Extended Mean Field Control Games with Moment Interactions: General Framework and Linear-Quadratic Model

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***Abstract*— Mean field control games (MFCGs) provide a framework for studying non-cooperative games involving groups of cooperative players in the limit as both the number of players and the number of groups go to infinity. This class of games includes mean field games (MFGs) and mean field control (MFC) problems as special cases, corresponding to purely non-cooperative and purely cooperative settings, respectively. We incorporate interactions through the mean of actions, hence the terminology of extended MFCGs. We first show that the equilibrium control can be described using a coupled system of partial differential equations consisting of a Hamilton-Jacobi-Bellman equation and Fokker-Planck equation. After introducing the general framework, we focus on a linear–quadratic (LQ) structure. We first show that the equilibrium can be reduced to a system of ODEs, generalizing those obtained for MFGs and MFC problems. We then provide two numerical examples illustrating specific features of MFCGs.**

I. INTRODUCTION

Mean Field Games (MFGs) and Mean Field Control (MFC) have been highly active areas of research since the foundational work of Lasry and Lions [20] and Huang, Caines, and Malhamé [19]. These frameworks capture the asymptotic behavior of large populations of interacting agents in purely non-cooperative (MFG) and purely cooperative (MFC) regimes. However, many real-world systems do not fall strictly into either extreme. Instead, they feature hierarchical structures where players coordinate their strategies within distinct groups, while the groups themselves compete non-cooperatively. When the number of groups is finite and each group is infinite, such games have been referred to as *mean field type games*; see e.g. [14], [6]. When the number of groups tends to infinity, this gives rise to MFGs in which each player solves a MFC problem. This motivates the study of *Mean Field Control Games* (MFCGs), introduced in [3], which represent a Nash equilibrium between collaborating groups. Recent literature has begun exploring models [4] and algorithms [5] for this intermediate regime. Along this line, [12] considered games that interpolate between MFC and MFG, and [13] proposed models with a mixture of cooperation and non-cooperation.

Concurrently, a second major evolution in mean field theory has been the development of "extended" models, also called "of controls" type. Standard mean field frameworks assume that agents interact exclusively through the empirical distribution of their states. In many applications, however, agents' costs and dynamics depend explicitly on the *actions* or *controls* of the broader population. To address this, extended MFGs were formalized [17], [1] to allow the Lagrangian to depend on the joint distribution of states and controls. This class of games have also been referred to as MFGs *of control* [11], [2]. This has sparked significant recent work on extended mean field problems from both theoretical [15], [16], [10], [9] and numerical perspectives [22], [23]. [21] studied linear-quadratic MFCGs with action distribution in a specific LQ setting.

In this paper, we bridge these developments by proposing a general framework for *Extended Mean Field Control Games* with moment-type interactions. We consider a generalized environment where groups of cooperative players interact in a non-cooperative game, but crucially, the interaction occurs through distinct generalized moments of the joint state-action distribution.

Specifically, we first formulate the general extended MFCG problem and characterize its equilibrium via a coupled system of partial differential equations, consisting of a Hamilton-Jacobi-Bellman (HJB) equation and a Fokker-Planck (FP) equation (see Theorem 1). Second, we specialize the framework to a Linear-Quadratic (LQ) structure. We demonstrate that in the LQ setting, the PDE system can be reduced to system of ordinary differential equations (ODEs) featuring Riccati-type equations (see Theorem 2). Finally, we propose a model for LQ MFCGs with an interpolation parameter that blends global and group-level benchmarks which we illustrate through numerical experiments.

II. EXTENDED MFCG MODEL

In this section, we formulate the extended MFCG problem mathematically.

General notation. Let d be the state dimension, k be the action dimension and m be the dimension of the interaction terms. $\mathcal{P}(E)$ denotes the set of probability measures on E.

Individual dynamics. The dynamics of a representative agent interacts with the population through a time-dependent interaction term $\bar{\varphi} : [0,T] \to \mathbb{R}^m$. If the agent uses a control $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}^k$, then we denote their state at time t by $X_t^{v,\bar{\varphi}} \in \mathbb{R}^d$. We denote by $m_t^{v,\bar{\varphi}} \in \mathcal{P}(\mathbb{R}^d)$ the density of $X_t^{v,\bar{\varphi}}$. We assume that the evolution of this state involves its own distribution through a term of the form:

$$\tilde{\varphi}_t^{v,\bar{\varphi}} = \mathbb{E}[\varphi(t, X_t^{v,\bar{\varphi}}, v(t, X_t^{v,\bar{\varphi}}))] = \int \varphi(t, x', v(t,x')) m^{v,\bar{\varphi}}(t,x') dx', \qquad (1)$$

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where $\varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^m$ is a given interaction function. The dynamics is:

$$dX_t^{v,\bar{\varphi}} = b(t, X_t^{v,\bar{\varphi}}, v(t, X_t^{v,\bar{\varphi}}), \tilde{\varphi}_t^{v,\bar{\varphi}}, \bar{\varphi}_t)dt + \sigma(t, X_t^{v,\bar{\varphi}})dW_t, \tag{2}$$

with initial condition $X_0^{v,\bar{\varphi}} \sim m_0$ for a given m_0. Note that this dynamics is of McKean-Vlasov (MKV) type.

The density $m_t^{v,\bar{\varphi}}$ of $X_t^{v,\bar{\varphi}}$ satisfies the following Kolmogorov-Fokker-Planck (KFP) equation:

$$\partial_t m^{v,\bar{\varphi}} + A^* m^{v,\bar{v}} + \operatorname{div}\left(b(t,x,v,\tilde{\varphi}_t^{v,\bar{\varphi}}, \bar{\varphi}_t) m^{v,\bar{\varphi}}\right) = 0, \tag{3}$$

with initial condition $m^{v,\bar{\varphi}}(0) = m_0$, where the second-order differential operator and its adjoint are defined by:

$$Au(t,x) = -\operatorname{Tr}(a(t,x) D_x^2 u(t,x)),$$

$$A^* u(t,x) = -\sum_{i,j}^{d} \partial_{ij}\left(a_{ij}(t,x) u(t,x)\right),$$

with $a(t,x) = \frac{1}{2}\sigma\sigma^\top(t,x)$.

Population dynamics. We assume that the interactions with the population are of the same form as (1). They could occur with a different function than φ but for simplicity we will consider the same function for individual and population interactions. Hence, if the whole population uses control $\bar{v}$, the population density, denoted by $m^{\bar{v}}$, solves the KFP:

$$\partial_t m^{\bar{v}} + A^* m^{\bar{v}} + \operatorname{div}\left(b(t,x,\bar{v},\tilde{\varphi}_t^{\bar{v},\bar{\varphi}^{\bar{v}}}, \bar{\varphi}_t^{\bar{v}}) m^{\bar{v}}\right) = 0,$$

with initial condition $m^{\bar{v}}(0) = m_0$, where

$$\bar{\varphi}_t^{\bar{v}} = \int \varphi(t, x', \bar{v}(t,x')) m^{\bar{v}}(t,x') dx'.$$

When the population uses control $\bar{v}$, the term $\bar{\varphi}_t$ in (2) is replaced by $\bar{\varphi}_t^{\bar{v}}$. Furthermore, to alleviate the notation, we denote $X_t^{v,\bar{v}} = X_t^{v,\bar{\varphi}^{\bar{v}}}$, $m_t^{v,\bar{v}} = m_t^{v,\bar{\varphi}^{\bar{v}}}$, $\tilde{\varphi}_t^{v,\bar{v}} = \tilde{\varphi}_t^{v,\bar{\varphi}^{\bar{v}}}$.

Cost function. If the population uses control $\bar{v}$ and the agent uses control v, the agent's cost is defined as follows, where the integrals with respect to $m^{v,\bar{v}}$ can be interpreted as expectations with respect to $X^{v,\bar{v}}$:

$$J(v;\bar{v}) = \int_0^T \int_{\mathbb{R}^n} f(t,x,v(t,x), \tilde{\gamma}_t^{v,\bar{v}}, \bar{\gamma}_t^{\bar{v}}) m^{v,\bar{v}}(t,x) dx dt + \int_{\mathbb{R}^n} g(x, \tilde{\psi}_T^{v,\bar{v}}, \bar{\psi}_T^{\bar{v}}) m^{v,\bar{v}}(T,x) dx, \tag{4}$$

where the interaction terms are:

$$\tilde{\gamma}_t^{v,\bar{v}} = \int_{\mathbb{R}^n} \gamma(t,x',v(t,x')) m^{v,\bar{v}}(t,x') dx',$$

$$\bar{\gamma}_t^{\bar{v}} = \int_{\mathbb{R}^n} \gamma(t,x',\bar{v}(t,x')) m^{\bar{v}}(t,x') dx',$$

$$\tilde{\psi}_T^{v,\bar{v}} = \int_{\mathbb{R}^n} \psi(x') m^{v,\bar{v}}(T,x') dx',$$

$$\bar{\psi}_T^{\bar{v}} = \int_{\mathbb{R}^n} \psi(x') m^{\bar{v}}(T,x') dx'$$

for some given functions $\gamma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^m$ and $\psi : \mathbb{R}^d \to \mathbb{R}^m$. The solution concept we study is the following, based on the idea of Nash equilibrium.

Definition 1 (Extended MFCG Equilibrium). *A control $\hat{v}$ is an equilibrium for the extended MFCG problem if $\hat{v}$ is a minimizer of $v \mapsto J(v;\hat{v})$.*

In other words, a control is an equilibrium if it is an optimal control for a player facing a population in which all the players use this control. This can be interpreted as a fixed point problem: first, given the population's evolution, we find an optimal control by solving an MFC problem; second, the population's evolution should be consistent with this optimal control.

III. MFCG EQUILIBRIUM

In this section, we characterize the MFCG equilibrium by a forward-backward PDE system.

When the population's evolution is given, the problem for a representative player in MFCG reduces to an MFC problem. However, this problem is of "extended" type in the sense that it involves the action distribution. To the best of our knowledge, extended MFC problems have not yet been studied using a PDE perspective. For this reason, we start by proving the following result.

Lemma 1 (Necessary condition of optimality for individual problem). *Let $\bar{v}$ be the control used by the population. Given $\bar{\varphi}^{\bar{v}}$, $\bar{\gamma}^{\bar{v}}$, $\bar{\psi}_T^{\bar{v}}$, if v^* is an optimal feedback control minimizing (4) subject to (3), then it satisfies the necessary optimality condition: for all (t,x),*

$$\begin{aligned} &\partial_v f(t,x,v^*(t,x), \tilde{\gamma}_t^{v^*,\bar{v}}, \bar{\gamma}_t^{\bar{v}}) \\ &+ D_x u(t,x) \cdot \partial_v b(t,x,v^*(t,x), \tilde{\varphi}_t^{v^*,\bar{v}}, \bar{\varphi}_t^{\bar{v}}) \\ &+ \partial_v \gamma(t,x,v^*(t,x)) \cdot K_t^\gamma + \partial_v \varphi(t,x,v^*(t,x)) \cdot K_t^\varphi = 0, \end{aligned} \tag{5}$$

where $(m^{v^*,\bar{v}}, u^{v^*,\bar{v}})$ solves PDE system:

$$\begin{cases} \partial_t m^{v^*,\bar{v}}(t,x) + A^* m^{v^*,\bar{v}}(t,x) \\ \qquad + \operatorname{div}(b(t,x,v^*(t,x), \tilde{\varphi}_t^{v^*,\bar{v}}, \bar{\varphi}_t^{\bar{v}}) m^{v^*,\bar{v}}(t,x)) = 0, \\ m^{v^*,\bar{v}}(0,x) = m_0(x), \\ -\partial_t u^{v^*,\bar{v}}(t,x) + A u^{v^*,\bar{v}}(t,x) \\ \qquad - b(t,x,v^*(t,x), \tilde{\varphi}_t^{v^*,\bar{v}}, \bar{\varphi}_t^{\bar{v}}) \cdot D_x u^{v^*,\bar{v}}(t,x) \\ \qquad - f(t,x,v^*(t,x), \tilde{\gamma}_t^{v^*,\bar{v}}, \bar{\gamma}_t^{\bar{v}}) \\ \qquad - \varphi(t,x,v^*(t,x)) K_t^\varphi - \gamma(t,x,v^*(t,x)) K_t^\gamma = 0, \\ u(x,T) = g(x, \tilde{\psi}_T^{v^*,\bar{v}}, \bar{\psi}_T^{\bar{v}}) + \psi(x) K_T^\psi, \end{cases} \tag{6}$$

with K_t^γ, K_t^φ, and K_T^ψ given by:

$$K_t^\gamma = \int_{\mathbb{R}^n} \partial_{\tilde{\gamma}} f(t,x',v^*(t,x'), \tilde{\gamma}_t^{v^*,\bar{v}}, \bar{\gamma}_t^{\bar{v}}) m^{v^*,\bar{v}}(t,x') dx',$$

$$K_t^\varphi = \int_{\mathbb{R}^n} D_x u^{v^*,\bar{v}}(t,x') \cdot \partial_{\tilde{\varphi}} b(t,x',v^*(t,x'), \tilde{\varphi}_t^{v^*,\bar{v}}, \bar{\varphi}_t^{\bar{v}}) m^{v^*,\bar{v}}(t,x') dx',$$

$$K_T^\psi = \int_{\mathbb{R}^n} \partial_{\tilde{\psi}} g(x', \tilde{\psi}_T^{v^*,\bar{v}}, \bar{\psi}_T^{\bar{v}}) m^{v^*,\bar{v}}(T,x') dx'.$$

Before providing the proof, a few remarks are in order.

Remark 1 (Terminology). Note that, although the equation for u is often referred to as the HJB equation in the MFC literature, it is *not* the Bellman equation of the MFC problem, and the function u is *not* the value function of the player; see e.g. [8] for more details on the connection between the Bellman equation and the HJB equation.

Remark 2 (Consistency with standard MFC). If the individual player does not interact with the population (i.e., the functions b, f, and g are constant wrt $\bar{\varphi}, \bar{\gamma}$, and $\bar{\psi}$ respectively), and if, furthermore, the action distribution is not involved (i.e., the functions φ, γ and ψ are constant with respect to the action), then the problem reduces to a standard MFC problem. Consistently with this, the PDE system in Lemma 1 reduces to the standard MFC PDE system given e.g. in [7, eq. (4.12)].

Proof of Lemma 1. Let v^* be an optimal control. Consider the perturbation $v^*+\theta w$ and compute the Gateaux derivative

$$
\begin{aligned}
&\frac{d}{d\theta}J(v^*+\theta w;\bar{v})\Big|_{\theta=0}\\
&=\int_0^T\int_{\mathbb{R}^n}(\partial_{\bar{\gamma}}f\cdot\check{\gamma}_t\cdot m^{v^*,\bar{v}}+\partial_v f\cdot w\cdot m^{v^*,\bar{v}}+f\cdot\check{m})dxdt\\
&\qquad+\int_{\mathbb{R}^n}(\partial_{\bar{\psi}}g\cdot\check{\psi}_T\cdot m^{v^*,\bar{v}}(T,x)+g\cdot\check{m}(T,x))dx,
\end{aligned}
$$

where $\check{m}$, $\check{\gamma}_t$, $\check{\varphi}_t$ and $\check{\psi}_T$ are first variations given by

$$
\begin{aligned}
&\partial_t\check{m}+A^*\check{m}+\mathrm{div}(b\check{m})\\
&\quad+\mathrm{div}\left((\partial_{\bar{\varphi}}b\cdot\check{\varphi}_t+\partial_v b\cdot w)m^{v^*,\bar{v}}\right)=0, \check{m}(0,x)=0,\\
&\check{\gamma}_t=\int_{\mathbb{R}^n}\gamma(t,\xi,v^*)\check{m}(t,\xi)d\xi\\
&\qquad+\int_{\mathbb{R}^n}\partial_v\gamma(t,\xi,v^*)w(t,\xi)m^{v^*,\bar{v}}(t,\xi)d\xi,\\
&\check{\varphi}_t=\int_{\mathbb{R}^n}\varphi(t,\xi,v^*)\check{m}(t,\xi)d\xi\\
&\qquad+\int_{\mathbb{R}^n}\partial_v\varphi(t,\xi,v^*)w(t,\xi)m^{v^*,\bar{v}}(t,\xi)d\xi,\\
&\check{\psi}_T=\int_{\mathbb{R}^n}\psi(\xi)\check{m}(T,\xi)d\xi.
\end{aligned}
\tag{7}
$$

Substituting the variation $\check{\gamma}_t$ and $\check{\psi}_T$ into the Gateaux derivative and using the definitions of K_t^{γ} and K_T^{ψ}, we have

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n}\partial_{\bar{\gamma}}f\cdot\check{\gamma}_t\cdot m^{v^*,\bar{v}}dxdt\\
&=\int_0^T\int_{\mathbb{R}^n}K_t^{\gamma}\left(\gamma(t,x,v^*)\check{m}+\partial_v\gamma(t,x,v^*)wm^{v^*,\bar{v}}\right)dxdt
\end{aligned}
$$

and $\displaystyle\int_{\mathbb{R}^n}\partial_{\bar{\psi}}g\cdot\check{\psi}_T\cdot m^{v^*}(T)dx=\int_{\mathbb{R}^n}K_T^{\psi}\psi(x)\check{m}(x,T)dx.$

It follows that

$$
\begin{aligned}
&\frac{d}{d\theta}J(v^*+\theta w;\bar{v})\Big|_{\theta=0}\\
&=\int_0^T\int_{\mathbb{R}^n}\left[(\partial_v f+K_t^{\gamma}\partial_v\gamma)\,wm^{v^*,\bar{v}}+(f+K_t^{\gamma}\gamma)\,\check{m}\right]dxdt\\
&\quad+\int_{\mathbb{R}^n}\left(g+K_T^{\psi}\psi\right)\check{m}(T,x)dx.
\end{aligned}
$$

To eliminate $\check{m}$, we take the inner product of the equation of u in (9) with $\check{m}$ over $\mathbb{R}^n\times[0,T]$. Applying integration by parts and substituting the equation of $\check{m}$ in (7) yields

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^n}(f+\gamma K_t^{\gamma})\,\check{m}dxdt&=-\int_{\mathbb{R}^n}u(x,T)\check{m}(T,x)dx\\
&+\int_0^T\int_{\mathbb{R}^n}(D_x u\cdot\partial_v b+K_t^{\varphi}\partial_v\varphi)\,wm^{v^*,\bar{v}}dxdt.
\end{aligned}
$$

Using this relation, the cost variation simplifies to

$$
\begin{aligned}
&\frac{d}{d\theta}J(v^*+\theta w;\bar{v})\Big|_{\theta=0}\\
&=\int_0^T\int_{\mathbb{R}^n}\big[\partial_v f+D_x u\cdot\partial_v b\\
&\quad+\partial_v\gamma K_t^{\gamma}+\partial_v\varphi K_t^{\varphi}\big]wm^{v^*,\bar{v}}dxdt\\
&\quad+\int_{\mathbb{R}^n}\left(g+K_T^{\psi}\psi-u(T,x)\right)\check{m}(x,T)dx.
\end{aligned}
$$

The terminal condition of the equation for u in (9) dictates that the second integral vanishes. Since v^* is optimal, $\frac{dJ}{d\theta}\big|_{\theta=0}=0$ for any arbitrary variation w. By the fundamental theorem of calculus of variations, the bracketed term must be identically zero almost everywhere, which yields the optimality condition (5). □

The following theorem characterizes the extended MFCG equilibrium by a system of PDEs.

Theorem 1. *If $\hat{v}$ is an equilibrium control for the MFCG, then it satisfies: for all (t,x),*

$$
\begin{aligned}
&\partial_v f(t,x,\hat{v}(t,x),\bar{\gamma}_t^{\hat{v}},\bar{\gamma}_t^{\hat{v}})\\
&+D_x u^{\hat{v}}(t,x)\cdot\partial_v b(t,x,\hat{v}(t,x),\bar{\varphi}_t^{\hat{v}},\bar{\varphi}_t^{\hat{v}})\\
&+\partial_v\gamma(t,x,\hat{v}(t,x))\hat{K}_t^{\gamma}+\partial_v\varphi(t,x,\hat{v}(t,x))\hat{K}_t^{\varphi}=0,
\end{aligned}
\tag{8}
$$

where $(m^{\hat{v}},u^{\hat{v}})$ solves the PDE system:

$$
\begin{cases}
\partial_t m^{\hat{v}}(t,x)+A^*m^{\hat{v}}(t,x)\\
\qquad+\mathrm{div}(g(t,x,\hat{v}(t,x),\bar{\varphi}_t^{\hat{v}},\bar{\varphi}_t^{\hat{v}})m^{\hat{v}}(t,x))=0,\\
m^{\hat{v}}(0,x)=m_0(x),\\
-\partial_t u^{\hat{v}}(t,x)+Au^{\hat{v}}(t,x)\\
\qquad-b(t,x,\hat{v}(t,x),\bar{\varphi}_t^{\hat{v}},\bar{\varphi}_t^{\hat{v}})\cdot D_x u^{\hat{v}}(t,x)\\
\qquad-f(t,x,\hat{v}(t,x),\bar{\gamma}_t^{\hat{v}},\bar{\gamma}_t^{\hat{v}})\\
\qquad-\varphi(t,x,\hat{v}(t,x))\hat{K}_t^{\varphi}-\gamma(t,x,\hat{v}(t,x))\hat{K}_t^{\gamma}=0,\\
u^{\hat{v}}(x,T)=g(x,\bar{\psi}_T^{\hat{v}},\bar{\psi}_T^{\hat{v}})+\psi(x)\hat{K}_T^{\psi},
\end{cases}
\tag{9}
$$

with $\hat{K}^{\gamma}_t$, $\hat{K}^{\varphi}_t$ and $\hat{K}^{\psi}_t$ given by:

$$\hat{K}^{\gamma}_t = \int_{\mathbb{R}^n} \partial_{\tilde{\gamma}} f(t,x',\hat{v}(t,x'),\bar{\gamma}^{\hat{v}}_t,\bar{\gamma}^{\hat{v}}_t) m^{\hat{v}}(t,x')dx',$$
$$\hat{K}^{\varphi}_t = \int_{\mathbb{R}^n} D_x u^{\hat{v}}(t,x') \cdot \partial_{\tilde{\varphi}} b(t,x',\hat{v}(t,x'),\bar{\varphi}^{\hat{v}}_t,\bar{\varphi}^{\hat{v}}_t) m^{\hat{v}}(t,x')dx',$$
$$\hat{K}^{\psi}_T = \int_{\mathbb{R}^n} \partial_{\tilde{\psi}} g(x',\bar{\psi}^{\hat{v}}_T,\bar{\psi}^{\hat{v}}_T) m^{\hat{v}}(T,x')dx'.$$

Proof. Let $\hat{v}$ be an equilibrium control. By the optimality condition in Definition 1, $\hat{v}$ is an optimal feedback control that minimizes (4) subject to (3). Applying Lemma 1, it follows that $\hat{v}$ satisfies (5) upon substituting v^* with $\hat{v}$. Furthermore, since both the population and the individual player use control $\hat{v}$, we have $\tilde{\varphi}^{v^*,\bar{v}}_t = \bar{\varphi}^{\hat{v},\hat{v}}_t = \bar{\varphi}^{\hat{v}}_t$, $\tilde{\gamma}^{v^*,\bar{v}}_t = \bar{\gamma}^{\hat{v},\hat{v}}_t = \bar{\gamma}^{\hat{v}}_t$, and $\tilde{\psi}^{v^*,\bar{v}}_T = \bar{\psi}^{\hat{v},\hat{v}}_T = \bar{\psi}^{\hat{v}}_T$, hence (5)–(6) rewrites as (8)–(9), and this completes the proof. $\square$

IV. Linear Quadratic Case

In this section, we consider the extended MFCG model in a linear quadratic framework. To alleviate the notation, we consider that the dynamics does not depend on mean field terms. We take: b linear in (x,v) and independent of $(\tilde{\gamma},\bar{\gamma})$, f quadratic in $(x,v,\tilde{\varphi},\bar{\varphi})$, φ linear in (x,v), g quadratic in $(x,\tilde{\psi},\bar{\psi})$, and ψ linear in x. To be specific, we consider the following model. The dynamics is given by

$$dX_t = (A_t X_t + B_t v_t)dt + (C_t X_t + D_t v_t)dW_t, \qquad (10)$$

with $X_0 \sim m_0$, where $v_t = v(t,X_t)$ is a feedback control at time t, with value in the action set $\mathbb{R}^k$, and A,B,C,D are given deterministic processes. Let $\tilde{X},\bar{X}$ and $\tilde{v},\bar{v}$ be the mean of the states and of the controls for agents *within* group and for agents of the *entire population*, respectively. Given the global population mean $\bar{X} = (\bar{X}_t)_{t\in[0,T]}$ and $\bar{v} = (\bar{v}_t)_{t\in[0,T]}$, a representative group regulator chooses v to minimize the total quadratic cost functional

$$J(v;\bar{X},\bar{v}) = \mathbb{E}\bigg[\int_0^T F(t,X_t,\tilde{X}_t,\bar{X}_t,v_t,\tilde{v}_t,\bar{v}_t)dt + G(X_T,\tilde{X}_T,\bar{X}_T)\bigg], \qquad (11)$$

where the running cost F and terminal cost G are given by

$$\begin{aligned} F(t,x,\tilde{x},\bar{x},v,\tilde{v},\bar{v}) &= Q_t x\cdot x + \tilde{Q}_t\tilde{x}\cdot\tilde{x} + \bar{Q}_t\bar{x}\cdot\bar{x} \\ &\quad + M_t\bar{x}\cdot x + \tilde{M}_t\bar{x}\cdot\tilde{x} + R_t v\cdot v + \tilde{R}_t\tilde{v}\cdot\tilde{v} \\ &\quad + \bar{R}_t\bar{v}\cdot\bar{v} + N_t\bar{v}\cdot v + \tilde{N}_t\bar{v}\cdot\tilde{v} + 2S_t x\cdot v \\ &\quad + 2\tilde{S}_t\tilde{x}\cdot\tilde{v} + 2\bar{S}_t\bar{x}\cdot\bar{v} + 2q_t\cdot x + 2\tilde{q}_t\cdot\tilde{x} \\ &\quad + 2\bar{q}_t\cdot\bar{x} + 2r_t\cdot v + 2\tilde{r}_t\cdot\tilde{v} + 2\bar{r}_t\cdot\bar{v}, \\ G(x,\tilde{x},\bar{x}) &= Hx\cdot x + \tilde{H}\tilde{x}\cdot\tilde{x} + \bar{H}\bar{x}\cdot\bar{x}, \end{aligned} \qquad (12)$$

with Q, $\tilde{Q}$, $\bar{Q}$, M, $\tilde{M}$, R, $\tilde{R}$, $\bar{R}$, N, $\tilde{N}$, S, $\tilde{S}$, $\bar{S}$, q, $\tilde{q}$, $\bar{q}$, r, $\tilde{r}$, $\bar{r}$ being deterministic, matrix-valued processes, and H, $\tilde{H}$, $\bar{H}$ being constant matrices. We denote by $\mathcal{S}^n$ the set of all $n\times n$ symmetric matrices with real entries. For $U\in\mathcal{S}^n$, $U\ge 0$ means U is positive semi-definite. The linear–quadratic framework studied in this paper differs from the control-average mean-field stochastic large-population system introduced in [21]. We derive the associated ODE system inspired by the results of [18]. Note that [18] studies MFC and MFG problems separately, whereas we consider the MFCG formulation, which encompasses both MFC and MFG as special cases.

Assumption 1. *The coefficient matrices satisfy*

1) $A,C\in L^\infty([0,T];\mathbb{R}^{d\times d})$, $B,D\in L^\infty([0,T];\mathbb{R}^{d\times k})$;
2) $Q,\tilde{Q},\bar{Q},M,\tilde{M}\in L^\infty([0,T];\mathcal{S}^d)$, $R,\tilde{R},\bar{R},N,\tilde{N}\in L^\infty([0,T];\mathcal{S}^k)$, $H,\tilde{H},\bar{H}\in\mathcal{S}^d$;
3) $H\ge 0$, $H+\tilde{H}\ge 0$, *and for some* $\delta_1\ge 0,\delta_2>0$, $Q,Q+\tilde{Q}\ge\delta_1 I$ *and* $R,R+\tilde{R}\ge\delta_2 I$;
4) $S,\tilde{S},\bar{S}\in L^\infty([0,T];\mathbb{R}^{k\times d})$; $q,\tilde{q},\bar{q}\in L^\infty([0,T];\mathbb{R}^d)$; $r,\tilde{r},\bar{r}\in L^\infty([0,T];\mathbb{R}^k)$;
5) $\|S\|^2_\infty,\|S+\tilde{S}\|^2_\infty<\delta_1\delta_2$ *if* $\delta_1>0$, *and* $S=\tilde{S}=0$ *otherwise.*

We then derive the system of ordinary differential equations (ODEs) associated with the linear quadratic MFCG equilibrium. We first introduce two ODEs:

$$\begin{cases} \dot{P}_t + A_t^\top P_t + P_t A_t + C_t^\top P_t C_t + Q_t \\ \qquad -\Lambda_0^\top(t)\Sigma_0^{-1}(t)\Lambda_0(t) = 0, \\ P_T = H, \end{cases} \qquad (13)$$

and

$$\begin{cases} \dot{\Pi}_t + A_t^\top \Pi_t + \Pi_t A_t + C_t^\top P_t C_t + Q_t + \tilde{Q}_t \\ \qquad -\Lambda_1^\top(t)\Sigma_1^{-1}(t)\Lambda_1(t) = 0, \\ \Pi_T = H + \tilde{H}, \end{cases} \qquad (14)$$

where

$$\begin{aligned} \Lambda_0(t) &= B_t^\top P_t + D_t^\top P_t C_t + S_t, \\ \Lambda_1(t) &= B_t^\top \Pi_t + D_t^\top P_t C_t + S_t + \tilde{S}_t, \\ \Sigma_0(t) &= D_t^\top P_t D_t + R_t, \\ \Sigma_1(t) &= D_t^\top P_t D_t + R_t + \tilde{R}_t. \end{aligned}$$

Now we are ready to give the result that characterizes the MFCG equilibrium.

Theorem 2. *Under Assumption 1, there exist unique solutions P,Π to equations (13) and (14). Moreover, if $(\hat{v}_t)_{t\in[0,T]}$ is an MFCG equilibrium control, then*

$$\hat{v}_t = -\Sigma_0^{-1}(t)\Lambda_0(t)(X_t - z_t) - a_t,$$

where (z,a) satisfies the ODE system:

$$\begin{cases} \dot{z}_t = \big(A_t - B_t\Sigma_1^{-1}(t)\Lambda_1(t)\big)z_t - B_t\Sigma_1^{-1}(t) \\ \qquad \big(\tfrac{1}{2}(N_t+\tilde{N}_t)a_t + r_t + \tilde{r}_t + B_t^\top\phi_t\big), \\ z_0 = \int_{\mathbb{R}^d} x m_0(x)dx, \\ a_t = -\Sigma_1^{-1}(t)\big(\Lambda_1(t)z_t + \tfrac{1}{2}(N_t+\tilde{N}_t)a_t \\ \qquad + r_t + \tilde{r}_t + B_t^\top\phi_t\big), \\ \phi_t = \int_0^t \Big\{(\Pi_s B_s + C_s^\top P_s D_s + S_s^\top + \tilde{S}_s^\top)\Sigma_1^{-1}(s) \\ \qquad \big(\tfrac{1}{2}(N_s+\tilde{N}_s)a_s + r_s + \tilde{r}_s\big) + \tfrac{1}{2}(M_s+\tilde{M}_s)z_s \\ \qquad + q_s + \tilde{q}_s\Big\}ds. \end{cases} \qquad (15)$$

This result could be derived from our general result in Theorem 1 after making a suitable ansatz. Due to space limitation, we present an alternative proof building upon ODEs derived by [18] for standard MFC.

Proof. The proof is divided into two steps.

Step 1: Solving the individual MFC with a fixed population evolution. In the linear quadratic framework, fixing the distribution flow means that we fix $(\bar{X}_t)_{t\in[0,T]}$ and $(\bar{v}_t)_{t\in[0,T]}$. Now viewing them as given processes, the MFCG problem becomes a MFC problem with slight modification for the coefficients of linear terms:

$$\begin{aligned}\check{q}_t(\bar{X}) &= \frac{1}{2}M_t\bar{X}_t + q_t, \quad \check{\tilde{q}}_t(\bar{X}) = \frac{1}{2}\tilde{M}_t\bar{X}_t + \tilde{q}_t, \\ \check{r}_t(\bar{v}) &= \frac{1}{2}N_t\bar{v}_t + r_t, \quad \check{\tilde{r}}_t(\bar{v}) = \frac{1}{2}\tilde{N}_t\bar{v}_t + \tilde{r}_t.\end{aligned} \tag{16}$$

The additional six terms in the cost functionals F and G (see (12)), namely, $\bar{Q}_t\bar{X}_t\cdot\bar{X}_t, \bar{R}_t\bar{v}_t\cdot\bar{v}_t, 2\bar{S}_t\bar{X}_t\cdot\bar{v}_t, 2\bar{q}_t\cdot\bar{X}_t, 2\bar{r}_t\cdot\bar{v}_t, \bar{H}\bar{X}_T\cdot\bar{X}_T$, are not affected by the individual player and hence do not affect the optimal control. by the results in [18, Theorem 2.6], the optimal control is given by

$$\begin{aligned}\hat{v}_t(\bar{X},\bar{v}) &= -\Sigma_0^{-1}(t)\Lambda_0(t)(X_t-\tilde{X}_t) \\ &\quad - \Sigma_1^{-1}(t)\big(\Lambda_1(t)\tilde{X}_t + \check{r}_t(\bar{v}) + \check{\tilde{r}}_t(\bar{v}) + B_t^\top\phi_t(\bar{X},\bar{v})\big),\end{aligned}$$

where

$$\begin{aligned}\phi_t(\bar{X},\bar{v}) = \int_0^t \Big\{&(\Pi_sB_s + C_s^\top P_sD_s + S_s^\top + \tilde{S}_s^\top)\Sigma_1^{-1}(s) \\ &\big(\check{r}_s(\bar{v}) + \check{\tilde{r}}_s(\bar{v})\big) + \check{q}_s(\bar{X}) + \check{\tilde{q}}_s(\bar{X})\Big\}ds,\end{aligned}$$

and P,Π are the unique solutions of the equations (13), (14) respectively, which is guaranteed by [18, Theorem 2.6], under Assumption 1.

Step 2: Applying the consistency condition at equilibrium. From Step 1, we know that under a given population mean field flow $(\bar{X},\bar{v})$, the flow of the mean of the optimal controlled inter-group state $\tilde{X}$ satisfies the ODE

$$\begin{cases}\dot{\tilde{X}}_t = \big(A_t - B_t\Sigma_1^{-1}(t)\Lambda_1(t)\big)\tilde{X}_t \\ \quad -B_t\Sigma_1^{-1}(t)\big(\check{r}_t(\bar{v}_t) + \check{\tilde{r}}_t(\bar{v}_t) + B_t^\top\phi_t\big), \\ \tilde{X}_0 = \int_{\mathbb{R}^d} xm_0(x)dx,\end{cases} \tag{17}$$

and the mean process of the optimal control is

$$\begin{aligned}\tilde{v} = \mathbb{E}[\hat{v}_t(\bar{X},\bar{v})] = -\Sigma_1^{-1}(t)\big(&\Lambda_1(t)\tilde{X}_t + \frac{1}{2}(N_t+\tilde{N}_t)\bar{v}_t \\ &+ r_t + \tilde{r}_t + B_t^\top\phi_t(\bar{X},\bar{v})\big)\end{aligned} \tag{18}$$

We use $(z_t)_{t\in[0,T]}, (a_t)_{t\in[0,T]}$ to represent the mean processes of the state and of the control in equilibrium, respectively. The consistency condition requires that for any $t\in[0,T]$,

$$\tilde{X}_t = \bar{X}_t = z_t, \qquad \tilde{v}_t = \bar{v}_t = a_t.$$

Hence, along with (16), (17) and (18), we obtain the coupled equation system (15). The associated equilibrium control $\hat{v}$ is

$$\hat{v}_t = -\Sigma_0^{-1}(t)\Lambda_0(t)(X_t - z_t) - a_t,$$

which concludes the proof. □

V. Model and Numerical experiments

We construct an MFCG through parameterization by $\lambda_1,\lambda_2\in[0,1]$ that allows us to recover MFG and MFC in extreme cases of the parameters. One way to do this is to consider a cost functional built around the following term:

$$\begin{aligned}&\frac{1}{2}\Big(X_t - (1-\lambda_1)\bar{X}_t - \lambda_1\tilde{X}_t\Big)^2 \\ &\qquad + \frac{1}{2}\Big(\alpha_t - (1-\lambda_2)\bar{\alpha}_t - \lambda_2\tilde{\alpha}_t\Big)^2\end{aligned}$$

This motivating form is consistent with the general LQ framework along the equilibrium path, where the consistency condition $\tilde{X}_t = \bar{X}_t = z_t$ allows the cross terms involving $X_t\tilde{X}_t$ and $\alpha_t\tilde{\alpha}_t$ to be absorbed into the M_t and N_t coefficients respectively. With $\lambda_1=\lambda_2=0$, the individual tracks the global state and control means $\bar{X}_t$ and $\bar{\alpha}_t$, treating them as fixed external benchmarks. This is the pure MFG assumption, where each agent takes the aggregate behavior of the entire population as given and outside of their control. Conversely, with $\lambda_1=\lambda_2=1$, the individual tracks the group-level state and control means $\tilde{X}_t$ and $\tilde{\alpha}_t$. This is the pure MFC assumption, where a planner accounts for the fact that coordinating group behavior shifts the benchmark itself. Choosing these extreme pairs collapses the three-level structure to the standard two-level structure required for purely MFG or MFC. There also exist asymmetric mixed states that can capture more complex behavior:

- Cooperative Output, Competitive Effort ($\lambda_1=1,\lambda_2=0$): Agents internalize their contribution to the group's collective state (e.g., firm output) but treat the global effort norm as a fixed external pressure. This can trigger a tragedy of the commons or an "internal rat race," where workers exert excessive, uncoordinated effort to hit coordinated production targets.
- Competitive Output, Cooperative Effort ($\lambda_1=0,\lambda_2=1$): Agents compete against the global population on their state (e.g., market share or individual wealth) but internalize the cost of their effort relative to the group. This can lead to "collusive slacking," where a group collectively throttles effort to minimize deviation penalties, resulting in the entire group underperforming against the broader competitive market.

For interior values $\lambda_1,\lambda_2\in(0,1)$, both pressures coexist. The agent partially benchmarks against the global norm and partially against the group norm across both dimensions. Furthermore, holding one parameter fixed at an extreme (for instance, setting $\lambda_1=1$ to assume fully internalized group output) while allowing $\lambda_2\in(0,1)$ to vary, isolates the marginal impact of external labor market competition on internal firm dynamics.

Note that this setting is different from the setting of interpolation between MFC and MFG in [12], where the mixture is done directly at the level of the costs. It is closer to the mixed individual model in [13], where the mixture between individual and population means appears inside the cost functional.

A. Numerical Experiment 1: A Verification Example

Take $d = 1$. We consider the dynamics (10) with $A_t = D_t = 0, B_t = C_t = 1$.

$$dX_t = \alpha_t dt + X_t dW_t,$$

To build the running cost, we first apply a standard quadratic penalty to the control by setting $R_t = 1$, while zeroing out all other control-dependent and cross-term coefficients ($N_t, \tilde{N}_t, S_t, \tilde{S}_t, \bar{S}_t, r_t, \tilde{r}_t, \bar{r}_t = 0$). Next, to generate the interpolated tracking penalty for the state, we set the state-dependent coefficients to $Q_t = \frac{1}{2}$, $\bar{Q}_t = \frac{1}{2}(1-\lambda)^2$, $\tilde{Q}_t = \frac{1}{2}\lambda^2 - \lambda$, $M_t = -(1-\lambda)$, and $\tilde{M}_t = \lambda(1-\lambda)$. The remaining state coefficients are set to zero ($q_t, \tilde{q}_t, \bar{q}_t = 0$). Because the tracking penalty is evaluated continuously over time, the terminal cost is entirely zero ($H, \tilde{H}, \bar{H} = 0$). The resulting cost functional (11) for this example simplifies to:

$$J(\alpha; \bar{X}, \bar{\alpha}) = \mathbb{E}\left\{\int_0^T \left[\alpha_t^2 + \frac{1}{2}\left(X_t - (1-\lambda)\bar{X}_t - \lambda\tilde{X}_t\right)^2\right] dt\right\}.$$

Applying these simplified coefficients to the general ODE system (13), (14), and (15), the auxiliary variables collapse to $\Lambda_0(t) = P_t$, $\Lambda_1(t) = \Pi_t$, and $\Sigma_0(t) = \Sigma_1(t) = 1$. The Riccati equations reduce to the following scalar ODEs: $\dot{P}_t + P_t + \frac{1}{2} - P_t^2 = 0$, $P_T = 0$, and $\dot{\Pi}_t + P_t + \frac{1}{2}(1-\lambda)^2 - \Pi_t^2 = 0$, $\Pi_T = 0$.

For the consistency condition (15), we evaluate the integrand for ϕ_t. Because $N_s, \tilde{N}_s, r_s, \tilde{r}_s, q_s, \tilde{q}_s = 0$, the integrand simplifies to $\frac{1}{2}(M_s + \tilde{M}_s)z_s$. Substituting our chosen coefficients $M_s = -(1-\lambda)$ and $\tilde{M}_s = \lambda(1-\lambda)$ yields:

$$\frac{1}{2}\big(-(1-\lambda) + \lambda(1-\lambda)\big)z_s = -\frac{1}{2}(1-\lambda)^2 z_s.$$

By differentiating the integral form of ϕ_t, we can express the coupled equilibrium system (15) as the following:

$$\begin{cases} \dot{z}_t = -\Pi_t z_t - \phi_t, \quad z_0 = X_0, \\ \dot{\phi}_t = -\frac{1}{2}(1-\lambda)^2 z_t, \quad \phi_0 = 0, \\ a_t = -\Pi_t z_t - \phi_t. \end{cases} \tag{19}$$

We set $T = 1$ and $X_0 = 1$ and solve the system (19) for five representative values $\lambda \in \{0, 0.25, 0.5, 0.75, 1\}$. The mean state z_t (Fig. 1(a)) starts at $X_0 = 1$ for all λ. Since $H = \tilde{H} = 0$, there is no terminal cost penalizing the final state, so agents have no incentive to drive z_T toward any particular value. The trajectories are therefore free to evolve purely under the running cost, and the most significant differences across values of λ appear in the mid-horizon regime. At $\lambda = 0$ (pure MFG), agents treat the population mean $\bar{X}_t$ as a fixed external benchmark and react to deviations from it, driving the mean state lower before recovering near T. As λ increases toward 1 (pure MFC), the group planner internalizes the effect of its own control on the benchmark, which reduces the incentive to correct the state. As a result, the mean state dips less from its initial value, with the $\lambda = 1$ curve remaining closest to $X_0 = 1$ throughout the horizon.

The mean control a_t (Fig. 1(b)) exhibits the same qualitative ordering. The MFG regime applies the most negative control early — a strong corrective push to reduce the state — followed by a large positive surge near T. This "overreaction" is a hallmark of non-cooperative behavior: each agent ignores the externality its control imposes on the group benchmark. The MFC regime, by contrast, produces a nearly flat control profile, reflecting the social planner's ability to coordinate effort smoothly across time.

Finally, the Price of Anarchy J_{MFCG}/J_{MFC} (Fig. 1(c)) decreases monotonically from approximately 2.25 at $\lambda = 0$ to 1 at $\lambda = 1$, with the curve being convex in λ. The ratio equals 1 at $\lambda = 1$ by construction, since $\lambda = 1$ corresponds to the cooperative optimum. The large value at $\lambda = 0$ quantifies the efficiency loss from non-cooperative behavior: the equilibrium aggregate cost is more than twice the socially optimal cost in the pure MFG regime.

B. Numerical Experiment 2: A "Two-Pillar" Corporate Rat Race

We propose a model of career concerns in a large labor market where workers are evaluated simultaneously on two performance dimensions: their *effort* relative to peers, and their *accumulated output* relative to peers. We consider a continuum of workers partitioned across multiple distinct firms, each accumulating human capital through effort while facing social pressure on both dimensions.

This model has three distinct levels of interaction:

- **Individual:** Representative worker's output $X_t \in \mathbb{R}$ and effort $\alpha_t \in \mathbb{R}$.
- **Group (Firm Mean):** The within-firm means $\tilde{\alpha}_t^\gamma = \mathbb{E}[\alpha_t^\gamma]$ and $\tilde{X}_t^\gamma = \mathbb{E}[X_t^\gamma]$.
- **Global (Economy Mean):** The economy-wide means $\bar{\alpha}_t$ and $\bar{X}_t$.

The representative worker's output evolves according to

$$dX_t = \alpha_t\, dt + X_t\, dW_t, \qquad X_0 = x_0,$$

which is the same as in the first example.

We define two independently parameterized interpolated benchmarks. The *output benchmark* is controlled by $\lambda_1 \in [0,1]$:

$$\mu_t^{\lambda_1} := (1-\lambda_1)\bar{X}_t + \lambda_1\tilde{X}_t^\gamma,$$

and the *effort benchmark* is controlled by $\lambda_2 \in [0,1]$:

$$\nu_t^{\lambda_2} := (1-\lambda_2)\bar{\alpha}_t + \lambda_2\tilde{\alpha}_t^\gamma.$$

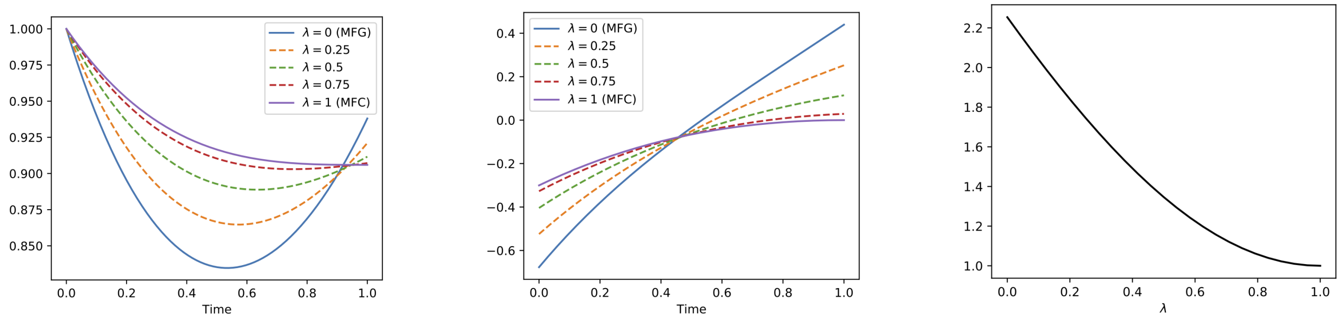
At $\lambda_1 = \lambda_2 = 0$, both benchmarks track the economy-wide averages (pure MFG); at $\lambda_1 = \lambda_2 = 1$, they track the firm's internal averages (pure MFC). The two parameters can be set independently, enabling the asymmetric regimes mentioned earlier.

The representative worker minimizes:

$$J(\alpha; \tilde{\alpha}^\gamma, \bar{\alpha}, \tilde{X}^\gamma, \bar{X}) = \mathbb{E}\left\{\int_0^T f(X_t, \alpha_t, \tilde{\alpha}_t^\gamma, \bar{\alpha}_t, \tilde{X}_t^\gamma, \bar{X}_t)\, dt\right\},$$

with $g \equiv 0$. The running cost takes the form:

$$f(x, \alpha, \tilde{\alpha}^\gamma, \bar{\alpha}, \tilde{X}^\gamma, \bar{X}) = \frac{C}{2}\alpha^2 + \frac{K_\alpha}{2}(\alpha - \nu_t^{\lambda_2})^2 + \frac{K_x}{2}(x - \mu_t^{\lambda_1})^2.$$



(a) Mean state z_t. (b) Mean control a_t. (c) Price of Anarchy J_{MFCG}/J_{MFC}.

Fig. 1: Numerical Experiment 1: Mean state z_t, mean control a_t, and Price of Anarchy as functions of $\lambda \in [0,1]$, interpolating between MFG ($\lambda = 0$) and MFC ($\lambda = 1$).

The three terms capture: physical effort cost ($\frac{C}{2}\alpha^2$); a "hustle" penalty for deviating from the effort norm ($\frac{K_\alpha}{2}(\alpha - \nu_t^{\lambda_2})^2$, controlled by λ_2); and a "KPI" penalty for falling behind peers' accumulated output ($\frac{K_x}{2}(x-\mu_t^{\lambda_1})^2$, controlled by λ_1). The model is governed by the following parameters:

- $C > 0$: quadratic effort cost.
- $K_\alpha > 0$: penalty for deviating from the effort norm $\nu_t^{\lambda_2}$.
- $K_x > 0$: penalty for falling behind the output norm $\mu_t^{\lambda_1}$.
- $\lambda_1, \lambda_2 \in [0,1]$: independent interpolation parameters for the output and effort benchmarks respectively.

This can be recovered from the general cost functional by taking the state-dependent coefficients as:

$$Q_t = \tfrac{K_x}{2}, \quad \tilde{Q}_t = K_x\left(\tfrac{\lambda_1^2}{2} - \lambda_1\right), \quad \bar{Q}_t = \tfrac{K_x(1-\lambda_1)^2}{2},$$
$$M_t = -K_x(1-\lambda_1), \quad \tilde{M}_t = K_x\lambda_1(1-\lambda_1), \quad q_t = \tilde{q}_t = 0,$$

and the control-dependent coefficients as:

$$R_t = \tfrac{C+K_\alpha}{2}, \quad \tilde{R}_t = K_\alpha\left(\tfrac{\lambda_2^2}{2} - \lambda_2\right), \quad \bar{R}_t = \tfrac{K_\alpha(1-\lambda_2)^2}{2},$$
$$N_t = -K_\alpha(1-\lambda_2), \quad \tilde{N}_t = K_\alpha\lambda_2(1-\lambda_2),$$

with all remaining coefficients zero. The auxiliary variables collapse to $\Lambda_0(t) = P_t$, $\Lambda_1(t) = \Pi_t$, $\Sigma_0(t) = \frac{C+K_\alpha}{2}$, and $\Sigma_1(t) = \frac{C+K_\alpha(1-\lambda_2)^2}{2}$. The Riccati equations (13) and (14) reduce to: $\dot{P}_t + P_t + \dfrac{K_x}{2} - \dfrac{2P_t^2}{C+K_\alpha} = 0$, $P_T = 0$, and $\dot{\Pi}_t + P_t + \dfrac{K_x(1-\lambda_1)^2}{2} - \dfrac{2\Pi_t^2}{C+K_\alpha(1-\lambda_2)^2} = 0$, $\Pi_T = 0$. The consistency condition (15) reduces to:

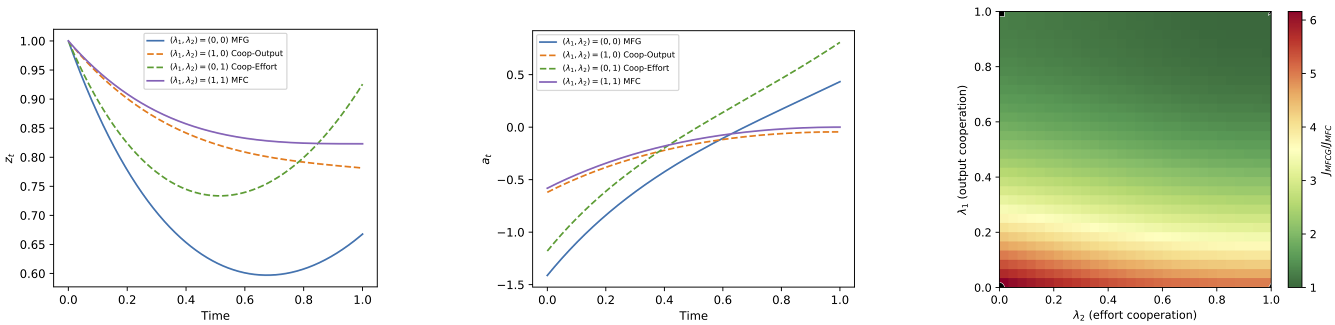
$$\begin{cases} \dot{z}_t = a_t, \quad z_0 = x_0, \\ \dot{\phi}_t = -\dfrac{K_\alpha(1-\lambda_2)^2\Pi_t}{C+K_\alpha(1-\lambda_2)^2}\, a_t \\ \qquad - \dfrac{K_x(1-\lambda_1)^2}{2}\, z_t, \quad \phi_0 = 0, \\ a_t = -\dfrac{2}{C}\big(\Pi_t z_t + \phi_t\big). \end{cases} \tag{20}$$

Notice that $\dot{z}_t = a_t$ follows directly from $A_t = 0$ and $B_t = 1$. Since the benchmark deviation penalties vanish along the symmetric equilibrium path ($\tilde{\alpha}_t = \bar{\alpha}_t = a_t$, $\tilde{X}_t = \bar{X}_t = z_t$), the equilibrium cost reduces to $J(\lambda_1,\lambda_2) = \int_0^T \frac{C}{2} a_t^2\, dt \geq 0$. The MFC optimum achieves the minimum of this cost, so the Price of Anarchy $\mathrm{PoA}(\lambda_1,\lambda_2) = J(\lambda_1,\lambda_2)/J(1,1) \geq 1$ is well-defined.

We set $T = 1$, $X_0 = 1$, $C = 2$, $K_\alpha = 3$, $K_x = 2$ and solve the system (20) for the four corner regimes $(\lambda_1,\lambda_2) \in \{(0,0),(1,0),(0,1),(1,1)\}$. The results are shown in Fig. 2. The mean state z_t (Fig. 2(a)) starts at $X_0 = 1$ for all regimes and declines throughout the horizon, since without an exogenous reward for building output, workers minimize conformity costs only. The MFG regime $(0,0)$ suffers the steepest decline, while MFC $(1,1)$ achieves the shallowest decline. The Coop-Output regime $(1,0)$ closely tracks MFC, confirming that output cooperation is the primary driver of efficiency. The Coop-Effort regime $(0,1)$ exhibits a non-monotone trajectory: workers collectively reduce output early in the horizon before a sharp late-horizon recovery. This is the "collusive slacking" effect described earlier — when workers internalize the effort norm but compete on output, they collectively suppress effort at first, then scramble to close the output gap near T.

With the mean control a_t (Fig. 2(b)), all curves start negative (workers reduce effort to conform to a declining norm) and rise toward zero at T, consistent with zero terminal cost. The MFG regime applies the most negative initial control and never fully recovers, reflecting non-cooperative overreaction to the global norm. MFC produces the flattest and least wasteful control profile.

The Price of Anarchy heatmap (Fig. 2(c)) illustrates the asymmetry between the values $\lambda_1, \lambda)2$. The PoA ranges from 1.0 at MFC $(1,1)$ to arboud 6 at MFG $(0,0)$. Moving from $(0,0)$ to $(1,0)$ by activating output cooperation alone collapses the PoA to 1.25, while activating effort cooperation alone $(0,1)$ only reduces it to 4.95. This asymmetry arises because the "KPI" penalty K_x is attached to the *state* X_t, which accumulates over the entire horizon $[0,T]$, compounding the competitive externality over a worker's career. The "hustle" penalty K_α, by contrast, is purely instantaneous. Consequently, output cooperation yields far greater efficiency gains than effort cooperation, even though $K_\alpha > K_x$ in this experiment. This is made possible by the two-parameter (λ_1,λ_2) structure of the extended MFCG framework.



(a) Mean state z_t. (b) Mean control a_t. (c) Price of Anarchy J_{MFCG}/J_{MFC}.

Fig. 2: Numerical Experiment 2: Mean state z_t, mean control a_t, and Price of Anarchy over the (λ_1, λ_2) parameter space for the four corner regimes: MFG $(0, 0)$, Coop-Output $(1, 0)$, Coop-Effort $(0, 1)$, and MFC $(1, 1)$. Parameters: $C = 2$, $K_\alpha = 3$, $K_x = 2$, $T = 1$, $X_0 = 1$.

VI. Concluding Remarks

In this paper, we formulate a framework for extended MFCG problems, which provides a unified model to study systems where cooperative groups interact in a non-cooperative environment, featuring interactions through the joint distribution of states and actions. To characterize the equilibrium, we derive the associated HJB-KFP system. Focusing on an LQ model, we provide optimality conditions through ODEs. We complement this study with numerical experiments. We note that our current LQ setting does not yet represent the most exhaustive case; therefore, future work will involve the study of more expansive LQ models from both theoretical and numerical perspectives. We will also study other models, beyond the LQ setting. In particular, we intend to numerically solve the PDE system (9). Furthermore, the incorporation of common noise and hierarchical network structures within and between groups remains a significant area for exploration. These directions are reserved for future research.

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