Midterm 2 (Practice) Solutions

There are 11 problems worth 10 points each. A score of above 100 is possible on this midterm. Show all of your work for full credit.

1. Solve the following logarithmic equation.

$$\log_2(x-3) + \log_2(x+1) = 5$$

Solution:

$$\begin{aligned} \log_2(x-3) + \log_2(x+1) = 5 &\implies \log_2[(x-3)(x+1)] = 5 \\ &\implies 32 = x^2 - 2x - 3 \\ &\implies x^2 - 2x - 35 = 0 \\ &\implies (x-7)(x+5) = 0 \end{aligned}$$

Of the solutions $x = 7, -5$, $x = -5$ is extraneous since it results in a negative argument in both of the logarithms of the original equation, so we have $x = 7$.

2. Solve the following exponential equation.

$$5^{x/3} = 3^{x+1}$$

Solution: There are many equivalent approaches. Take $\log_5$ of both sides to get

$$\begin{aligned} \log_5(5^{x/3}) = \log_5(3^{x+1}) &\implies \frac{x}{3} = (x+1)\log_5(3) \\ &\implies \frac{x}{3} - x\log_5 3 = \log_5 3 \\ &\implies x = \frac{\log_5 3}{\frac{1}{3} - \log_5 3} \end{aligned}$$

Take $\log_3$ of both sides to get

$$\begin{aligned} \log_3(5^{x/3}) = \log_3(3^{x+1}) &\implies \frac{x}{3}\log_3 5 = (x+1) \\ &\implies \frac{x}{3}\log_3 5 - x = 1 \\ &\implies x = \frac{1}{\frac{1}{3}\log_3 5 - 1} \end{aligned}$$

Taking log with an unspecified base gets us

$$\begin{aligned} \log(5^{x/3}) = \log(3^{x+1}) &\implies \frac{x}{3}\log 5 = (x+1)\log 3 \\ &\implies \frac{x}{3}\log 5 - x\log 3 = \log 3 \\ &\implies x = \frac{\log 3}{\frac{1}{3}\log 5 - \log 3} \end{aligned}$$

All these answer are equivalent and acceptable. In the third answer, we can take the base to be equal to 5 or 3 to recover the first two answers.

3. Use the laws of logarithms to combine the expression into a single logarithm.

$$\log_3 x - 2\log_3(x+1) + 3\log_3 y$$

Solution:

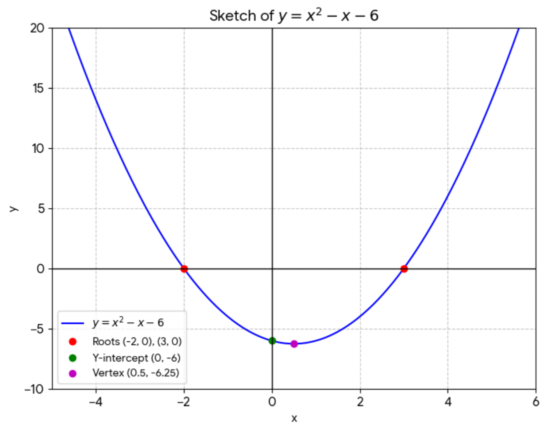
$$\begin{aligned} \log_3 x - 2\log_3(x+1) + 3\log_3 y &= \log_3 x - \log_3[(x+1)^2] + \log_3 y^3 \\ &= \log_3\left[\frac{xy^3}{(x+1)^2}\right] \end{aligned}$$

4. Express the quadratic function $f(x) = x^2 - x - 6$ in vertex form $a(x-h)^2 + k$, sketch its graph, and state the domain and range.

   Solution: Completing the square gives us

$$x^2 - x - 6 = x^2 - x + \frac{1}{4} - 6 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$$

   The domain is all real numbers. The range is $\left[-\frac{25}{4}, \infty\right)$.



5. Let $P(x) = 2x^3 - 5x^2 - 4x + 3$

   a) List all possible rational zeros of P.

   b) Find the complete factorization of P.

   c) Find the zeros of P.

   d) Sketch the graph of P.

   (This is probably a bit too long to be a single 10 point question on the midterm, but since this is just practice I'll leave it as is).

   Solution: If we have a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients, then the rational roots theorem says that the potential rational roots of the polynomials are the factors of a_0 over the factors of a_n. So by the rational roots theorem the possible zeros are $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

   $P(1) = 2 - 5 - 4 + 3 = -4 \neq 0$, so $x = 1$ is not a zero. $P(-1) = -2 - 5 + 4 + 3 = 0$ so $x = -1$ is a zero. We need to either do long division or synthetic division to divide out the factor of $x - 1$ from $P(x)$.

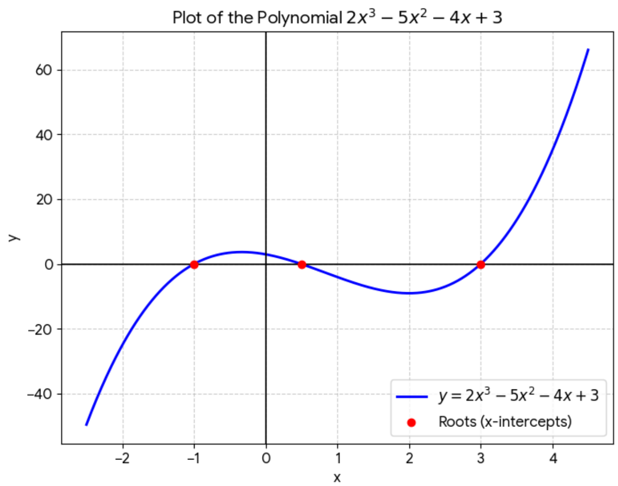
$$\begin{array}{r|l}
 & 2X^2 - 7X + 3 \\
\hline
X + 1 & 2X^3 - 5X^2 - 4X + 3 \\
 & -2X^3 - 2X^2 \\
\hline
 & -7X^2 - 4X \\
 & 7X^2 + 7X \\
\hline
 & 3X + 3 \\
 & -3X - 3 \\
\hline
 & 0
\end{array}$$

$$
\begin{array}{r|rrrr}
 & 2 & -5 & -4 & 3 \\
-1 & & -2 & 7 & -3 \\
\hline
 & 2 & -7 & 3 & 0
\end{array}
$$

So we have

$$P(x) = 2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3) = (x + 1)(2x - 1)(x - 3)$$

The zeros are $x = -1, \frac{1}{2}, 3$.



6. Consider the following rational functions

$$r(x) = \frac{2x - 1}{x^2 - x - 2}, \quad s(x) = \frac{x^3}{x^2 + 4}, \quad t(x) = \frac{x^3 - 9x}{x + 2}$$

$$u(x) = \frac{x^2 + x - 6}{x^2 - 25}, \quad w(x) = \frac{x^3 + 6x^2 + 9x}{x + 3}$$

You do not need to justify your answers for full credit, but if you want partial credit you should.

a) Which of these rational functions has a horizontal asymptote?

b) Which of these functions has a slant asymptote?

c) Which of these functions has **no** vertical asymptote?

d) Which of these functions has a "hole" (removable discontinuity)?

Solution:

a) In order to have a horizontal asymptote, the degree of the denominator has to be greater than or equal to the degree of the numerator. So $r(x), u(x)$ have horizontal asymptotes.

b) In order to have a slant asymptote, the degree of the numerator must be one greater than the degree of the numerator. So $s(x)$ has a slant asymptote.

c) In order to have a vertical asymptote, the degree must have a zero that is not a degree of the numerator. The denominator of $r(x) = x^2 - x - 2 = (x - 2)(x + 1)$, and neither $x = 2, -1$ are zeros of the numerator, so they are both vertical asymptotes. The denominator of $s(x)$ does not have any zeros. The zero of the denominator $t(x)$ is $x = -2$ which is not a zero of the numerator so it is a vertical asymptote. The zeros of the denominator $u(x)$ are $x = \pm 5$ which are not zeros of the numerator. The zero of the denominator of $w(x)$ $x = -3$ is a zero of the numerator. So $s(x)$ has no vertical asymptote.

d) Per the above, the function $w(x)$ has a hole (or removable discontinuity).

7. Solve the inequality.

$$x \leq \frac{6-x}{2x-5}$$

Solution: Move everything to one side and combine:

$$\begin{aligned} x \leq \frac{6-x}{2x-5} &\implies \frac{x(2x-5)}{2x-5} - \frac{6-x}{2x-5} \leq 0 \\ &\implies \frac{2x^2-5x-6+x}{2x-5} \leq 0 \\ &\implies \frac{2x^2-4x-6}{2x-5} \leq 0 \\ &\implies \frac{2(x^2-2x-3)}{2x-5} \leq 0 \\ &\implies \frac{2(x-3)(x+1)}{2x-5} \leq 0 \end{aligned}$$

So you need to make the sign chart with $x = 3, -1, \frac{5}{2}$ the intervals $(-\infty, 1)$ and $(5/2, 3)$ are strictly negative. We can include the endpoints $x = -1, 3$ since they will result in equality, but we must exclude $\frac{5}{2}$ because that will result in division by zero. So our final answer is $(-\infty, -1] \cup \left(\frac{5}{2}, 3\right]$.

8. The point $P(x, y)$ is on the unit circle in Quadrant IV. If $x = \frac{\sqrt{11}}{5}$. (Hint you should know: Quadrant IV has positive x and negative y).

Solution:

$$\begin{aligned} x^2 + y^2 = 1 &\implies \frac{11}{25} + y^2 = 1 \\ &\implies y^2 = \frac{14}{25} \\ &\implies y = -\frac{\sqrt{14}}{5} \end{aligned}$$

where we take the negative square root since we are in quadrant IV.

9. The point P in the figure has the y coordinate $\frac{4}{5}$. Find $\tan t$ and $\sec t$.

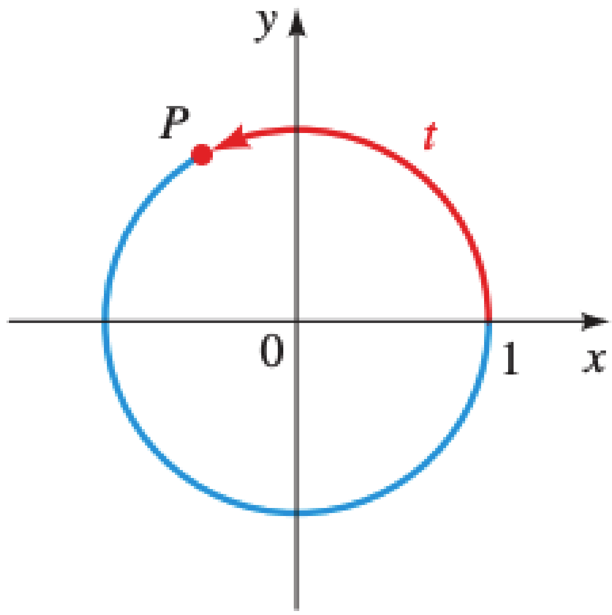
Solution: We have that $\cos t = -\frac{3}{5}, \sin t = \frac{4}{5}$. So

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

$$\sec t = \frac{1}{\cos t} = -\frac{5}{3}$$

10. Find the exact value of the following expressions

$$\sin\left(\frac{7\pi}{6}\right) \qquad \cos\left(\frac{13\pi}{4}\right) \qquad \tan\left(-\frac{5\pi}{3}\right) \qquad \csc\left(\frac{3\pi}{2}\right)$$



Solution:

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$
$$\cos\left(\frac{13\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$\tan\left(-\frac{5\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$
$$\csc\left(\frac{3\pi}{2}\right) = \frac{1}{\sin\left(\frac{3\pi}{2}\right)} = \frac{1}{-1} = -1$$

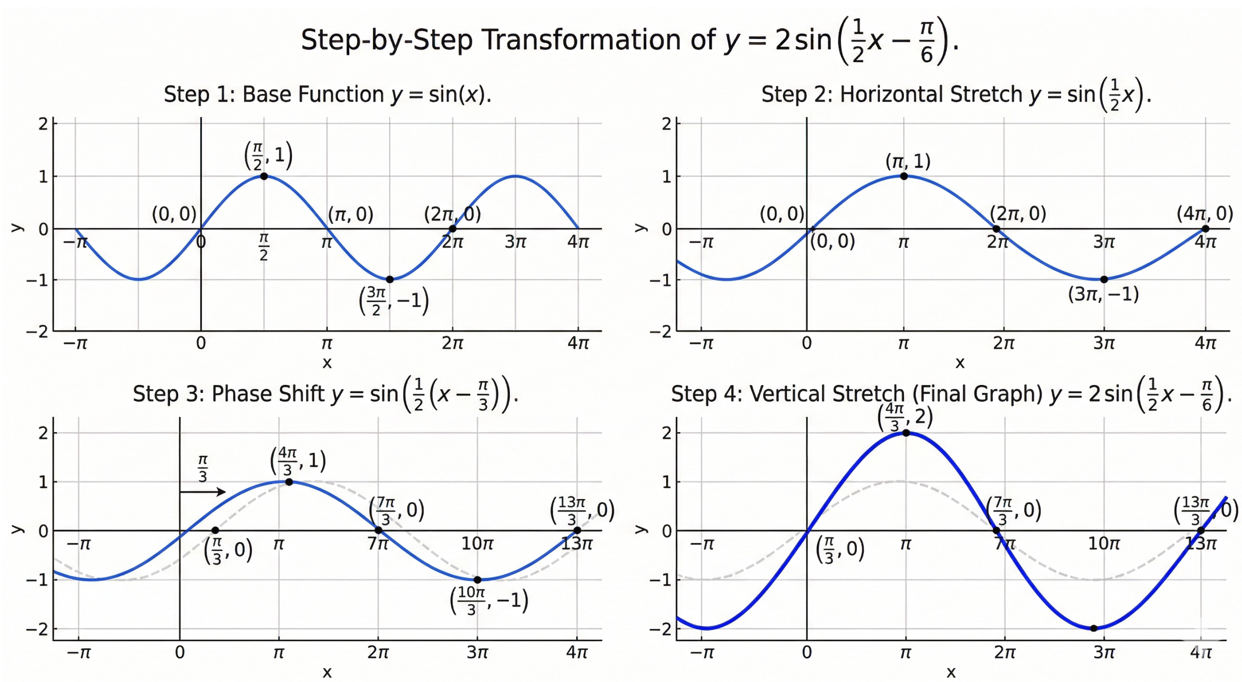
11. Sketch (at least) one period of the following function. Label at least three points with rational coordinates on the same period.
$$2\sin\left(\frac{1}{2}x - \frac{\pi}{6}\right)$$

Solution: The quantity in the parenthesis is probably as nasty as it gets. Factor to get

$$2\sin\left(\frac{1}{2}x - \frac{\pi}{6}\right) = 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right)$$

So the transformations are

$$\sin(x) \to \sin\left(\frac{1}{2}x\right) \to \sin\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) \to 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right)$$


Step-by-Step Transformation of $y = 2\sin\left(\frac{1}{2}x - \frac{\pi}{6}\right)$.
Step 1: Base Function $y = \sin(x)$.
$\left(\frac{\pi}{2}, 1\right)$
$(0, 0)$
$(\pi, 0)$
$(2\pi, 0)$
$\left(\frac{3\pi}{2}, -1\right)$
Step 2: Horizontal Stretch $y = \sin\left(\frac{1}{2}x\right)$.
$(\pi, 1)$
$(0, 0)$
$(2\pi, 0)$
$(4\pi, 0)$
$(3\pi, -1)$
Step 3: Phase Shift $y = \sin\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right)$.
$\frac{\pi}{3}$
$\left(\frac{4\pi}{3}, 1\right)$
$\left(\frac{7\pi}{3}, 0\right)$
$\left(\frac{13\pi}{3}, 0\right)$
$\left(\frac{\pi}{3}, 0\right)$
$\left(\frac{10\pi}{3}, -1\right)$
Step 4: Vertical Stretch (Final Graph) $y = 2\sin\left(\frac{1}{2}x - \frac{\pi}{6}\right)$.
$\left(\frac{4\pi}{3}, 2\right)$
$\left(\frac{7\pi}{3}, 0\right)$
$\left(\frac{13\pi}{3}, 0\right)$
$\left(\frac{\pi}{3}, 0\right)$
x
y