pi}{2}\right) &= \frac{1}{\cos\left(\frac{\pi}{2}\right)} = \frac{1}{0} \text{ so undefined.} \\
\cot\left(\frac{\pi}{2}\right) &= \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0
\end{aligned}$$

The previous shows that the domains of trig functions are not always all real numbers. There are some values they are not defined at.

sin and cos are defined for all values of t. However the other functions are not defined when the denominator is 0. $\cos = 0$ at precisely the values $\frac{\pi}{2} + \pi k$ where k is any integer, so the functions that are divided by cos, tan and sec are not defined at $\frac{\pi}{2} + \pi k$. Similarly $\sin = 0$ at precisely the values πk where k is any integer, so the functions that are divided by sin, csc and cot are not defined at πk.

Example 5.7: Evaluate trig functions II

Find each value.

a) $\cos \frac{2\pi}{3}$

b) $\tan\left(-\frac{\pi}{3}\right)$

c) $\sin \frac{19\pi}{4}$

Solution:

a) The point corresponding to $\frac{2\pi}{3}$ is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. So cos is the x-coordinate.

$$\cos \frac{2\pi}{3} = -\frac{1}{2}$$

b) I prefer to add 2π to get the equivalent $t = \frac{5\pi}{3}$. The terminal point corresponding to that value of t is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. So

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

c) I prefer to subtract 4π to get the equivalent $t = \frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{16\pi}{4} = \frac{3\pi}{4}$. The terminal point corresponding to that value of t is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\sin\frac{19\pi}{4} = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

Now we want to determine whether these trig functions are even or odd. Let's start with sin and cos.

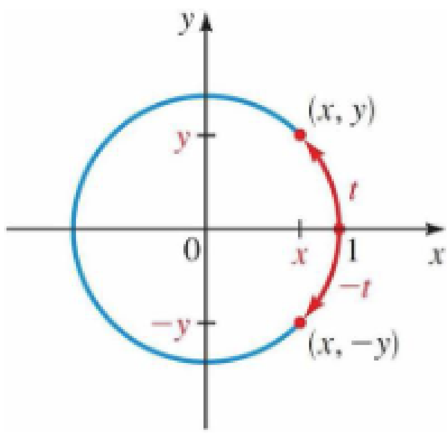


Figure 67: The terminal points at t and $-t$

We can see that $\sin(-t) = -y = -\sin t$ and $\cos(-t) = x = \cos t$. Then we have $\tan(-t) = \frac{-y}{x} = -\frac{y}{x} = -\tan(t)$. So we have that sin and tan are odd whereas cos is even.

The reciprocal of an even function is even, and the reciprocal of an odd function is odd. So we have that $\sin, \csc, \tan, \cot$ are odd and cos and sec are even.

Example 5.8: Evaluate trig functions III

Use even-odd properties to evaluate the following.

a) $\sin\left(-\frac{\pi}{6}\right)$

b) $\cos\left(-\frac{\pi}{4}\right)$

Solution:

a) Since sin is odd, we have $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

b) Since cos is even, we have $\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

Finally, we have the identity $\sin^2\theta + \cos^2\theta = 1$. This is not an equation – equations are only true for some values of the variable. This is an identity – it holds for any value of θ. This is because $\sin\theta$ is the y-coordinate on the unit circle and $\cos\theta$ is the x-coordinate on the unit circle, so we have $x^2 + y^2 = 1$ which gives rise to

$\sin^2\theta + \cos^2\theta = 1.$

To get the other two identities, first divide both sides of the equation by $\sin^2\theta$ to get

$$\begin{aligned}\sin^2\theta + \cos^2\theta = 1 &\implies \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \\ &\implies 1 + \cot^2\theta = \csc^2\theta\end{aligned}$$

then divide by $\cos^2\theta$ to get

$$\begin{aligned}\sin^2\theta + \cos^2\theta = 1 &\implies \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \\ &\implies \tan^2\theta + 1 = \sec^2\theta\end{aligned}$$

If the value of one trigonometric function is known, then the values of all the others can be computed.

Example 5.9: Find all trig values

If $\cos t = \frac{3}{5}$ and t is in quadrant IV, find the values of all the trigonometric functions of t

Solution: From the first pythagorean identity, we have

$$\begin{aligned}\sin^2 t + \cos^2 t &= 1 \\ \sin^2 t + \left(\frac{3}{5}\right)^2 &= 1 \\ \sin^2 t &= 1 - \frac{9}{25} = \frac{16}{25} \implies \sin t = \pm\frac{4}{5}\end{aligned}$$

Since t is in quadrant IV, the y-coordinate is negative so sin must take on the negative value

$$\sin t = -\frac{4}{5}$$

Now that sin and cos are known, the values of the other trigonometric functions can be found using the reciprocal identities.

$$\begin{aligned}\sin t &= -\frac{4}{5} \\ \cos t &= \frac{3}{5} \\ \tan t &= \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3} \\ \csc t &= \frac{1}{\sin t} = \frac{1}{-\frac{4}{5}} = -\frac{5}{4} \\ \sec t &= \frac{1}{\cos t} = \frac{1}{\frac{3}{5}} = \frac{5}{3} \\ \cot t &= \frac{1}{\tan t} = \frac{1}{-\frac{4}{3}} = -\frac{3}{4}\end{aligned}$$

5.3 Trigonometric Graphs §5.3

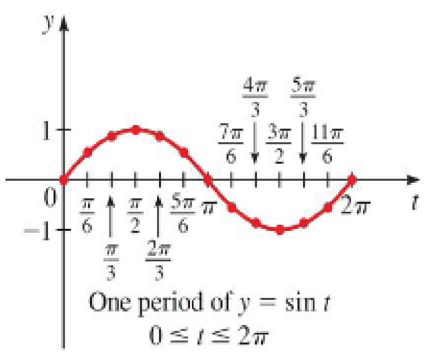
FOR THIS LECTURE - USE THE POWERPOINT

Today we graph sin and cos and transformations of these functions.

We first observe that sin and cos repeat after regular intervals. In other words,

$$\sin(t + 2\pi k) = \sin t \text{ for any integer } k$$
$$\cos(t + 2\pi k) = \cos t \text{ for any integer } k$$

In fact, we say a function is periodic with period p if $f(t + p) = f(t)$ for every t. The least such number (if it exists) is the **period** of f. So we say sin and cos have period 2π. So in order to sketch them, we sketch one period and then "repeat".
So let's graph sin and cos from $[0, 2\pi]$. Note that sin is odd and cos is even as we determined last time.



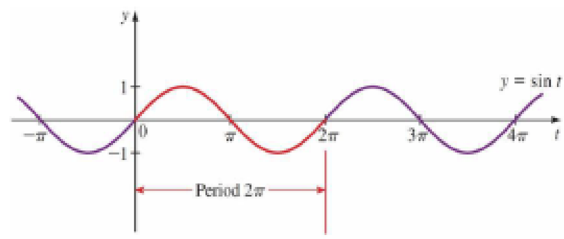
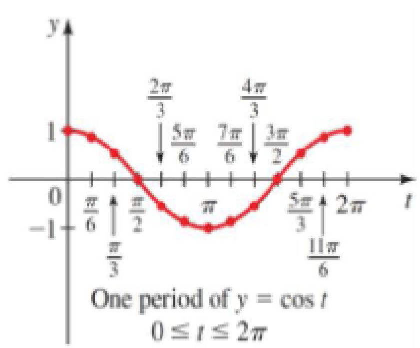


Figure 68: Graph of sin



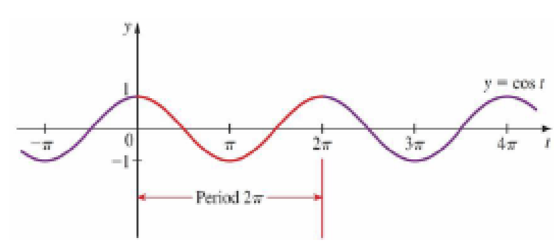


Figure 69: Graph of cos

Now we transform sin and cos as we did in §2.6.

Example 5.10: Transform sin and cos

Sketch the graph of each function:

a) $f(x) = 2 + \cos x$

b) $g(x) = -\cos x$

Solution: Drawn in class. See textbook or Wolfram Alpha/Desmos.

Now consider vertically stretching and shrinking sin or cos. In general, for the functions $y = a \sin x$ and $y = a \cos x$, the number $|a|$ is called the amplitude of these functions.

Example 5.11: Stretch cos

Find the amplitude of $y = -3 \cos x$ and sketch its graph.

Solution: Amplitude of 3. Drawn in class. See textbook or Wolfram Alpha/Desmos.

The vertical and horizontal stretching and shrinking is the difficult part. Since $\sin, \cos$ have period 2π, the functions

$$y = a \sin kx \qquad y = a \cos kx \qquad (k > 0)$$

complete one period as kx varies from 0 to 2π, that is, for

$$0 \leq kx \leq 2\pi \implies 0 \leq x \leq \frac{2\pi}{k}$$

So these functions complete one period from 0 to $\frac{2\pi}{k}$ and this have period $\frac{2\pi}{k}$.

Example 5.12: Horizontal stretching/shrinking sin and cos

Sketch the graph of each function:

a) $y = 4 \cos 3x$

b) $y = -2 \sin \left(\frac{1}{2}x\right)$

Solution: Drawn in class. See textbook or Wolfram Alpha/Desmos.

The curves

$$y = a \sin(k(x - b)) \quad y = a \cos(k(x - b))$$

have amplitude $|a|$, period $\frac{2\pi}{k}$ and horizontal shift b. One complete period is on the interval $\left[b, b + \frac{2\pi}{k}\right]$

Example 5.13: Fully transformed sin and cos

Sketch the graph of each function:

a) $y = 3 \sin \left(2 \left(x - \frac{\pi}{4}\right)\right)$

b) $y = \frac{3}{4} \cos \left(2x + \frac{2\pi}{3}\right)$

Solution: Drawn in class. See textbook or Wolfram Alpha/Desmos.

5.4 More Trigonometric Graphs §5.4

FOR THIS LECTURE - USE THE POWERPOINT

Today we will graph the other four trig functions: $\tan, \csc, \sec, \cot$.

5.5 Inverse Trigonometric Functions and their Graphs §5.5

FOR THIS LECTURE - USE THE POWERPOINT

6 Trigonometric Functions: Right Triangle Approach §6

6.1 Angle Measure §6.1

Typically radians are used to measure angles. But we can also use degrees. If one revolution around the unit circle is 2π radians, then we can also say it takes $360°$. One can convert radians to degrees by multiplying by $\dfrac{180}{\pi}$. One can convert degrees to radians by multiplying by $\dfrac{\pi}{180}$.

Example 6.1: Radians and degrees

a) Convert $60°$ to radians.

b) Convert $\frac{\pi}{6}$ radians to degrees.

Solution:

a)

$$60° = 60\left(\frac{\pi}{180}\right) \text{ rad } = \frac{\pi}{3} \text{ rad.}$$

b)

$$\frac{\pi}{6} \text{ rad } = \left(\frac{\pi}{6}\right)\left(\frac{180}{\pi}\right) = 30°$$

Coterminal angles are those that differ by $360°$ and correspond to the same point on the unit circle.

Example 6.2: Coterminal Angles

Find an angle with measure between $0°$ and $360°$ that is coterminal with the angle of $1290°$.

Solution: Subtract $360°$ as many times as you need.

$$1290° - 360° = 930°$$

$$930° - 360° = 570°$$

$$570° - 360° = 210°$$

So $210°$.

Now we also want to consider lengths of circle arcs. Suppose we have a circle of radius r and a "slice" of θ. Then the length of that arc is given by

$$s = \frac{\theta}{2\pi}(2\pi r) = \theta r$$

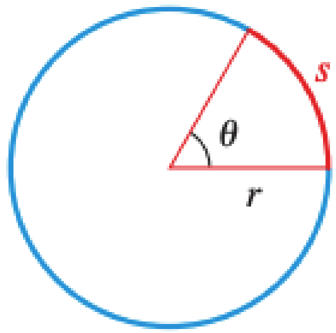


Figure 70: Length of circular arcs

Example 6.3: Arc lengths

a) Find the length of an arc of a circle with radius 10 that subtends a central angle of 30°.

b) A central angle θ is a circle of radius 4 is subtended by an arc of 6. Find the measure of θ in radians.

Solution:

a) We have $30° = \frac{\pi}{6}$ rad. So the length of the arc is given by

$$s = r\theta = 10\frac{\pi}{6} = \frac{5\pi}{3}$$

b) We have

$$\frac{\theta}{2\pi}(2\pi r) = \frac{\theta}{r} = 6 \implies \theta = \frac{3}{2}$$

Now we also want to consider areas of circular sectors. Suppose we have a circle of radius r and a "slice" of θ. Then the areas of that arc is given by

$$A = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta$$

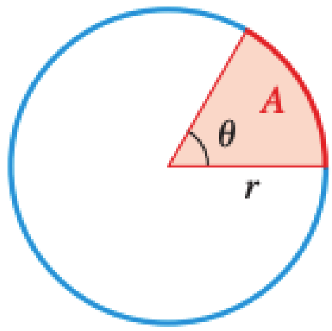


Figure 71: Area of circular sectors

Example 6.4: Arc sectors

FInd the area of a circle with central angle 60° if the radius of the circle is 3m.

Solution: The area of the circle is given by $\pi r^2 = 9\pi\ m^2$. We want $\frac{60^\circ}{350^\circ} = \frac{1}{6}$ of that, so the area of the arc is $\frac{3}{2}\pi\ m^2$.

7 Analytic Trigonometry §7

7.1 Trigonometric Identities §7.1

Up until now we have studied equations like $x + 2 = 5$ which is only true for the value $x = 3$.
We also have the concept of identities, such as

$$(x+1)^2 = x^2 + 2x = 1$$

$$\sin^2 t + \cos^2 t = 1$$

which are true for all values of the variable.

We use identities to rewrite the same expression in different ways. It is often possible to rewrite a complicated-looking expression as a much simpler one. To simplify algebraic expressions, we used factoring, common denominators.

Example 7.1: Trig Simplify I

Simplify the expression $\cos t + \tan t \sin t$.

Solution: Rewrite in terms of sin and cos:

$$\begin{aligned}\cos t + \tan t \sin t &= \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t \\ &= \frac{\cos^2 t + \sin^2 t}{\cos t} \\ &= \frac{1}{\cos t} \\ &= \sec t\end{aligned}$$

Example 7.2: Trig Simplify II

Simplify the expression $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$

Solution: Combine using a common denominator

$$\begin{aligned}\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} &= \frac{\sin\theta(1+\sin\theta) + \cos^2\theta}{\cos\theta(1+\sin\theta)} \\ &= \frac{\sin\theta + \cos^2\theta + \sin^2\theta}{\cos\theta(1+\sin\theta)} \\ &= \frac{\sin\theta + 1}{\cos\theta(1+\sin\theta)} \\ &= \frac{1}{\cos\theta} = \sec\theta\end{aligned}$$

Now we go on to proving trigonometric identities. To verify a trigonometric equation is an identity, we transform one side of the equation into the other side by a series of steps. This is not like some algorithm – it takes some judgment and experience.

1. Start with one side and try to transform it to the other. It is usually better to start with the side that looks more complicated or has a simplification you can recognize.
2. Use known identities like common denominators, factoring, trig identities.
3. Rewrite trig functions as sin and cos.

WARNING: To prove an identity, preforming the same operation on both sides of the equation is not sufficient. For example, if we start with an equation that is not an identity

$$\sin x = -\sin x$$

and square it, we end up with something that is true

$$\sin^2 x = \sin^2 x$$

Only operations that are reversible will transform an identity into an identity.

Here is an example where we rewrite in terms of sin and cos.

Example 7.3: Trig Identity I

Verify the identity

$$\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$$

Solution:

$$\begin{aligned}\cos\theta(\sec\theta - \cos\theta) &= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right)\\ &= 1 - \cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Here is an example by combining fractions.

Example 7.4: Trig Identity II

Verify the identity

$$2\tan x \sec x = \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$$

Starting with the right hand side. *Solution:*

$$\begin{aligned}\frac{1}{1-\sin x} - \frac{1}{1+\sin x} &= \frac{(1+\sin x) - (1-\sin x)}{(1-\sin x)(1+\sin x)}\\ &= \frac{2\sin x}{1-\sin^2 x}\\ &= \frac{2\sin x}{\cos^2 x}\\ &= 2\frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right)\\ &= 2\tan x \sec x\end{aligned}$$

Here is an example of where we “introduce something extra”

Example 7.5: Trig Identity III

Verify the identity

$$\frac{\cos u}{1-\sin u} = \sec u + \tan u$$

Starting with the LHS.

Solution:

$$\begin{aligned}
\frac{\cos u}{1-\sin u} &= \frac{\cos u}{1-\sin u}\cdot\frac{1+\sin u}{1-\sin u} \\
&= \frac{\cos u(1+\sin u)}{1-\sin^2 u} \\
&= \frac{\cos u(1+\sin u)}{\cos^2 u} \\
&= \frac{1+\sin u}{\cos u} \\
&= \frac{1}{\cos u}+\frac{\sin u}{\cos u}
\end{aligned}$$

Finally, we could work with both sides independently – like getting both sides to “meet in the middle”.

Example 7.6: Trig Identity IV

Verify the identity

$$\frac{1+\cos\theta}{\theta} = \frac{\tan^2\theta}{\sec\theta - 1}$$

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{1+\cos\theta}{\cos\theta} = \frac{1}{\cos\theta}+\frac{\cos\theta}{\cos\theta} = \sec\theta + 1 \\
\text{RHS} &= \frac{\tan^2\theta}{\sec\theta-1} = \frac{\sec^2\theta-1}{\sec\theta-1} = \frac{1}{\sec\theta-1}
\end{aligned}$$

This can take a lot of practice however the number of tricks that exist are finite – luckily the book has a huge number of practice problems.

7.2 Addition and Subtraction Formulas §7.2

We now derive identities for trigonometric functions of sums and differences.

$$\begin{aligned}
\sin(s+t) &= \sin s \cos t + \cos s \sin t \\
\sin(s-t) &= \sin s \cos t - \cos s \sin t \\
\cos(s+t) &= \cos s \cos t - \sin s \sin t \\
\cos(s-t) &= \cos s \cos t + \sin s \sin t \\
\tan(s+t) &= \frac{\tan s + \tan t}{1 - \tan s \tan t} \\
\tan(s-t) &= \frac{\tan s - \tan t}{1 + \tan s \tan t}
\end{aligned}$$

Before I forget – you will be provided with this on the quiz and final exam.

The book gives a rigorous proof for the justification of the first and third identities. I don't love doing this, but I am just going to give this formula without any motivation.

However, we can derive the second from the first, fourth from the sixth, third from one and two, and six from three and for. Recall that sin is an off function, so $\sin(-t) = -\sin t$. cos is an even function, so $\cos(-t) = \cos(t)$. If we rewrite $\sin(s-t)$ as $\sin(s+(-t))$, we can use the first formula to get

$$\begin{aligned}
\sin(s-t) = \sin(s+(-t)) &= \sin s \cos(-t) + \cos s \sin(-t) \\
&= \sin s \cos t - \cos s \sin t
\end{aligned}$$

We can also derive the formula for tan by dividing sin by cos.

$$\begin{aligned}
\tan(s+t) &= \frac{\sin(s+t)}{\cos(s+t)} \\
&= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t} \\
&= \frac{\frac{\sin s}{\cos s} + \frac{\sin t}{\cos t}}{1 - \tan x \tan t} \\
&= \frac{\tan s + \tan t}{1 - \tan s \tan t}
\end{aligned}$$

where in the third equality, we divided the numerator and denominator by $\cos s \cos t$.

Example 7.7: Addition and Subtraction formulas

a) $\cos 75°$

b) $\cos \frac{\pi}{12}$

Solution: The key here is to rewrite as a sum or difference of two angles for which we know the value of cos. We know the values of sin and cos for multiples of $30° \left(\frac{\pi}{6}\right)$ and $45° \left(\frac{\pi}{4}\right)$.

The first one is quite clear. $75° = 45° + 30°$ Applying the formula gives us:

$$\begin{aligned}
\cos 75° &= \cos(45° + 30°) \\
&= \cos 45° \cos 30° - \sin 45° \sin 30° \\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}
\end{aligned}$$

As for the second, let's look at $\frac{\pi}{4}$ and $\frac{\pi}{6}$ with the least common denominator $\frac{3\pi}{12}$ and $\frac{2\pi}{12}$. This makes it clear that $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$.

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\end{aligned}$$

Example 7.8: Recognize sin formula

Evaluate $\sin 20° \cos 40° + \cos 20° + \sin 40°$

Solution: We can recognize this as the addition formula for sin so we have $\sin 20° \cos 40° + \cos 20° + \sin 40° = \sin(20° + 40°) = \sin(60°) = \frac{\sqrt{3}}{2}$

Example 7.9: Another Trig Identity

Verify the identity

$$\frac{1 + \tan x}{1 - \tan x} = \tan\left(\frac{\pi}{4} + \pi\right)$$

Solution: Use the formula on the RHS

$$\begin{aligned}\tan\left(\frac{\pi}{4} + \pi\right) &= \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x}\\ &= \frac{1 + \tan x}{1 - \tan x}\end{aligned}$$

7.3 Double-Angle, Half-Angle, and Product-Sum formulas §7.3

We now introduce the double angle formulas, which are just special cases of the addition formulas.

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ \cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1\\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

Example 7.10: Use of double angle formula

If $\cos x = -\frac{2}{3}$ and x is in Quadrant II, find $\cos 2x$ and $\sin 2x$.

Solution: Using the double angle formula for cos, we get

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1\\ &= 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} = -\frac{1}{9}\end{aligned}$$

To use the formula for $\sin 2x = 2\sin x\cos x$, we need to find $\sin x$ first.

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(-\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

where we have used the positive square root because $\sin x$ is positive in Quadrant II. Thus

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) &= -\frac{4\sqrt{5}}{9}\end{aligned}$$

Example 7.11: A triple angle formula

Rewrite $\cos 3x$ in terms of $\cos x$.

Solution:

$$\begin{aligned}\cos 3x &= \cos(2x+x)\\ &= \cos 2x\cos x - \sin 2x\sin x\\ &= (2\cos^2 x - 1)\cos x - (2\sin x\cos x)\sin x\\ &= 2\cos^3 x - \cos x - 2\sin^2 x\cos x\\ &= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)\\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

Example 7.12: Trig Identity - double angle

Prove the identity

$$\frac{\sin 3x}{\sin x\cos x} = 4\cos x - \sec x$$

Solution:

$$\begin{aligned}
\frac{\sin 3x}{\sin x\cos x} &= \frac{\sin(x+2x)}{\sin x\cos x}\\
&= \frac{\sin x\cos 2x + \cos x\sin 2x}{\sin x\cos x}\\
&= \frac{\sin x(2\cos^2 x - 1) + \cos x(2\sin x\cos x)}{\sin x\cos x}\\
&= \frac{\sin x(2\cos^2 x - 1)}{\sin x\cos x} + \frac{\cos x(2\sin x\cos x)}{\sin x\cos x}\\
&= \frac{2\cos^2 x - 1}{\cos x} + 2\cos x\\
&= 2\cos x - \frac{1}{\cos x} + 2\cos x\\
&= 4\cos x - \sec x
\end{aligned}$$

We want half angle formulas as well, but first we start with these "formulas for lowering powers".

$$\begin{aligned}
\sin^2 x &= \frac{1-\cos(2x)}{2}\\
\cos^2 x &= \frac{1+\cos(2x)}{2}\\
\tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x}
\end{aligned}$$

These are nothing new. Just some algebra on the double angle formulas.

Example 7.13: Lowering Powers

Express $\sin^2 x\cos^2 x$ in terms of the first power of cosine.

Solution:

$$\begin{aligned}
\sin^2 x\cos^2 x &= \left(\frac{1-\cos x)}{2}\right)\left(\frac{1+\cos x)}{2}\right)\\
&= \frac{1-\cos^2(2x)}{4}\\
&= \frac{1}{4} - \frac{1}{4}cos^2 2x\\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos(2\cdot 2x)}{2}\right)\\
&= \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8}\\
&= \frac{1}{8}(1-\cos 4x)
\end{aligned}$$

A simpler way to do this would be to recall that

$$\sin 2x = 2\sin x\cos x \implies \sin x\cos x = \frac{1}{2}\sin 2x$$

$$\begin{aligned}
\sin^2 x\cos^2 x &= \frac{1}{4}\sin^2(2x)\\
&= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)\\
&= \frac{1}{8}(1-\cos 4x)
\end{aligned}$$

We move on to the half-angle formulas.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$
$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

Example 7.14: Half-Angle application

Find $\sin 22.5°$.

Solution: Since $22.5°$ is in the first quadrant, we pick the positive sign in the half angle formula.

$$\begin{aligned}\sin\frac{45°}{2} &= \sqrt{\frac{1-\cos 45°}{2}} \\ &= \sqrt{\frac{1-\sqrt{2}/2}{2}} \\ &= \sqrt{\frac{2-\sqrt{2}}{4}} \\ &= \frac{1}{2}\sqrt{2-\sqrt{2}}\end{aligned}$$

Example 7.15: Half-Angle application again

Find $\tan\frac{u}{2}$ if $\sin u = \frac{2}{5}$ and u is in quadrant II.

Solution: To use the half angle formula for tangent, we need to find $\cos u$. We have

$$\begin{aligned}\cos u &= -\sqrt{1-\sin^2 u} \\ &= -\sqrt{1-\left(\frac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}\end{aligned}$$

where the negative sign in front of the square root is because we are in quadrant II.

So plug into the half angle formula and get

$$\begin{aligned}\tan\frac{u}{2} &= \frac{1-\cos u}{\sin u} \\ &= \frac{1+\sqrt{21}/5}{\frac{2}{5}} = \frac{5+\sqrt{21}}{2}\end{aligned}$$

7.4 Basic Trigonometric Equations §7.4

We have been dealing with identities such as

$$\sin^2\theta + \cos^2\theta = 1$$

which are true for any value of θ.
Now we turn our attention to trigonometric equations that are only true for some versions of θ such as

$$2\sin\theta - 1 = 0$$

Example 7.16: Trig Equation I

Solve the equation

$$\sin\theta = \frac{1}{2}$$

Solution: We first look for for solutions where $\theta \in [0, 2\pi]$, which are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. However, we know that adding multiples of 2π to these are also valid solutions, so we write the answer as

$$\theta = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$$

where k is any integer.

Example 7.17: Trig Equation II

Solve the equation

$$\cos\theta = -\frac{\sqrt{2}}{2}$$

Solution: Very similar to the last one. We first look for for solutions where $\theta \in [0, 2\pi]$, which are $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$. However, we know that adding multiples of 2π to these are also valid solutions, so we write the answer as

$$\theta = \frac{3\pi}{4} + 2\pi k, \frac{5\pi}{4} + 2\pi k$$

where k is any integer.

Example 7.18: Trig Equation III

Solve

$$\tan^2\theta - 3 = 0$$

Solution: We must have $\tan\theta = \pm\sqrt{3}$. Tangent has period π. We look for solutions in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which are $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$, so we give the solution as

$$\theta = \frac{\pi}{3} + \pi k, -\frac{\pi}{3} + \pi k$$

Now we add another wrinkle and look at equations which require factoring.

Example 7.19: Trig Equation IV

Solve the equation

$$2\cos^2\theta - 7\cos\theta + 3 = 0$$

Solution: Factor

$$2\cos^2\theta - 7\cos\theta + 3 = 0 \implies (2\cos\theta - 1)(\cos\theta - 3) = 0$$

So we require $\cos\theta = \frac{1}{2}$ or $\cos\theta = 3$. The latter is not possible, so

$$\theta = \frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k$$

7.5 More Trigonometric Equations §7.5

Now we move onto equations that require the use of things like the double angle formulas, the fundamental trig identity, etc.

Example 7.20: Trig Equation V

Solve the equation

$$1 + \sin\theta = 2\cos^2\theta$$

Solution: Substitute $\cos^2\theta = 1 - \sin^2\theta$ so we have

$$\begin{aligned} 1 + \sin\theta &= 2(1 - \sin^2\theta) \\ 2\sin^2\theta + \sin\theta - 1 &= 0 \\ (2\sin\theta - 1)(\sin\theta + 1) &= 0 \end{aligned}$$

So we have $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ which gives us

$$\theta = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k, \frac{3\pi}{2} + 2\pi k$$

Example 7.21: Trig Equation VI

Solve the equation

$$\sin 2\theta - \cos\theta = 0$$

Solution: Substitute $\sin 2\theta = 2\sin\theta\cos\theta$ so we can factor as

$$2\sin\theta\cos\theta - \cos\theta = 0 \implies (\cos\theta)(2\sin\theta - 1) = 0$$

So we have $\sin\theta = \frac{1}{2}$ or $\cos\theta = 0$ which gives us

$$\theta = \frac{\pi}{2} + 2\pi k, \frac{2\pi}{2} + 2\pi k, \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$$

Example 7.22: Trig Equation VII

Solve the equation

$$2\sin 3\theta - 1 = 0$$

Solution: We require $\sin 3\theta = \frac{1}{2}$, which implies

$$3\theta = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k \implies \theta = \frac{\pi}{18} + \frac{2}{3}\pi k, \frac{5\pi}{18} + \frac{2}{3}\pi$$

8 Polar Coordinates, Parametric Equations, and Vectors §8

Not covered.

9 Systems of Linear Equations §9

9.1 Systems of Linear Equations in Two Variables §9.1

A **linear equation** takes the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where a_i, b are all constants. This is a linear equation with n-variables $x_1, x_2, \ldots x_n$. These are examples of linear equations

1. $3x + 4y = 5$
2. $2x_1 + \pi x_2 + ex_3 = 10^{10}$
3. $0 = 3$

Even though the last equation is clearly false, it is still a linear equation. with all the constants $a_i = 0$ and $b = 3$. Here are some examples of nonlinear equations

1. $x^2 + 1 = 0$
2. $3x + 4y + 6z = e^x$
3. $3\sin(x) + 4y + 5z = 6$

A **system of linear equations** is a set of equations that involve the same variables. Here is a system of two equations and two unknowns.

$$\begin{aligned} 2x - y &= 5 \\ x + 4y &= 7 \end{aligned} \tag{1}$$

Here is a system of three equations and four unknowns.

$$\begin{aligned} w + x - 3y + 5z &= 2 \\ -3w + 2x - 2y + z &= 4 \\ w + 2x + 3y + 4z &= 5 \end{aligned} \tag{2}$$

A **solution** to a system is an assignment of values for the variables that makes each equation in the system true. For example, we can see that $x = 3, y = 1$ is a solution to the system (System 1).

$$\begin{aligned} 2(3) - 1 &= 5 \quad \checkmark \\ 3 + 4(1) &= 7 \quad \checkmark \end{aligned}$$

In the rest of this week, we are going to

1. Discuss the existence and uniqueness of solutions for 2×2 systems.
2. Solve 2×2 systems.
3. Extend the same ideas to general $n \times n$ systems.

We first discuss existence and uniqueness of solutions to 2×2 linear systems. In an equation with two variables $ax + by = c$, we have the interpretation that this equation is a line. So if we have two equations in two unknowns, we have three possibilities.

1. The two lines intersect exactly once (linear system has exactly one solution).
2. The two lines are parallel but distinct from each other (the linear system has no solutions).
3. The two lines are identical (the linear system has infinitely many solution).

So for any linear system, we have that there exists a solution if either case 1) or case 3) holds. Furthermore, the solution is unique if only 1) holds.

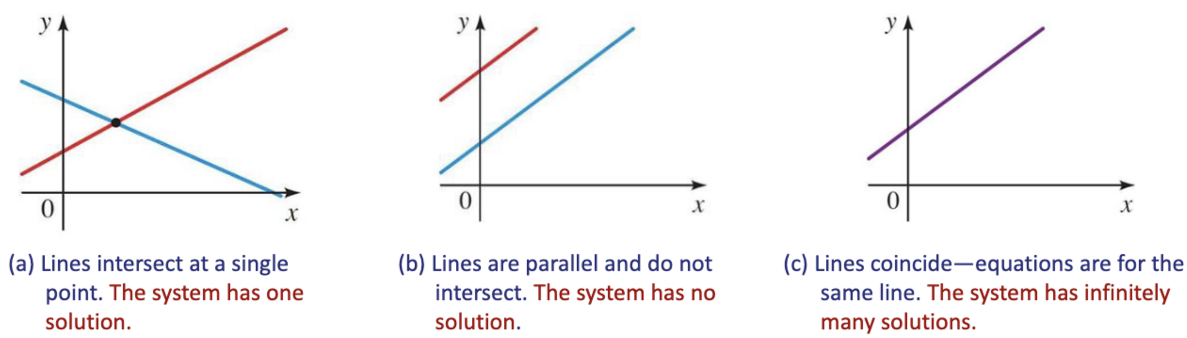


Figure 72: Three possibilites for the solution of 2×2 linear systems.

For example, here is the graphical representation for system 1 in Figure 9.1.

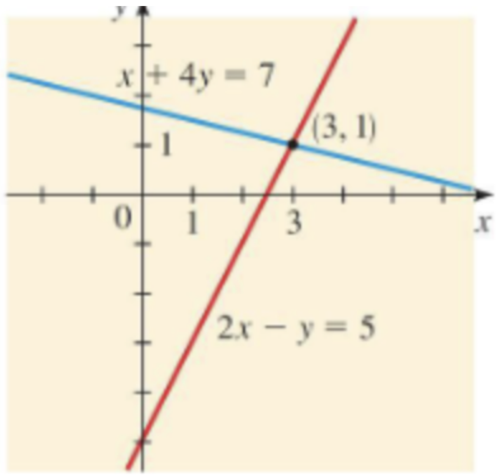


Figure 73: The linear system $x + 4y = 7, 2x - y = 5$ with solution $(3, 1)$.

Even without graphing, we can determine that the first $x + 4y = 7$ has the slope $m = -\frac{1}{4}$ and the second equation $2x - y = 5$ has the slope $m = 2$. So the lines must intersect at some point and the system has a unique solution.

Suppose we are now given the system

$$\begin{aligned} -12x + 3y &= 7 \\ 8x - 2y &= 5 \end{aligned} \tag{3}$$

We can check that each line has the slope $m = 4$.

$$\begin{aligned} -12x + 3y = 7 &\implies 3y = 12x + 7 \implies y = 4x + \frac{7}{3} \\ 8x - 2y = 5 &\implies 2y = 8x - 5 \implies y = 4x - \frac{5}{2} \end{aligned}$$

Moreover they have different y-intercepts so they are not the same line, but merely parallel. So this system does not have a solution.

Finally, we consider the system

$$\begin{aligned} 3x - 6y &= 12 \\ 4x - 8y &= 16 \end{aligned} \tag{4}$$

Multiply the first equation by 4 and the second equation by 3 to get

$$\begin{aligned} 12x - 24y &= 48 \\ 12x - 24y &= 48 \end{aligned} \tag{5}$$

and we can see that they are the same equation. (Alternatively you could have put them both in slope-intercept form to observe they are the same). In terms of solutions, this means that a solution of the first equation would be a solution of the second equation (and vice-versa). So since these two lines coincide (are

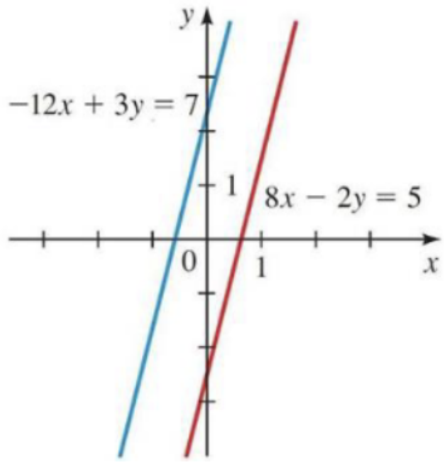


Figure 74: The linear system $-12x + 3y = 7, 8x - 2y = 5$ with no solution.

the same), there are an infinite number of solutions.

So up until now, given a 2×2 system, we can determine whether they 1) have a unique solution, 2) have no solution, or 3) have an infinite number of solutions.

We now seek to solve linear equations.
Consider System 1 again.

$$\begin{aligned} 2x - y &= 5 \\ x + 4y &= 7 \end{aligned}$$

In our quest to solve linear equations, we can do three things which we denote as row operations.

1. Swap the order of linear equations
2. Multiply one linear equation by a nonzero scalar
3. Add a multiple of one equation to the other.

By doing any of these three things, we may turn the system into something where the solution is much more apparent without affecting the solution set.

Suppose I seek to eliminate the variable y. I could take 4 times equation 1 and add it to equation 2. and take the result and replace equation 2 by it.

$$\begin{aligned} 2x - y &= 5 \\ 9x &= 27 \end{aligned}$$

So the second equation immediately implies $x = 3$, and we can plug into the first equation to get

$$2(3) - y = 5 \implies y = 1$$

So the solution is $x = 3, y = 1$, which is exactly what we got earlier.

Another way we could have proceeded is to eliminate the x variable. So by taking -2 times the second equation and add it to the first equations, we end up with

$$\begin{aligned} -9y &= -9 \\ x + 4y &= 7 \end{aligned}$$

So the first equation immediately applies $y = 1$, and we can plug into the second equation to get

$$x + 4(1) = y \implies x = 3$$

So the solution is $x = 3, y = 1$, which is exactly what we got earlier.

So this is what we call the elimination method to solve linear systems, where we attempt to eliminate one (or more) variables in an equation by row operations to uncover the answer.

Note another possibility would have been to eliminate x in the second equation by multiplying the first equation by $-\frac{1}{2}$ and adding it to the second, but this is rarely our first instinct as human beings because we dislike working with fractions when at all possible.

Now suppose we have the system

$$\begin{aligned} 2x - 3y &= 7 \\ 3x + 4y &= 2 \end{aligned}$$

Unlike before, I can't scale just one equation up to eliminate the variable in another without resorting to fractions. For example I could take $-\frac{3}{2}$ times equation 1 and add it to equation 2 to eliminate the x-variable. Alternatively, I could multiply both equations by an acceptable scalar to make the elimination possible. For example, if I multiply the first equation by 3 and the second equation by 2, we end up with

$$\begin{aligned} 6x - 9y &= 21 \\ 6x + 8y &= 4 \end{aligned}$$

Subtract the second equation from the first to get

$$-17y = 17 \implies y = -1$$

and then plugging in to either equation gives us $x = 2$. It is always a good idea to check your work by plugging in to the original system

$$\begin{aligned} 2(2) - 3(-1) &= 7 \quad \checkmark \\ 3(2) + 4(-1) &= 2 \quad \checkmark \end{aligned}$$

Now we consider solving the system

$$\begin{aligned} -12x + 3y &= 7 \\ 8x - 2y &= 5 \end{aligned}$$

Recall previously we determined this system has no solution.

We proceed by elimination. Take 2 times equation 1 and add it to 3 times equation 2, and replace equation 2 by the result.

$$\begin{aligned} -24x + 6y &= 14 \\ 24x - 6y &= 15 \end{aligned} \implies \begin{aligned} -24x + 6y &= 14 \\ 0 &= 29 \end{aligned}$$

What's going on here? Clearly the fact that $0 = 29$ is complete nonsense. We found this by performing row operations that do not change the solution of the equation. Clearly there is no choice of x_1, x_2 that can make $0 = 29$. So we say that this equation is no solution, or **inconsistent**.

Finally, we consider solving the System 4

$$\begin{aligned} 3x - 6y &= 12 \\ 4x - 8y &= 16 \end{aligned}$$

We multiply the first equation by 4 and the second equation by 3

$$12x - 24y = 48$$
$$12x - 24y = 48$$

They are the same equation. If you insist on subtracting one from the other you get

$$12x - 24y = 48$$
$$0 = 0$$

No matter what choice of x, y we pick the second equation will always hold. $0 = 0$ always. So any choice of x, y that satisfies $12x - 24y = 48$ is a solution. So there are an infinite number of solutions, which is precisely the all points on the line $12x - 24y = 48$.

So now we can solve 2×2 equations via elimination, and in the course of doing so can also identify systems with no solutions (inconsistent systems) as well as systems with infinitely many solutions.

9.2 Systems of Linear equations in Several Variables §9.2

Now we extend this idea to $n \times n$ systems (though in practice we will typically be working with 3×3 systems. Let's talk about a generic 3×3 system:

$$
\begin{aligned}
a_1x_1 + a_2x_2 + a_3x_3 &= d_1 \\
b_1x_1 + b_2x_2 + b_3x_3 &= d_2 \\
c_1x_1 + c_2x_2 + c_3x_3 &= d_3
\end{aligned}
$$

Each linear equation in 3-variables is now a plane instead of a line. Visualization gets much harder in 3D compared to 2D (and is basically impossible for higher dimensions), but we will still attempt to do so here.

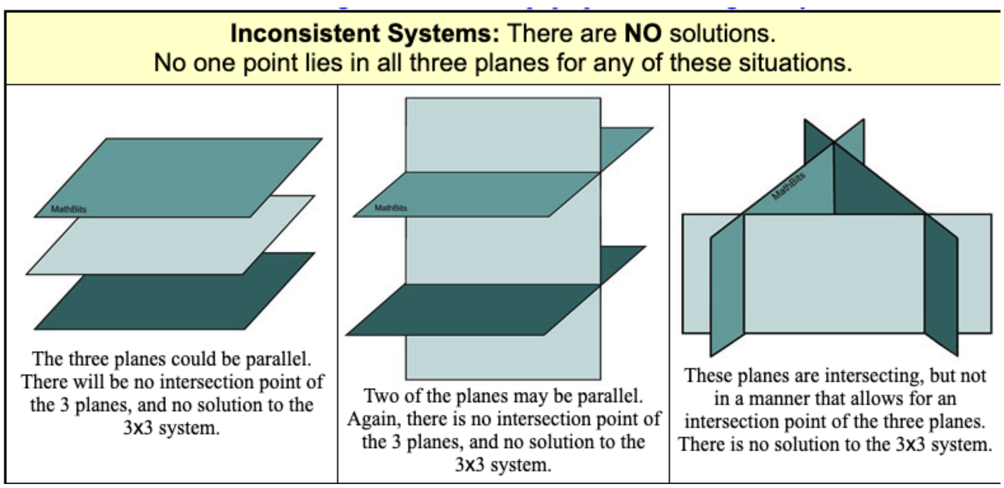


Figure 75: Visualization of an inconsistent system in 3D.

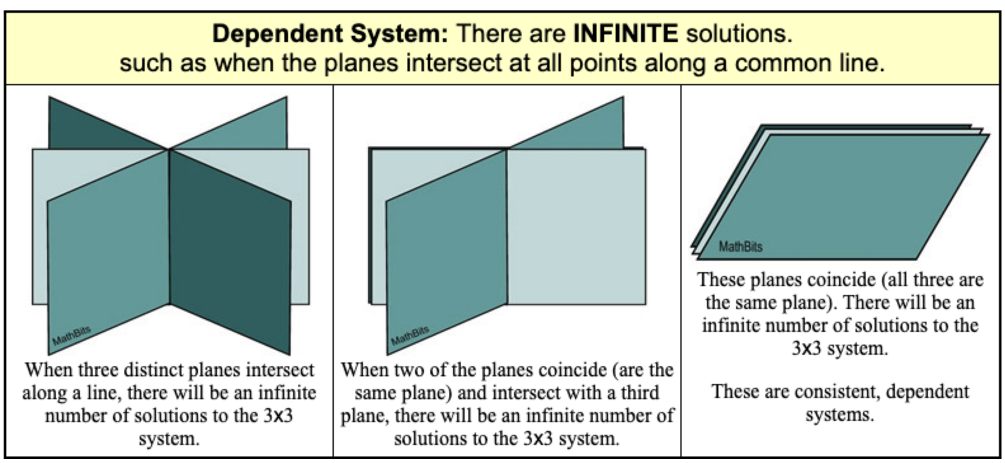


Figure 76: Visualization of a system in 3D with infinitely many solutions.

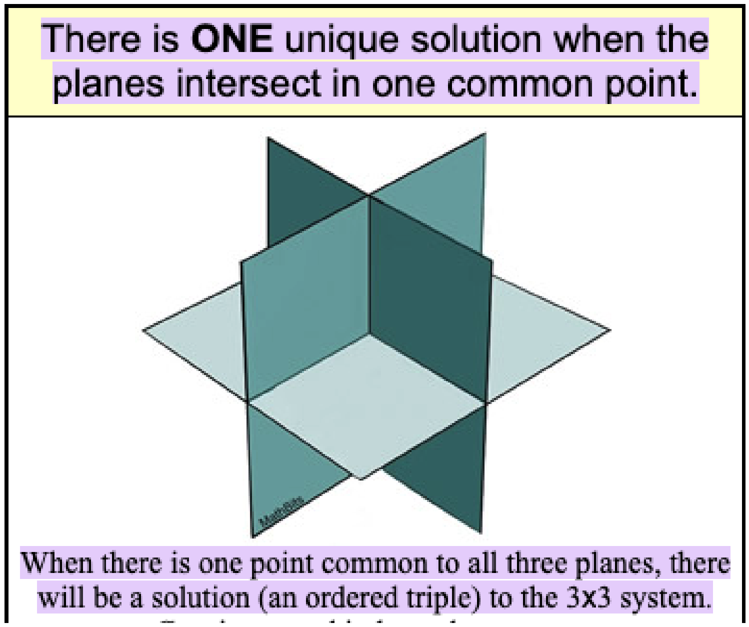


Figure 77: Visualization of a system in 3D with a unique solution.

Let's first consider a system in **triangular form**.

$$\begin{aligned} x - 2y - z &= 1 \\ y + 2z &= 5 \\ z &= 3 \end{aligned}$$

These are quite easy to solve via **back substitution**. The last equation tells us $z = 3$. Substituting that value into the second equation gives us $y = -1$. Substituting both of these values into the first equation gives us $x = 2$.

As the systems get larger and larger, we have to be more systematic unlike in the 2×2 system where we could pick any method we want based off ease of doing arithmetic. We seek to apply the three row operations to get a system in upper triangular form.

Example 9.1: Solving a 3×3 system

Solve the system

$$\begin{aligned} x - 2y + 3z &= 1 \\ x + 2y - z &= 13 \\ 3x + 2y - 5z &= 3 \end{aligned}$$

Solution:. I take -1 times the first equation and add it to the second, and take -3 times the first equation and add it to the third.

$$\begin{aligned} x - 2y + 3z &= 1 \\ 4y - 4z &= 12 \\ 8y - 14z &= 0 \end{aligned}$$

I then take -2 times the second equation and add it to the third.

$$\begin{aligned} x - 2y + 3z &= 1 \\ 4y - 4z &= 12 \\ -6z &= -24 \end{aligned}$$

From here we can back substitute to get $x = 3, y = 7, z = 4$.

Example 9.2: A 3×3 system with no solutions

Solve the system

$$\begin{aligned} x + 2y - 2z &= 1 \\ 2x + 2y - z &= 6 \\ 3x + 4y - 3z &= 5 \end{aligned}$$

Solution: I take -2 times the first equation and add it to the second, and take -3 times the first equation and add it to the third.

$$\begin{aligned} x + 2y - 2z &= 1 \\ -2y + 3z &= 4 \\ -2y + 3z &= 2 \end{aligned}$$

I can do the next row operation, but from here I can already tell that this system has no solutions, or other words this system is inconsistent.

Example 9.3: A 3×3 system with infinitely many solutions

Solve the system

$$\begin{aligned} x - y + 5z &= -2 \\ 2x + y + 4z &= 2 \\ 2x + 4y - 2z &= 8 \end{aligned}$$

Solution: I take -2 times the first equation and add it to the second, and take -1 times the first equation and add it to the third.

$$\begin{aligned} x - y + 5z &= -2 \\ 3y - 6z &= 6 \\ 6y - 12z &= 12 \end{aligned}$$

I take -2 times the second equation and add it to the third to get.

$$\begin{aligned} x - y + 5z &= -2 \\ 3y - 6z &= 6 \\ 0 &= 0 \end{aligned}$$

The new third equation is certainly true, but it gives us no information. So what this tells us is that z can be any number, and from that the second equation will determine y in terms of z, and the first equation will determine x in terms of y and z. So let's solve the two equations in terms of z.

$$3y - 6z = 6 \implies y - 2z = 2 \implies y = 2z + 2$$

$$x - y + 5z = -2 \implies x = y - 5z - 2 \implies x = 2z + 2 - 5z - 2 = -3z$$

So we have infinitely many solutions of the form $(-3z, 2z + 2, z)$.