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1 Fixed Point Iteration

1.1 What it is and Motivation

Consider some function \( g(x) \) (we are almost always interested in continuous functions in this class). Define a fixed point of \( g(x) \) to be some value \( p \) such that \( g(p) = p \). Say we want to find a fixed point of a given \( g(x) \). One obvious thing to do is to try fixed point iteration. Pick some starting value \( x_0 \), and continue to iterate

\[
x_{n+1} = g(x_n)
\]


... until (hopefully) this eventually converges to a fixed point.

This is all fine and good, but you should find this very strange. Sure, this is a very simple and straightforward idea, and it’s great if it works. But you should have no reason to expect that this would work in general!! The following example is meant to illustrate this.

**Example 1** (From BFB p60) Consider the equation \( f(x) = x^3 + 4x^2 - 10 \). By the Intermediate Value Theorem there exists a root in the interval \([1, 2]\). There are many ways to change the equation \( f(x) = 0 \) to a fixed point iteration of the form \( x = g(x) \).

Here are 5 such examples (which you can verify are equivalent to \( f(x) = 0 \) yourself with algebra):

a) \( x = g_1(x) = x - x^3 - 4x^2 + 10 \)

b) \( x = g_2(x) = \left( \frac{10}{x} - 4x \right)^{1/2} \)

c) \( x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2} \)

d) \( x = g_4(x) = (\frac{10}{x+x})^{1/2} \)

e) \( x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \)

Take \( p_0 = 1.5 \), Table 4 lists the results for each of the fixed point iterations above. If one of these fixed point iterations \( g_i(x) \) converges to a fixed point, it must (by design) be a root of \( f(x) \).

Note that these all behave very differently. This should convince you that finding a “good” fixed point iteration is no easy task. An important practical note for this class is that we won’t worry about the problem of finding a good fixed point function. We will always take them as given and analyze them.

**Comment:** Note that in these five examples, we changed a rootfinding problem \( f(x) = 0 \) to a fixed point iteration \( x_{n+1} = g(x_n) \) by doing algebra on \( f(x) = 0 \). Newton’s method is also a fixed point iteration of the form \( x_{n+1} = g(x_n) \), where \( g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \). But we didn’t get this fixed point iteration by algebra like the 5 in the example, we got it from the idea “linearize and solve exactly”.

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### Table 1: Fixed Point Iteration

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<th>Iter (n)</th>
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<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
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### 1.2 Two Main Theorems

Say somebody gives you a fixed point iteration and a starting point, and you begin iterating. The first thing you would hope for is that your iteration will converge (question of existence). The next thing you would wonder is what value does this iteration converge to (question of uniqueness)? Does it converge to a unique fixed point, or will different starting values converge to different fixed points?

#### 1.2.1 Existence

**Theorem 1** If \( g(x) \) is a continuous function that defines our fixed point iteration and there exists an interval \([a, b]\) such that \( x \in [a, b] \implies g(x) \in [a, b] \), then \( g(x) \) has at least one fixed point in \([a, b]\).

**Proof:** We know that \( g(a) > a \) and \( g(b) < b \). Let’s consider the new function \( h(x) := g(x) - x \), which is continuous. Since \( h(a) = g(a) - a > 0 \) and \( h(b) = g(b) - b < 0 \), by the Intermediate Value Theorem there exists some point \( p \in [a, b] \) such that \( h(p) = 0 \), or equivalently \( g(p) = p \).

#### 1.2.2 Uniqueness

**Theorem 2** If \( g'(x) \) exists on the interval \((a, b)\) and there exists some \( k < 1 \) such that \( |g'(x)| \leq k \ \forall x \in (a, b) \), then there is exactly one fixed point in \([a, b]\).

**Proof:** We proceed with the standard method for proving uniqueness by assuming two such fixed points exist and reach a contradiction. Suppose \( p, q \in [a, b] \) are fixed points of
\( g(x) \). Then by the Mean Value Theorem, there exists some \( \xi \in (p, q) \subset [a, b] \) such that

\[
g'(\xi) = \frac{g(p) - g(q)}{p - q}
\]

Thus

\[
|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \leq k|p - q| < |p - q|
\]

So we have concluded that \( |p - q| < |p - q| \) with strict inequality, and this is absurd. So our assumption must be false and there is a unique fixed point in the interval.

**Warning:** We need to show existence first before we can even begin to discuss uniqueness. What this means practically is that if you find some \( k < 1 \) such that \( |g'(x)| \leq k < 1 \), you cannot conclude this iteration has a unique fixed point based off this alone. You need this in combination with existence.

**Comment:** I want to note that we proved both parts of the main theorem of this section with very basic machinery (Intermediate Value Theorem and Mean Value Theorem), which is why I repeat them here. In the course there will be some proofs that are going to require a lot of ingenuity which you should just try to follow and appreciate, but there is no way you would have discovered them yourself. But with these, I really want you to feel like with a deep understanding of the problem and Calculus, you could have come up with these.

### 1.3 Examples

**Example 2** Consider the iteration

\[
p_n = \frac{p_{n-1}^2 + 3}{5}, \quad n = 1, 2, 3...
\]

a) Show this fixed point iteration converges to a unique point for any initial \( p_0 \in [0, 1] \).

b) What value \( p \) does this iteration converge to?

c) Estimate how many iterations \( n \) are required to obtain an absolute error \( |p_n - p| \) less than \( 10^{-4} \) where \( p_0 = 1 \). (No numerical value needed, just give an expression for \( n \)).

**Solution:**

a) Let \( g(x) \equiv \frac{x^2 + 3}{5} \). We want to show \( g(x) \in [0, 1] \) for \( x \in [0, 1] \). Since \( g(0) = \frac{3}{5} \), \( g(1) = \frac{4}{5} \), and \( g \) is an increasing function, this shows the existence of a fixed point. (There are other ways to show existence, I’m just showing you something different). We also want to show that \( |g'(x)| \leq k < 1 \) for some \( k \). Since \( g'(x) = \frac{2x}{5} \leq \frac{2}{5} = k < 1 \), this shows that the fixed point is unique.

b) Let \( \lim_{n \to \infty} p_n = p \). Taking limits on both sides of the iteration:

\[
\lim_{n \to \infty} p_n = \lim_{n \to \infty} \frac{p_{n-1}^2 + 3}{5} \implies p = \frac{p^2 + 3}{5}.
\]

Solve this equation with the quadratic formula to get \( p = \frac{5 \pm \sqrt{13}}{2} \), and discard the root not in the interval \([0, 1]\). So this iteration converges to \( p = \frac{5 - \sqrt{13}}{2} \).
Example 3 Consider the iteration
\[ p_n = 2 + \frac{1}{p_{n-1}}, \quad n = 1, 2, 3... \quad p_0 = 2 \]

a) Show that \( p_n \in [2, 2.5] \) \( \forall n \) and that this iteration has a unique fixed point.

b) What value \( p \) does this iteration converge to?

Solution:

a) Since \( p_n \geq 0 \) \( \forall n \), this implies that \( \frac{1}{p_n} \geq 0 \implies 2 + \frac{1}{p_n} \geq 2. \) For the other side, since \( p_n \geq 2, \) this implies that \( \frac{1}{p_n} \leq \frac{1}{2} \implies 2 + \frac{1}{p_n} \leq 2.5. \) (Again, there are other ways to show existence, but here is yet another approach that works sometimes).

Define \( g(x) \equiv 2 + \frac{1}{x}, \max_{[2,2.5]} |g'(x)| = \max_{[2,2.5]} \frac{1}{x^2} \leq \frac{1}{4} = k < 1. \)

b) Take limits and then \( p = 1 \pm \sqrt{2}. \) Discard the one that’s negative so \( p = 1 + \sqrt{2}. \)

Example 4 Let \( g(x) = \frac{1}{x} + \frac{x}{4}. \)

a) Prove that \( g(x) \) has a unique fixed point \( p \in [1, 2]. \)

b) Let \( p_0 = \frac{3}{2}, p_{n+1} = g(p_n). \) How many iterations are required to guarantee \( |p_n - p| \leq 10^{-7} ? \) (Write your answer as a ratio of logarithms).

Solution:

a) \( g'(x) = -\frac{1}{x^2} + \frac{1}{4} = 0 \) when \( x = \pm 2. \) So the only critical point in \([1, 2]\) is an endpoint. \( g(1) = 1 + \frac{1}{4} = \frac{5}{4} \) (the max), and \( g(2) = \frac{1}{2} + \frac{1}{2} = 1 \) (the min). \( g \) maps \([1, 2] \) to \([1, 2]\) since both extrema are in this range.

\( g''(x) = \frac{2}{x^3} > 0, \) so \( g'(x) \) has its extreme at an endpoint. \( g'(1) = -1 + \frac{1}{4} = -\frac{3}{4}, g'(2) = 0 \implies -\frac{3}{4} \leq g'(x) \leq 0. \) So \( |g'(x)| \leq \frac{3}{4} \) for \( x \in [1, 2]. \)

b) \( |p_n - p| \leq k^n \max(p_0 - a, b - p_0) = \left(\frac{3}{4}\right)^n \left(\frac{1}{2}\right) \leq 10^{-7}. \) This implies that \( n \geq \frac{\ln(2 \times 10^{-7})}{\ln(3/4)} \)

Comment: These are really all the same problem, but note that you can take different approaches on different \( g(x). \)

Example 5 (Pretty Hard) Let \( g(x) = x - \tan(x) \) and consider the fixed point iteration \( p_{n+1} = g(p_n). \) Show that if \( |p_0| \leq \frac{\pi}{6}, \) then \( p_n \to 0 \) at least quadratically. (Useful Hints: \( \frac{d}{dx} \tan(x) = \sec^2(x), 1 \leq \sec^2(x) \leq \frac{4}{3} \) for \( |x| \leq \frac{\pi}{6}, \) and that \( \tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}} < \frac{\pi}{6}. \)
Solution: Since \( g(x) = x - \tan(x) \) and \( g'(x) = 1 - \sec^2(x) \), we can show that \( 0 \leq \sec^2(x) - 1 \leq \frac{1}{3} \) for \(-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}\). This tells us that \( 0 \geq g'(x) \geq -\frac{1}{3} \), which implies \( g(x) \) is decreasing on \(-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}\). This gives us that \( |g'(x)| \leq \frac{1}{3} < 1 \).

Compute \( g(-\frac{\pi}{6}) = -\frac{\pi}{6} - \tan(-\frac{\pi}{6}) = -\frac{\pi}{6} + \frac{1}{\sqrt{3}} < \frac{\pi}{6} \). Then compute \( g(\frac{\pi}{6}) = \frac{\pi}{6} - \tan(\frac{\pi}{6}) = \frac{\pi}{6} - \frac{1}{\sqrt{3}} > -\frac{\pi}{6} \). Since \( g(x) \) is decreasing, \( g(x) \) maps \([-\frac{\pi}{6}, \frac{\pi}{6}]\) to itself.

So we have shown that if \( |p_0| \leq \frac{\pi}{6}, p_n \to 0 \).

As for the quadratically part, we have that \( g(0) = 0 \) and \( g'(0) = 0 \). This implies the convergence is at least quadratic. (See the first slide of the Week 4 Lecture notes for this theorem and the proof that connects this with the definition of order of convergence).