POLYNOMIALLY EFFECTIVE EQUIDISTRIBUTION FOR UNIPOTENT ORBITS IN PRODUCTS OF SL_2 FACTORS

E. LINDENSTRAUSS, A. MOHAMMADI, AND L. YANG

ABSTRACT. We sketch the proof of an effective equidistribution theorem for one-parameter unipotent subgroups in S-arithmetic quotients arising from $\mathbf{K}$-forms of $\mathrm{SL}_2^{\mathsf{n}}$ where $\mathbf{K}$ is a number field. This gives an effective version of equidistribution results of Ratner and Shah with a polynomial rate.

The key new phenomenon is the existence of many intermediate groups between the SL_2 containing our unipotent and the ambient group, which introduces potential local and global obstruction to equidistribution.

Our approach relies on a Bourgain-type projection theorem in the presence of obstructions, together with a careful analysis of these obstructions.

1. Introduction

Let $\mathbf{K}$ be a number field, and let $\mathbf{G}$ be a semisimple $\mathbf{K}$-group satisfying that $\mathbf{G}(\mathbb{C}) \simeq \prod_{i=1}^{\mathsf{n}} \mathrm{SL}_2(\mathbb{C})$. Let S be a finite set of places of $\mathbf{K}$, and let $\mathcal{O}_S$ be the set of S-integers in $\mathbf{K}$. Let

$$G = \mathbf{G}(\mathbf{K}_S),$$

and let Γ be commensurable with $\mathbf{G}(\mathcal{O}_S)$. Put $X = G/\Gamma$, and let m_X denote the probability Haar measure on X.

Let $\mathbf{F}$ be a local field of characteristic zero, and let $\iota : \mathrm{SL}_2(\mathbf{F}) \to G$ be an embedding. Set $H = \iota(\mathrm{SL}_2(\mathbf{F}))$.

Fix a right invariant metric on G. This metric induces a metric d_X on X, and natural volume forms on X and its analytic submanifolds. For every $\eta > 0$, we let X_η denote the set of points $x \in X$ where the injectivity radius at x is at least η.

If $\mathbf{F} = \mathbb{R}, \mathbb{C}$, we set $\varpi_{\mathbf{F}} = e^{-1}$; otherwise, fix a uniformizer $\varpi_{\mathbf{F}} \in \mathbf{F}$. Let us put $|\varpi_{\mathbf{F}}| = 1/\mathsf{q}$. For all $t \in \mathbb{N}$ and $r \in \mathbf{F}$, let a_t and u_r be the images of

$$\begin{pmatrix} \varpi_{\mathbf{F}}^{-t} & 0 \\ 0 & \varpi_{\mathbf{F}}^{t} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

in H, respectively.

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1.1. **Theorem.** *For every $x_0 \in X$ and large enough R (depending logarithmically on the injectivity radius at x_0), for any $T \geq A_1 R$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$, we have
$$\Big|\int_{B_1^{\mathbf{F}}} \varphi(a_T u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi)\mathsf{q}^{-\kappa_1 R}$$
where $B_1^{\mathbf{F}} = \{r \in \mathbf{F} : |r| \leq 1\}$ and $\mathcal{S}(\varphi)$ is a certain Sobolev norm.

(2) There exists $x \in X$ and a subgroup $H \leq M \leq G$ such that Mx is periodic with $\mathrm{vol}(Mx) \leq \mathsf{q}^R$, and
$$d_X(x, x_0) \leq T^{A_1}\mathsf{q}^{-2T+A_1 R}$$

The constants A_1 and κ_1 are positive, and depend on X but not on x_0.

The general strategy of the proof is similar in spirit to that developed in recent works by the authors, including joint works with Zhiren Wang; see [LMWY25] and the references therein. The argument proceeds in three main phases. First, we obtain a small but positive initial dimension via an effective closing lemma. In the second phase, we use a certain projection theorem to improve this dimension to nearly full dimension. Finally, in the third phase, we exploit the spectral gap on the ambient space to upgrade the nearly full dimensional result to equidistribution, in the spirit of Venkatesh [Ven10].

The key difference between the present work and [LMWY25] is that we are able to address possible obstructions in the second phase. These obstructions a priori arise as subrepresentations of H in $\mathrm{Lie}(G)$ and a posteriori as intermediate subgroups as in part (2) of Theorem 1.1. This is accomplished through two key new ingredients. The first is a Bourgain-type projection theorem (Theorem 2.1), which is established and systematically exploited in this paper. The second is an intricate analysis of the resulting obstructions: by assembling the local obstructions furnished by Theorem 2.1, we construct a global obstruction. The argument at this stage is reminiscent of Ratner's proof of her measure classification theorem for quotients of semisimple groups in [Rat90], and can also be viewed as related to the use of entropy in the proof by Margulis and Tomanov of this measure classification result in [MT94].

In this paper, we give a brief outline of the main steps of the proof; full details will appear elsewhere.

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2. A Bourgain-type projection theorem

As mentioned above, one of the main ingredients in this work is a projection theorem. In this section, we discuss this theorem, which is of independent interest. This projection result is in the spirit of Bourgain's projection theorem [Bou10]. Other important contributions in this direction can be found e.g. in Shmerkin's paper [Shm20] and in the works [He19, He20] by Weikun He. Another useful reference in this context is the paper [SG20] by Salehi-Golsefidy which has a discretized sum-product result over $\mathbb{Q}_p$; in this paper we use sum-product like results in both the archimedean and non-archimedean cases.

The projection like results we need are similar in nature to the results of Bénard and He in [BH24]. The main difference is that we are interested in cases where the assumptions of [BH24] regarding the richness of set of projected directions (e.g. as in [BH24, Thm. 2.1, item (ii)]) are not satisfied (and indeed, in our case there is a possible obstruction to obtaining a dimension increment).

Let $\mathbf{F}$ be a local field of characteristic zero. For every $\delta > 0$ and any $v \in \mathbf{F}^{\mathsf{m}}$, let $B_\delta^{\mathbf{F}^{\mathsf{m}}}(v)$ denote the ball of radius δ centered at v. We denote $B_\delta^{\mathbf{F}^{\mathsf{m}}}(0)$ simply by $B_\delta^{\mathbf{F}^{\mathsf{m}}}$.

Let Φ_0 be a $\mathsf{d}+1$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{F})$. Put $\Phi = \oplus_{i=1}^{\mathsf{m}} \Phi_0$. We will identify Φ with $\Phi_0 \otimes \mathbf{F}^{\mathsf{m}}$, and the action of $\mathrm{SL}_2(\mathbf{F})$ on $\mathrm{Mat}_{(\mathsf{d}+1)\times\mathsf{m}}(\mathbf{F})$ is given by left matrix multiplication.

Let $\{e_0, \ldots, e_{\mathsf{d}}\} \subset \Phi_0$ denote a collection of unit pure weight vectors: $\|e_i\| = 1$ and $a_t e_i = e^{(\mathsf{d}-2i)t} e_i$. Let $\Phi_0^{\mathrm{hw}} = \mathbf{F} \cdot e_0$ denote the line spanned by a highest weight vector in Φ_0, and put $\Phi^{\mathrm{hw}} = \oplus_{i=1}^{\mathsf{m}} \Phi_0^{\mathrm{hw}}$. Let $\pi^+ : \Phi \to \Phi^{\mathrm{hw}}$ denote the projection parallel to $(\mathbf{F}\cdot e_1 \oplus \cdots \oplus \mathbf{F}\cdot e_{\mathsf{d}}) \otimes \mathbf{F}^{\mathsf{m}}$.

By a *box* in $\mathbf{F}^{\mathsf{m}}$, we mean $v + L$ where

$$L = \{\textstyle\sum r_i u_i : r_i \in B_1^{\mathbf{F}}\}$$

where $B_1^{\mathbf{F}} = \{r \in \mathbf{F} : |r| \le 1\}$ and $\{u_0, \ldots, u_k\}$ is an *orthogonal* set.

By a *representation box* in Φ, we mean $v + V$ for

$$v \in \Phi \quad \text{and} \quad V = \left\{\textstyle\sum r_{ij} e_i \otimes u_j : 0 \le i \le \mathsf{d}, 0 \le j \le k, r_{ij} \in B_1^{\mathbf{F}}\right\}$$

where $\{u_0, \ldots, u_k\} \subset \mathbf{F}^{\mathsf{m}}$ is an orthogonal set, and $B_1^{\mathbf{F}} = \{r \in \mathbf{F} : |r| \le 1\}$.

2.1. **Theorem.** *Let $0 < \alpha < \mathsf{m}(\mathsf{d}+1)$ and $0 < \delta < 1$. Let $\Theta \subset B_1^{\Phi}$ satisfy*

$$\mathcal{N}_\delta(\Theta) \ge \delta^{-\alpha}$$

where $\mathcal{N}_\delta(\bullet)$ denote the δ-covering number of $\bullet$. Then at least one of the following holds for all small ϵ and large C.

(1) There exists a subset $\mathsf{B} \subset B_1^{\mathbf{F}}$ with $|B_1^{\mathbf{F}} \setminus \mathsf{B}| \le \delta^{\epsilon}$ so that

$$\mathcal{N}_\delta(\pi^+(u_r\Theta)) \ge \delta^{-\frac{\alpha}{\mathsf{d}+1}-\epsilon} \quad \textit{for all } r \in \mathsf{B}.$$

(2) There is a representation box $v + V$ satisfying

$$\delta^{-\alpha+C\epsilon} \ll \mathcal{N}_\delta(V) \ll \delta^{-\alpha-C\epsilon}$$

$$\mathcal{N}_\delta(\Theta \cap \mathsf{Nhd}_{C\delta}(v+V)) \gg \delta^{-\alpha+C\epsilon}$$

The proof of this theorem is based on Balog–Szemerédi–Gowers theorem, e.g., in the form given by Bourgain in [Bou10, Prop. (∗∗)], and the following metric sum-product theorem.

2.2. **Theorem.** *Let* $\Theta_1, \Theta_2 \subset B_1^{\mathbf{F}^{\mathsf{m}}}$ *be two subsets,* $0 < \hat\alpha < \mathsf{m}$*, and* $0 < \delta < 1$*. Assume that*

$$\mathcal{N}_\delta(\Theta_i) \geq \delta^{-\hat\alpha} \quad \textit{for } i = 1, 2.$$

For every $\epsilon_1 > 0$*, put*

$$\mathsf{Exc}_{\epsilon_1}(\Theta_1, \Theta_2) = \left\{ r \in B_1^{\mathbf{F}} : \mathcal{N}_\delta(\Theta_1 + r\Theta_2) < \delta^{-\hat\alpha - \epsilon_1} \right\}$$

The following holds for all small enough ϵ_1 *and* ϵ_2*. If* $|\mathsf{Exc}_{\epsilon_1}(\Theta_1, \Theta_2)| > \delta^{\epsilon_2}$*, then there are boxes* $v_i + L \subset \mathbf{F}^{\mathsf{m}}$ *so that both of the following hold*

$$\delta^{-\hat\alpha + C\epsilon} \ll \mathcal{N}_\delta(L) \ll \delta^{-\hat\alpha - C\epsilon}$$

$$\mathcal{N}_\delta\big(\Theta_i \cap \mathsf{Nhd}_{C\delta}(v_i + L)\big) \gg \delta^{-\hat\alpha + C\epsilon} \qquad \textit{for } i = 1, 2$$

where $\epsilon = \max(\epsilon_1, \epsilon_2)$ *and* C *depends only on* $\mathbf{F}$ *and* m*.*

We will use Theorem 2.1 with $\Phi = \mathfrak{sl}_2(\mathbf{F}) \otimes \mathbf{F}^{\mathsf{m}}$. That is: when Φ_0 is the 3-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{F})$. Indeed, our argument will also use the following trivial estimate in this case.

Let $\pi^0 : \Phi \to (\mathbf{F}e_0 \oplus \mathbf{F}e_1) \otimes \mathbf{F}^{\mathsf{m}}$ denote the projection onto the space of non-negative weights. If $\mathcal{N}_\delta(\Theta) \geq \delta^{-\alpha}$, then for all $r \in B_1^{\mathbf{F}}$, except a set of measure $\leq \delta^{\star\epsilon}$, we have

$$\mathcal{N}_\delta(\pi^0(u_r\Theta)) \geq \delta^{-\frac{2\alpha}{3} + \epsilon}. \tag{2.1}$$

We emphasize that (2.1), as well as its analogue for π^+ (with exponent $-\alpha/3$), hold for all sets, regardless of the existence of obstructions, see e.g., [BH24, §2].

It is also worth mentioning that in the proof of Theorem 1.1, we actually need a strengthening of Theorems 2.1 and 2.2 for subrings of a product of possibly different local fields $\prod_{i=1}^{\mathsf{n}} \mathbf{F}_i$. These generalizations are proved using similar basic strategy.

3. Proof of Theorem 1.1

In this section, we provide a more detailed outline of the proof of Theorem 1.1. To simplify the discussion, we will focus on the case where $\mathbf{K} = \mathbb{Q}$, $S = \{\infty\}$, and assume that $\mathbf{G}$ is such that

$$G = \mathbf{G}(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

We further set $H = \{(g, g, g) : g \in \mathrm{SL}_2(\mathbb{R})\}$. For all $t, r \in \mathbb{R}$, let a_t and u_r denote the images of

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

in H, respectively.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Then $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{R}^3$ as H-representation, see §2. Moreover, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ where $\mathfrak{r} \simeq \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{R}^2$ is H-invariant. Note that $\mathfrak{r}$ is not an H-irreducible representation. This constitutes a key difference between the setting considered in [LMWY25] and the problem at hand.

3.1. **Initial dimension and a closing lemma.** As it was mentioned, the proof follows the same general steps as in [LMWY25]. Indeed, assuming part (2) in Theorem 1.1 does not hold, we first use an argument relying on Margulis functions for periodic orbits, to show the following. For all $\tau \geq t_1 := T - O(R)$ and all but a set with measure $\ll e^{-\star R}$ of $r \in [0,1]$,

$$d_X(x, a_\tau u_r x_0) \gg e^{-D_0 R} \tag{3.1}$$

for all $x \in X$ so that $\mathrm{vol}(Mx) \leq e^R$ for some $H \leq M \leq G$, where D_0 depends on G and the implied constants depend on X, see [LMWY25, Prop. 4.4].

Then we use the arithmeticity of Γ to show that for any $x_1 = a_{t_1} u_r x_0$ satisfying (3.1) the points in $\{a_{\star R} u_r x_1 : r \in [0,1]\}$ — possibly after removing an exceptional set of measure $e^{-\star R}$ — are separated and in general position with respect to all intermediate subgroups, see [LMWY25, Prop. 4.6]. When H has a centralizer in G this requires also using the techniques of [LMM+24] (cf. also [ELMW25, Prop. 7.1]).

We interpret this separation as a (small) positive dimension at controlled scales; see (3.4) for a more precise formulation.

3.2. **Improving the dimension and obstructions.** The basic strategy in the next, and most involved, phase is to show that we can improve this dimension to nearly full dimension unless there is a global obstruction as in part (2) of Theorem 1.1. A basic tool here is Theorem 2.1.

Let $s \geq t_1$. We first show that if the convolution of the uniform measure on $\{a_s u_r x_0 : r \in [0,1]\}$ with the measure

$$\nu_\ell(\varphi) = \int_0^1 \varphi(a_\ell u_r)\,\mathrm{d}r \qquad \text{for all } \varphi \in C_c(H),$$

for an appropriate choice of ℓ, fails to produce incremental dimension improvement, then the the uniform measure on $\{a_s u_r x_0 : r \in [0,1]\}$ exhibits local, but *multi-scale*, obstructions. Then using the dynamics of the action of a_t, we promote this local obstruction to *global* obstructions along stable and unstable leaves. Finally, we use the structure along stable and unstable leaves to show that a significant part of the measure in near a local orbit of an intermediate group M. This, in view of the aforementioned closing lemma, implies that part (2) in Theorem 1.1 holds unless $M = G$.

In what follows κ denotes a small parameter and C a large parameter, depending on X, whose exact value may differ from one line to another.

We begin by giving a precise definition of a single scale version of the type of obstructions which arise in our analysis.

Definition 3.3. Suppose the parameters $0 < \alpha \leq \dim \mathfrak{g}$, $0 < \epsilon' < 1$, and $\Upsilon \geq 1$, are fixed. Let μ be a measure supported on X, and let $0 < \delta \leq \delta_2 <$

$\delta_1 \leq e^{-\kappa R}$. We say μ is (δ_2, δ_1)-*focused at* $y \in X_{e^{-\kappa R}}$ with parameters α, Υ, ϵ' if there exists a representation box $v + V \subset B^{\mathfrak{g}}_{2\delta_1}$ with

$$\Upsilon^{-1}(\tfrac{\delta_1}{\delta_2})^{\alpha-\epsilon'} \leq \mathcal{N}_{\delta_2}(V) \leq \Upsilon(\tfrac{\delta_1}{\delta_2})^{\alpha+\epsilon'}, \quad \text{and} \tag{3.2a}$$

$$\mu\big(\mathsf{B}^X_{\delta_1}(y) \cap \mathsf{Nhd}_{\mathsf{A}\delta_2}(\exp(v+V).y)\big) \geq \Upsilon^{-1}(\tfrac{\delta_1}{\delta_2})^{-\epsilon'}\mu\big(\mathsf{B}^X_{\delta_1}(y)\big) \tag{3.2b}$$

We say μ is (δ_2, δ_1)-*exactly focused at* y if in addition to (3.2a) and (3.2b), we also have

$$\mu\big(\mathsf{B}^X_{\delta_2}(z)\big) \geq \Upsilon^{-1}(\tfrac{\delta_1}{\delta_2})^{-\epsilon'}\delta_2^{\alpha}, \quad \text{for all } z \in \mathsf{B}^X_{\delta_1}(y) \cap \operatorname{supp}\mu \tag{3.3}$$

Here and in what follows $\mathsf{B}^X_b(y) = \exp(B^{\mathfrak{g}}_b(0))y$. We also remark that ϵ' in Definition 3.3 will be chosen to be a (large) multiple of ϵ in Theorem 2.1.

An elementary, but important, observation for our purposes is that if μ is (δ_2, δ_1)-exactly focused at y, and moreover $\mu(\mathsf{B}^X_b(z)) \leq \Upsilon b^{\alpha}$ for all $\delta_2 \leq b \leq \delta_1$ and all $z \in \mathsf{B}^X_{\delta_1}(y)$, then α is necessarily very close to an integer multiple of 3 (in the case of $G \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, one of $\mathsf{d} = 3, 6, 9$) and the box $v + V$ has size nearly δ_1 in d-directions and has size roughly comparable to δ_2 in the complementary directions.

Let us write $\mu_t = \nu_t * \delta_{x_0}$ for all t. Our initial dimension takes the following form: For all $s \geq t_1$, we can write $\mu_s = \mu_{s,0} + \mu_{s,1}$ where $\mu_{s,1}(X) \leq \delta^{\kappa}$ and for all $y \in X_{\delta^{\kappa}}$ we have

$$\mu_{s,0}\big(\mathsf{B}^X_b(y)\big) \leq b^{\alpha} \quad \text{for all } \delta < b \leq \delta' \tag{3.4}$$

where $0 < \alpha_{\text{ini}} \leq \alpha < \dim \mathfrak{g}$ and $\delta = e^{-\star R}$.

Let $s \geq t_1$. Using Theorem 2.1, for every $e^{2\ell}\delta \leq b \leq e^{-2\ell}\delta'$ we can decompose μ_s as follows

$$\mu_s = \mu^{(b)}_{s,\text{ip}} + \mu^{(b)}_{s,\text{fs}} + \mu^{(b)}_{s,1}, \tag{3.5}$$

where $\mu^{(b)}_{s,1}(X) \ll \delta^{\kappa}$. The measure $\mu^{(b)}_{s,\text{fs}}$ is supported on the $(e^{-2\ell}b, b)$-focused set and for all but an exceptional set with measure $\ll \delta^{\kappa}$ of $r \in [0, 1]$,

$$a_\ell u_r.(\mu^{(b)}_{s,\text{ip}} + \mu^{(b)}_{s,\text{fs}})(\mathsf{B}^X_{\hat{b}}(y)) \leq \Upsilon \hat{b}^{\alpha} \quad \text{for all } e^{2\ell}\delta \leq \hat{b} \leq \delta', \tag{3.6a}$$

$$a_\ell u_r.\mu^{(b)}_{s,\text{ip}}\big(\mathsf{B}^X_b(y)\big) \leq \Upsilon e^{-\epsilon\ell} b^{\alpha} \tag{3.6b}$$

where $\Upsilon \ll \delta^{-\star\kappa}$.

The indices ip and fs stand for improvement and focused, respectively. In particular, (3.6a) states that the dimension is preserved at all scales and (3.6b) (specifically, the $e^{-\epsilon\ell}$ factor in that equation) states that the dimension is improved for $\mu^{(b)}_{s,\text{ip}}$ at scale b. Altogether, we have decomposed the measure into a negligibly small piece $\mu^{(b)}_{s,1}$, a piece $\mu^{(b)}_{s,\text{fs}}$ which is focused in the sense of Definition 3.3, and a piece $\mu^{(b)}_{s,\text{ip}}$ where the dimension is improved at certain scale. The constant ϵ in (3.6b) is sufficiently small so that Theorem 2.1 holds for all $\epsilon' \leq \epsilon$.

Roughly speaking, the dimension estimate for $a_\ell u_r.\mu_s$ at scale $\hat{b}$ is obtained as follows: if $u_r.\mu_s$ has dimension α_1 at scale $\hat{b}$ and dimension α_2 at scale $e^{-2\ell}\hat{b}$, then $a_\ell u_r.\mu_s$ has dimension at least $\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ at scale $\hat{b}$.

Suppose now for some $s \geq t_1$ and $0 < \alpha_{\mathrm{ini}} \leq \alpha < \dim \mathfrak{g}$, the measure μ_s admits a decomposition $\mu_s = \mu_{s,0} + \mu_{s,1}$ where $\mu_{s,1}(X) \leq \delta^\kappa$, and

$$\mu_{s,0}\big(\mathsf{B}^X_{\hat{b}}(y)\big) \leq \Upsilon \hat{b}^\alpha \qquad \text{for all } \delta \leq \hat{b} \leq \delta' \text{ and all } y \in X_{\delta^\kappa}$$

Let $s' \sim \log(1/\delta)$, and assume that for some $e^{6s'}\delta \leq b \leq \delta'$

$$\mu_t\big(\{y : \mu_t(\mathsf{B}^X_b(y)) \geq \Upsilon b^{\alpha+\hat{\epsilon}}\}\big) > e^{-\kappa R} \tag{3.7}$$

where $t = s + 3s'$ and $\hat{\epsilon}$ is a small multiple of ϵ in (3.6b).

Define ℓ by $e^{2\ell} b = \delta^\kappa$, and for $1 \leq i \leq 4$, set $b_i = e^{2(1-i)\ell} b$. Note that $b_1 = b$, and b_i for $2 \leq i \leq 4$ is comparable to b^i, up to a factor of size $\delta^{-\star\kappa}$. For $j = 1, 2, 3$, we will investigate

$$\mu_{t-j\ell} \quad \text{at scales} \quad \{b_1, \ldots, b_{j+1}\}.$$

Applying an ip-fs-negligible decomposition, as in (3.5), to the measures $\mu_{t-j\ell}$, we show that if (3.7) holds, there will be a set $\mathcal{F} \subset X_{\delta^\kappa}$ with $\mu_{t-3\ell}(\mathcal{F}) \gg \delta^{\star\kappa}$, so that $\mu_{t-3\ell}$ is simultaneously (b_4, b_3) and (b_2, b_1)-exactly focused on $\mathcal{F}$. We conclude from this that

- (A-1) For all $r \in [0,1]$ except a set with measure $\ll \delta^{\star\kappa}$, the measures $\mu_{t-2\ell}$ is (b_2, b_1)-exactly focused on $a_\ell u_r.\mathcal{F}$,
- (A-2) Similarly, for all $r \in [0,1]$ except a set with measure $\ll \delta^{\star\kappa}$, the measure $\mu_{t-\ell}$ is (b_2, b_1)-exactly focused on $a_{2\ell} u_r \mathcal{F}$ — we use the fact that $\mu_{t-3\ell}$ is (b_4, b_3)-exactly focused on $\mathcal{F}$ to show that $\mu_{t-\ell}$ has exact dimension at scale b_2 on $a_{2\ell} u_r.\mathcal{F}$.

Using (A-1) and (A-2) we will conclude that the measure $\mu_{t-2\ell}$ has a *global* obstruction as discussed in the beginning of §3.2 by using the argument outlined below.

For every $y \in \mathcal{F}$, let V_y denote the linear subspace corresponding to the long sides (i.e. those of size roughly b_1) of the box provided by Definition 3.3 applied with the measure $\mu_{t-3\ell}$, the point y, and scales (b_2, b_1). Also let $r \in [0,1]$ be outside the exceptional set in (A-1) above. By investigating the directions of $a_\ell u_r V_{y'}$ for all $y' \in \mathcal{F}$ within the b_1-neighborhood of a piece of the stable leaf (with size δ^κ) through y, we obtain the following. There exists a set $\mathcal{F}' \subset a_\ell u_r.\mathcal{F}$ with $\mu_{t-2\ell}(\mathcal{F}') \gg \delta^{\star\kappa}$, so that $\mu_{t-2\ell}$ is (b_2, b_1)-exactly focused on $\mathcal{F}'$. Moreover, for every $z \in \mathcal{F}'$ there exists a subspace

$$L^+_z \subset \mathbb{R}^3 \simeq G^+$$

(namely $V_{y'} \cap \mathfrak{g}^+$ where $a_\ell u_r y' \in \mathsf{B}^X_{b_1}(z)$) satisfying the following. The intersection of $\mathcal{F}'$ with the b_1-neighborhood of a piece of the *unstable leaf* (with size δ^κ) through z is contained in $\delta^{-\star\kappa} b_1$-neighborhood of $E^+_z.z$, where $E^+_z = B^{L^+_z}_{\delta^\kappa}$. Moreover, the b_1-covering number of this intersection is $\gg \delta^{\star\kappa} b_1^{-\alpha/3}$.

More careful analysis actually gives that the intersection of $\mathcal{F}'$ with a tube around such a δ^κ-piece of an unstable leaf of size b_1 in the $\mathfrak{g}^0$-direction and b_2 in the $\mathfrak{g}^-$-direction (a "b_1, b_2-tube") is of distance at most $\delta^{-\star\kappa}b_1, b_2, b_2$ in the $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$ directions respectively of $\exp(B_{\delta^\kappa}^{\mathfrak{g}^+\oplus\mathfrak{g}^0} \cap V_{y'}).z$, y' as above.

Similarly, for $\hat{y} \in a_{2\ell}u_r.\mathcal{F}$, let $V_{\hat{y}}$ be a subspace given by Definition 3.3 applied with $\mu_{t-\ell}$, the point $\hat{y}$, and scales (b_2, b_1). Investigating $a_{-\ell}u_r.V_{\hat{y}'}$ for $\hat{y}' \in a_{2\ell}u_r.\mathcal{F}$ within the b_1-neighborhood of a piece of the unstable leaf (with size δ^κ) through $\hat{y}$, we obtain the following. Trimming $\mathcal{F}'$ if necessary, for every $z \in \mathcal{F}'$ there is a subspace $L_z^- \subset \mathbb{R}^3 \simeq G^-$ with $\dim L_z^- = \dim L_z^+$ so that the following holds. The intersection of $\mathcal{F}'$ with the b_1-neighborhood of a piece of the *stable leaf* (with size δ^κ) through any $z \in \mathcal{F}'$ is contained in a $\delta^{-\star\kappa}b_1$-neighborhood of $E_z^-.z$, where $E_z^- = B_{\delta^\kappa}^{L_z^-}$. Moreover, the b_1-covering number of this intersection is $\gg \delta^{\star\kappa}b_1^{-\alpha/3}$, and again one also get more precise information when intersecting with a b_1, b_2-tube as above; in the present case, however, the roles of $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are interchanged.

Next we show that up to errors of size $\delta^{-\star\kappa}b_1$, the subspaces L_z^+ and L_z^- are respectively the unstable and stable subspaces of the same H-invariant subspace. To see this, let $z \in \mathcal{F}'$ and let $y \in \mathcal{F}$ be so that $a_\ell u_r y \in \mathsf{B}_{b_1}^X(z)$. Let V_y be as above, similarly, let V_z denote the subspaces provided by Definition 3.3 applied with the measure $\mu_{t-2\ell}$, the point z, and scales (b_2, b_1). By comparing the center directions of V_z and V_y, and using the relationship between L_z^+ and the unstable direction of V_y, we conclude that L_z^+ is within $\delta^{-\star\kappa}b_1$ neighborhood of V_z. Similarly, comparing the center directions of V_z and $V_{\hat{y}}$ for $\hat{y} \in a_{2\ell}u_r.\mathcal{F}$, and using the relationship between the stable direction of V_z and L_z^-, we have L_z^- is within $\delta^{-\star\kappa}b_1$ neighborhood of V_z. Furthermore, the discussion above also yields that α is nearly equal to $\dim V_z$, see the remark following Definition 3.3.

We now note that

$$\mathcal{N}_{b_1}\big(E_z^- E_z^+ E_z^- E_z^+\big) \gg \delta^{\star\kappa} b_1^{-\mathsf{k}}$$

where $\mathsf{k} \geq \hat{\kappa} + \dim V_z$ unless V_z is within $b_1^{1-\star\hat{\kappa}}$ of $\mathrm{Lie}(M_z)$ for some $H \leq M_z \leq G$, where $\hat{\kappa}$ is small (in a way that depends only on G).

On the other hand, the preceding discussion of the structure of $\mathcal{F}'$ along stable and unstable directions implies that the b_1-covering number of

$$\mathcal{F}' \cap \mathsf{Nhd}_{\delta^{-\star\kappa}b_1}\big(E_z^- E_z^+ E_z^- E_z^+.z\big)$$

is $\gg \delta^{\star\kappa}b_1^{-\mathsf{k}}$. Since $\mathcal{N}_{b_1}(\mathcal{F}') \ll b_1^{-\alpha}$ and α is nearly equal to $\dim V_z$, by choosing $\hat{\kappa}$ appropriately, we conclude (after trimming $\mathcal{F}'$ if necessary) that for all $z \in \mathcal{F}'$, we have

$$V_z \subset \mathsf{Nhd}_{b_1^{1/2}}\big(\mathrm{Lie}(M_z)\big)$$

for a subgroup $H \leq M_z \leq G$.

Recall now that $M_z^+ M_z^- M_z^+$ is Zariski open and dense in M_z. Using the above structure of $\mathcal{F}'$ along stable and unstable directions again, we conclude that the b_1-covering number of

$$\mathcal{F}' \cap \mathsf{Nhd}_{Cb_1^{1/2}}\big(B_{\delta^\kappa}^{M_z^+} B_{\delta^\kappa}^{M_z^-} B_{\delta^\kappa}^{M_z^+}.z\big)$$

is $\gg \delta^{\star\kappa} b_1^{-\alpha}$. Therefore, there exists some $z \in \mathcal{F}'$ such that

$$\mu_{t-2\ell}\big(\mathsf{Nhd}_{Cb_1^{1/2}}(B_{\delta^\kappa}^{M_z}.z)\big) \gg \delta^{\star\kappa}.$$

In view of the closing lemma discussed in §3.1, this implies that either part (2) in Theorem 1.1 holds or $M_z = G$.

3.4. **From high dimension to equidistribution.** The previous step implies that μ_{T-cR} has dimension close to $\dim \mathfrak{g}$ at scales $e^{-10cR} \le b \le \delta'$ unless part (2) in Theorem 1.1 holds. Thus assuming the latter does not hold, in this final phase we use the exponential mixing of a_t with respect to m_X to conclude that μ_T equidistributes with respect to m_X.

This step is carried out using similar arguments as in [LMWY25, §9].

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(E.L.) Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA
and
The Einstein Institute of Mathematics, Edmund J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
Email address: elonl@ias.edu

A.M.: Department of Mathematics, University of California, Berkeley, CA 94720
Email address: amirmo@math.berkeley.edu

L.Y.: Department of Mathematics, National University of Singapore, 119076, Singapore
Email address: lei.yang@nus.edu.sg