EFFECTIVE EQUIDISTRIBUTION IN RANK 2 HOMOGENEOUS SPACES AND VALUES OF QUADRATIC FORMS

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Abstract. We establish effective equidistribution theorems, with a polynomial error rate, for orbits of unipotent subgroups in quotients of quasi-split, almost simple Linear algebraic groups of absolute rank 2.

As an application, inspired by the results of Eskin, Margulis and Mozes, we establish quantitative results regarding the distribution of values of an indefinite ternary quadratic form at integer points, giving in particular an effective and quantitative proof of the Oppenheim Conjecture.

Contents

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Part 0. Introduction

A landmark result of Margulis is his proof of the Oppenheim Conjecture by showing that every $\mathrm{SO}(2,1)$ orbit in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ is either periodic or unbounded (or both — in our terminology an orbit $L.x$ is periodic if it supports a finite L invariant measure; this implies $L.x$ is closed but does not preclude it being noncompact). Subsequently, Dani and Margulis showed that any $\mathrm{SO}(2,1)$-orbit in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ is either periodic or dense, and also classified possible orbit closures for a one-parameter unipotent subgroup of $\mathrm{SO}(2,1)$.

More precise information regarding the behavior of orbits of one-parameter unipotent subgroups in quotients of real Lie groups was provided by Ratner in [Rat90, Rat91a, Rat91b]. These remarkable theorems have been highly influential and have had a plethora of applications, many of them quite unexpected.

This paper is a step in a program to make Ratner's equidistribution theorem effective. Previously, the first three authors proved effective equidistribution results for unipotent flows in $G = \mathrm{SL}_2(\mathbb{C})$ or $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and the last named author for $G = \mathrm{SL}_3(\mathbb{R})$ and u_t a singular one-parameter unipotent group; here we consider the generic one-parameter group in any quasi-split group of absolute rank 2.

As an application, we give here an effective and quantitative equidistribution result for the values of an indefinite ternary quadratic form at integer points in large balls. This gives in particular an effective and quantitative proof of the Oppenheim Conjecture. An effective proof of the Oppenheim Conjecture was given by Margulis and the first named author in [LM14], but the rates we give here (say of the smallest nonzero value of $|Q(v)|$ with v an integer vector of norm at most T) are polynomial, which is the right kind of dependency, vs. a polylogarithmic rate in [LM14].

Moreover, the quantitative equidistribution result for the values of an indefinite ternary quadratic form is new — no effective or explicit rate was previously known. Here we follow Eskin, Margulis and Mozes who gave a beautiful argument proving a qualitative[1] result of this type for all indefinite quadratic forms of signature (p, q) with $p \geq 3$ and subsequently also for forms of signature $(2, 2)$. The techiniques of Eskin, Margulis and Mozes were recently extended to forms of signature $(2, 1)$ by Wooyeon Kim in [Kim24].

1. Effective equidistribution

In this section we state the main equidistribution theorems of this paper, Theorem 1.1 and Theorem 1.2. These theorems will be proved in Part 1 of the paper.

In §2, we will discuss applications of these equidistribution theorems to values of quadratic forms, viz. Oppenheim conjecture. Part 2 of the paper, is devoted to the proof of these results using the equidistribution result Theorem 1.1 as well as an analysis of the cusp excursions for certain orbits which is closely related to the works of Eskin Margulis and Mozes in [EMM98, EMM05] and Wooyen Kim [Kim24].

Throughout Part 1 of the paper, G denotes the connected component of identity (as a Lie group) of the real points of an $\mathbb{R}$-algebraic group isogenous to one of the following

$$\mathrm{SL}_2(\mathbb{C}), \quad \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}), \quad \mathrm{SL}_3(\mathbb{R}), \quad \mathrm{SU}(2,1), \quad \mathrm{Sp}_4(\mathbb{R}), \quad \mathbf{G}_2(\mathbb{R}).$$

Put differently, G is the finite index subgroup $\mathbf{G}(\mathbb{R})^+ \subset \mathbf{G}(\mathbb{R})$ where $\mathbf{G}$ is a semisimple, connected, algebraic $\mathbb{R}$-group which is $\mathbb{R}$-quasi split and has absolute rank 2, and $\mathbf{G}(\mathbb{R})^+$ is the subgroup generated by unipotent one-parameter subgroups [Mar91, Ch. I]. For $G = \mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, Theorem 1.1 and Theorem 1.2 where established in [LMW22]; the proof we give here in particular gives a somewhat more streamlined proof of effective equidistribution also in these cases, but up to minor cosmetic improvements in Theorem 1.2 we do not provide any new results in those cases.

Let H denote the image of a principal $\mathrm{SL}_2(\mathbb{R})$ in G. In particular, H is a maximal connected subgroup which is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, and $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathfrak{r}$ decomposes as sum of two *irreducible* representations of H, see §3.1.

For all $t, r \in \mathbb{R}$, let a_t and u_r denote the images of

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

in H, respectively. Then a_t and u_r are *regular* one parameter diagonalizable and unipotent subgroups of G, respectively. With this normalization, if $v \in \mathfrak{r}$ is a highest weight vector for a_t, then

$$\mathrm{Ad}(a_t)v = e^{\mathsf{m}t}v \qquad \text{where } \dim \mathfrak{r} = 2\mathsf{m} + 1.$$

[1]Eskin, Margulis and Mozes called their result a quantitative version of the Oppenheim conjecture. It is quantitative in the sense that it counts the number of lattice points in a large ball for which $Q(v)$ is in a given interval, but not quantitative in the sense it does not say how large the ball has to be before this asymptotical behaviour begins to hold, and is not effective.

Let $\Gamma \subset G$ be an arithmetic lattice. By Margulis' arithmeticity theorem, any lattice $\Gamma \subset G$ is arithmetic, except possibly when G is isogenous to $\mathrm{SU}(2,1)$ and $\mathrm{SL}_2(\mathbb{C})$, or G is isogenous to $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$ and Γ is reducible, where non-arithmetic lattices exist.

Let $X = G/\Gamma$. Let d be the right invariant metric on G which is defined using the Killing form and the Cartan involution. This metric induces a metric d_X on X, and natural volume forms on X and its submanifolds. Let m_X denote the probability Haar measure on X

1.1. **Theorem.** *For every $x_0 \in X$ and large enough R (depending explicitly on x_0), for any $T \geq R^{A_1}$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$, we have

$$\Big|\int_0^1 \varphi(a_{\log T}u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi)R^{-\kappa_1}$$

where $\mathcal{S}(\varphi)$ is a certain Sobolev norm.

(2) There exists $x \in X$ such that Hx is periodic with $\mathrm{vol}(Hx) \leq R$, *and*

$$d_X(x, x_0) \leq R^{A_1}(\log T)^{A_1}T^{-\mathsf{m}}$$

where $\dim \mathfrak{r} = 2\mathsf{m} + 1$.

The constants A_1 and κ_1 are positive, and depend on X but not on x_0.

The strategy for the proof of Theorem 1.1 is similar to the general strategy developed in [LM23, LMW22]. A significant simplification is achieved here thanks to the use of higher dimensional energy. This in turn is made possible by using stronger projection theorems proved by Gan, Guo, Wang [GGW22].

Quantitative and effective versions of the aforementioned rigidity theorems in homogeneous dynamics have been sought after for some time, we refer to [LM23, LMW22, Yan22] for a more detailed discussion of this problem and some recent progress.

As it was done in [LMW22], combining Theorem 1.1 and the Dani–Margulis linearization method [DM91] (cf. also Shah [Sha91]), that allows to control the amount of time a unipotent trajectory spends near invariant subvarieties of a homogeneous space, we also obtain an effective equidistribution theorem for long pieces of unipotent orbits (more precisely, we use a sharp form of the linearization method taken from [LMMS19]).

1.2. **Theorem.** *For every $x_0 \in X$ and large enough R (depending explicitly on X), for any $T \geq R^{A_2}$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$, we have

$$\Big|\frac{1}{T}\int_0^T \varphi(u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi)R^{-\kappa_2}$$

where $\mathcal{S}(\varphi)$ is a certain Sobolev norm.

(2) *There exists* $x \in G/\Gamma$ *with* $\mathrm{vol}(H.x) \leq R^{A_2}$, *and for every* $r \in [0,T]$ *there exists* $g(r) \in G$ *with* $\|g(r)\| \leq R^{A_2}$ *so that*

$$d_X(u_s x_0, g(r)H.x) \leq R^{A_2}\left(\frac{|s-r|}{T}\right)^{1/A_3} \qquad \text{for all } s \in [0,T].$$

(3) *There is a parabolic subgroup* $P \subset G$ *and some* $x \in G/\Gamma$ *satisfying that* $\mathrm{vol}(R_u(P).x) \leq R^{A_2}$, *and for every* $r \in [0,T]$ *there exists* $g(r) \in G$ *with* $\|g(r)\| \leq R^{A_2}$ *so that*

$$d_X(u_s x_0, g(r)R_u(P).x) \leq R^{A_2}\left(\frac{|s-r|}{T}\right)^{1/A_3} \qquad \text{for all } s \in [0,T].$$

In particular, X *is not compact.*

The constants A_2, A_3, *and* κ_2 *are positive and depend on* X *but not on* x_0.

Remarks: In options (2) and (3), one can be more explicit about the properties of the element $g(r)$. For instance in option (2), since $u_s x_0$ needs to stay close to $g(r)H.x$ for all s in long intervals around r it follows from (2) that $g(r)$ is very close to $C_G(U)$ (there is of course some play between what "stay close" and "long interval" means — if one uses a stricter interpretation of what close means, then one must be more lenient on what long means and vice versa). An analogous statement for $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ turned out to be useful in the work of Forni, Kanigowski, Radziwiłł studying equidistribution of orbits at nearly prime times [FKR24]

2. Applications to the Oppenheim conjecture

We now discuss applications of Theorem 1.1 to values of quadratic forms in the context of the Oppenheim conjecture.

2.1. **Theorem.** *Let* Q *be an indefinite ternary quadratic form with* $\det Q = 1$. *For all* R *large enough, depending on* $\|Q\|$, *and all* $T \geq R^{A_4}$ *at least one of the following holds.*

(1) *For every* $s \in [-R^{\kappa_3}, R^{\kappa_3}]$, *there exists a primitive vector* $v \in \mathbb{Z}^3$ *with* $0 < \|v\| \leq T$ *so that*

$$|Q(v) - s| \leq R^{-\kappa_3}$$

(2) *There exists some* $Q' \in \mathrm{Mat}_3(\mathbb{R})$ *with* $\|Q'\| \leq R$ *so that*

$$\left\|Q - \lambda Q'\right\| \leq R^{A_4}(\log T)^{A_4} T^{-2} \qquad \text{where } \lambda = (\det Q')^{-1/3}.$$

The constants A_4 *and* κ_3 *are absolute.*

Since algebraic numbers cannot be well approximated by rationals, one concludes the following corollary from Theorem 2.1.

2.2. **Corollary.** *Let* Q *be a reduced, indefinite, ternary quadratic form which is not proportional to an integral form but has algebraic coefficients. Then* *for all T large enough, depending on the degrees and heights of the coefficients of Q, we have the following: for any*

$$s \in [-T^{\kappa_4}, T^{\kappa_4}]$$

there exists a primitive vector $v \in \mathbb{Z}^3$ with $0 < \|v\| \leq T$ so that

$$|Q(v) - s| \leq T^{-\kappa_4}.$$

The constant κ_4 depends on the degrees of the coefficients of Q.

As was mentioned, Theorem 2.1 and Corollary 2.2 are an effective version of a celebrated theorem of Margulis [Mar89], see also [DM90]. An effective version, with a polylogarithmic rate, was proved by the first named author and Margulis [LM14]. The proofs in [Mar89, DM90] and [LM14] rely on establishing a special case of Raghunathan's conjecture for unipotent flows — albeit an effective version in the case of [LM14].

Similarly, our proof of Theorem 2.1 is based on Theorem 1.1. Indeed, using Theorem 1.1, for $G = \mathrm{SL}_3(\mathbb{R})$, $H = \mathrm{SO}(Q)^\circ$, and adapting the arguments developed by Dani and Margulis [DM91] and by Eskin, Margulis, and Mozes [EMM98, EMM05], and recent advances by W. Kim [Kim24], we obtain the following theorem.

2.3. **Theorem.** *Let Q be an indefinite ternary quadratic form with $\det Q = 1$, and put*

$$\mathsf{C}_Q = \int_L \frac{\mathrm{d}\sigma}{\|\nabla Q\|}$$

where $L = \{v \in \mathbb{R}^3 : \|v\| \leq 1, Q(v) = 0\}$ and $\mathrm{d}\sigma$ is the area element on L.

Let $a < b$ and $A \geq 10^3$. There are constants T_0 and C depending on A and $\|Q\|$, absolute constants $N \geq 1$ and $0 < \delta_0 < 1$, and for every $0 < \delta \leq \delta_0$ some $\kappa = \kappa(\delta, A)$ so that the following holds.

Assume that for $T \geq T_0$ and all integral forms Q' with $\|Q'\| \leq T^\delta$

$$\big\|Q - \lambda Q'\big\| \geq \|Q'\|^{-A} \qquad \text{where } \lambda = (\det Q')^{-1/3},$$

then the following is satisfied: If

$$\Big|\#\big\{v \in \mathbb{Z}^3 : \|v\| \leq T, a \leq Q(v) \leq b\big\} - \mathsf{C}_Q(b-a)T\Big| \geq \\ + C(1 + |a| + |b|)^N T^{1-\kappa},$$

there are at most 4 lines $L_1, \ldots, L_4$ and at most 4 planes $P_1, \ldots, P_4$ so that

$$\#\big\{v \in \mathbb{Z}^3 : \|v\| \leq T, a \leq Q(v) \leq b\big\} = \\ \mathsf{C}_Q(b-a)T + \mathcal{R}_T + O\big((1 + |a| + |b|)^N T^{1-\kappa}\big)$$

where $\mathcal{R}_T = \#\{v \in (\cup_i L_i) \bigcup (\cup_i P_i) : \|v\| \leq T, a \leq Q(v) \leq b\}$.

More precisely, these exceptional lines and planes satisfy the following: $L_i \cap \mathbb{Z}^3 = \mathrm{Span}\{v_i\}$ satisfying that

$$\|v_i\| \leq T^\delta \quad \text{and} \quad |Q(v_i)| \leq T^{-2+\delta} \tag{2.1}$$

for every $1 \le i \le 4$.

Also $P_i \cap \mathbb{Z}^3 = \mathrm{Span}\{w_{i,1}, w_{i,2}\}$ *satisfying*

$$\|w_{i,1}\|, \|w_{i,2}\| \le T^\delta \quad \textit{and} \quad |Q^*(w_{i,1} \wedge w_{i,2})| \le T^{-2+\delta} \tag{2.2}$$

for every $1 \le i \le 4$.

Note that the main term in part (1) captures the asymptotic behavior of the volume of the solid given by $Q(v) = a$, $Q(v) = b$, and $\|v\| \le T$. We also remark that there is a dense family of irrational quadratic forms which are very well approximated by rational forms so that

$$\#\{v \in \mathbb{Z}^3 : \|v\| \le T, a \le Q(v) \le b\} \gg T(\log T)^{1-\varepsilon},$$

see [EMM98, §3.7].

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The landmark work of G.A. Margulis on the Oppenheim Conjecture and more generally in homogeneous dynamics and its applications to number theory has been a continued source of inspiration for us. In particular, E.L. and A.M. have had joint works closely related to this paper with Margulis, and are very grateful for all we have learned from him. His insights feature in many places in this paper, sometimes implicitly.

Part 1. Polynomially effective equidistribution theorems

In this chapter we prove Theorem 1.1 and Theorem 1.2. As noted earlier, the overall strategy for the proofs aligns with the approach developed in [LMW22]. A key simplification in the proof of Theorem 1.1 is accomplished through the use of higher-dimensional energy, made possible by stronger projection theorems established by Gan, Guo, and Wang [GGW22].

3. Notation and preliminary results

Let $\mathbf{G}$ be a semisimple, connected, $\mathbb{R}$-group with absolute rank 2 which is $\mathbb{R}$-quasi split. Let $G = \mathbf{G}(\mathbb{R})^+$, where $\mathbf{G}(\mathbb{R})^+$ is the subgroup generated by unipotent one-parameter subgroups. In other words G is the connected component of the identity in the Lie group $\mathbf{G}(\mathbb{R})$, see [Mar91, Ch. I].

More explicitly, $\mathbf{G}$ is isogenous to one the following groups

$$\mathrm{SO}(3,1), \quad \mathrm{SL}_2 \times \mathrm{SL}_2, \quad \mathrm{SL}_3, \quad \mathrm{SU}(2,1), \quad \mathrm{Sp}_4, \quad \mathbf{G}_2.$$

Indeed, $\mathbf{G}$ is $\mathbb{R}$-split except when it is isogenous to $\mathrm{SO}(3,1)$ or $\mathrm{SU}(2,1)$, in which case $\mathbf{G}$ is $\mathbb{R}$-quasi split but not split. We assume $G \subset \mathrm{SL}_{\mathsf{N}}(\mathbb{R})$ for some N which is fixed throughout the paper.

3.1. **The principal $\mathrm{SL}_2(\mathbb{R})$ in G.** There is a homomorphism $\rho : \mathrm{SL}_2(\mathbb{R}) \to G$ with finite central kernel so that

$$\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathfrak{r}$$

where $H = \rho(\mathrm{SL}_2(\mathbb{R}))$ and $\mathfrak{r}$ is an irreducible representation of H with dimension $2\mathsf{m} + 1$ where

- $\mathsf{m} = 1$ if $\mathbf{G}$ is isogeneous to $\mathrm{SO}(3,1)$ or $\mathrm{SL}_2 \times \mathrm{SL}_2$
- $\mathsf{m} = 2$ if $\mathbf{G}$ is isogeneous to SL_3 or $\mathrm{SU}(2,1)$
- $\mathsf{m} = 3$ if $\mathbf{G}$ is isogeneous to Sp_4
- $\mathsf{m} = 5$ if $\mathbf{G}$ is isogeneous to $\mathbf{G}_2$

see [And75] and the references therein, in particular, see [And75, Prop. 6].

For all $t, r \in \mathbb{R}$, let a_t, u_r, and u_s^- denote the images of

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \qquad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

in H, respectively. Then a_t and u_r are *regular* one parameter diagonalizable and unipotent subgroups of G, respectively. Similarly, u_s^- is a regular unipotent subgroups of G. We let $A = \{a_t\}$, $U = \{u_r\}$, and $U^- = \{u_s^-\}$.

All regular unipotent one-parameter groups, and hence the corresponding groups H, are conjugated to each other under $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$, the adjoint form of $\mathbf{G}$, see [And75, Thm. 1].

Note also that with the above normalization, if $v \in \mathfrak{r}$ is a highest weight vector for a_t, then $\mathrm{Ad}(a_t)v = e^{\mathsf{m}t}v$, recall that $\dim \mathfrak{r} = 2\mathsf{m} + 1$.

For any A-invariant subspace $V \subset \mathfrak{sl}_{\mathsf{N}}(\mathbb{R})$, let V^0 denote the space of a_t-fixed vectors and

$$V^{\pm} = \{z \in V : \lim_{t \to \mp\infty} \mathrm{Ad}(a_t)z = 0\}.$$

We assume the embedding $G \subset \mathrm{SL}_{\mathsf{N}}(\mathbb{R})$ is fixed so that $\mathfrak{g}^0 \oplus \mathfrak{g}^+ = \mathfrak{g} \cap \mathfrak{b}_{\mathsf{N}}$, the subalgebra of upper triangular matrices in $\mathfrak{gl}_{\mathsf{N}}$.

Lie algebras and norms. Recall that $G \subset \mathrm{SL}_{\mathsf{N}}(\mathbb{R})$. Let $|\ |$ denote the usual absolute value on $\mathbb{R}$. Let $\|\ \|$ denote the maximum norm $\mathrm{Mat}_{\mathsf{N}}(\mathbb{R})$, with respect to the standard basis.

We fix a norm on $\mathfrak{h}$ by taking the maximum norm where the coordinates are given by fixing unit basis vectors for the lines $\mathrm{Lie}(U)$, $\mathrm{Lie}(U^-)$, and $\mathrm{Lie}(A)$. Since $\mathfrak{r}$ is $\mathrm{Ad}(H)$-irreducible, each weight space is one dimensional. Fix a norm on $\mathfrak{r}$ by taking the max norm with respect to a basis consisting of unit pure weight (with respect to a_t) vectors. By taking maximum of these two norms we get a norm on $\mathfrak{g}$, which will also be denoted by $\|\ \|$.

Let $C_1 \geq 1$ be so that

$$\|hw\| \leq C_1 \|w\| \quad \text{for all } \|h - I\| \leq 2 \text{ and all } w \in \mathfrak{g}. \tag{3.1}$$

For all $0 < \delta < 1$, we define

$$\mathsf{B}_\delta^H := \{u_s^- : |s| \leq \delta\} \cdot \{a_t : |t| \leq \delta\} \cdot \{u_r : |r| \leq \delta\} \tag{3.2}$$

Define B^L_δ for $\delta > 0$ and $L = U, U^-$ and A similarly, and put

$$\mathsf{B}^{s,H}_\delta = \{u^-_s : |s| \le \delta\} \cdot \{a_t : |t| \le \delta\} = \mathsf{B}^{U^-}_\delta \cdot \mathsf{B}^A_\delta.$$

Note that for all $h_i \in (\mathsf{B}^H_\delta)^{\pm 1}$, $i = 1, \ldots, 5$, we have

$$h_1 \cdots h_5 \in \mathsf{B}^H_{100\delta}. \tag{3.3}$$

We also define $\mathsf{B}^G_\delta := \mathsf{B}^H_\delta \cdot \exp(B_{\mathfrak{r}}(0,\delta))$ where $B_{\mathfrak{r}}(0,\delta)$ denotes the ball of radius δ in $\mathfrak{r}$ with respect to $\|\ \|$.

Given an open subset $\mathsf{B} \subset L$, with L any of the above, and $\delta > 0$, put $\partial_\delta \mathsf{B} = \{\mathsf{h} \in \mathsf{B} : \mathsf{B}^L_\delta.\mathsf{h} \not\subset \mathsf{B}\}$.

We fix Haar measures m_G on G and m_H on H. Let $\Gamma \subset G$ be an arithmetic lattice, and let m_X denote the probability Haar measure on $X = G/\Gamma$.

For all $x \in X$, define $\mathrm{inj}(x)$ as follows

$$\mathrm{inj}(x) = \min\bigl\{0.01, \sup\bigl\{\delta : g \mapsto gx \text{ is injective on } \mathsf{B}^G_{10N^2\delta}\bigr\}\bigr\}. \tag{3.4}$$

For every $\eta > 0$, let $X_\eta = \bigl\{x \in X : \mathrm{inj}(x) \ge \eta\bigr\}$.

Commutation relations. Let us record the following two lemmas.

3.2. **Lemma** ([LM23], Lemma 2.1)**.** *There exist δ_0 and C_2 depending on m so that the following holds. Let $0 < \delta \le \delta_0$, and let $w_1, w_2 \in B_{\mathfrak{r}}(0,\delta)$. There are $h \in H$ and $w \in \mathfrak{r}$ which satisfy*

$$\tfrac{2}{3}\|w_1 - w_2\| \le \|w\| \le \tfrac{3}{2}\|w_1 - w_2\| \quad \textit{and} \quad \|h - I\| \le C_2\delta\|w\|$$

so that $\exp(w_1)\exp(-w_2) = h\exp(w)$. More precisely,

$$\|w - (w_1 - w_2)\| \le C_2\delta\|w_1 - w_2\|$$

3.3. **Lemma** ([LM23], Lemma 2.2)**.** *There exists δ_0 so that the following holds for all $0 < \delta \le \delta_0$. Let $x \in X_\delta$ and $w \in B_{\mathfrak{r}}(0,\delta)$. If there are $h, h' \in \mathsf{B}^H_{2\delta}$ so that $\exp(w')hx = h'\exp(w)x$, then*

$$h' = h \quad \textit{and} \quad w' = \mathrm{Ad}(h)w.$$

Moreover, we have $\|w'\| \le 2\|w\|$.

The set $\mathsf{E}_{\eta,t,\beta}$. For all $\eta, t, \beta > 0$, set

$$\mathsf{E}_{\eta,t,\beta} := \mathsf{B}^{s,H}_\beta \cdot a_t \cdot \{u_r : r \in [0,\eta]\} \subset H. \tag{3.5}$$

Then $m_H(\mathsf{E}_{\eta,t,\beta}) \asymp \eta\beta^2 e^t$ where m_H denotes our fixed Haar measure on H.

Throughout the paper, the notation $\mathsf{E}_{\eta,t,\beta}$ will be used only for $\eta, t, \beta > 0$ which satisfy $e^{-0.01t} < \beta \le \eta^2$, even if this is not explicitly stated.

For all $\eta, \beta, \tau > 0$, put

$$\mathsf{Q}^H_{\eta,\beta,\tau} = \bigl\{u^-_s : |s| \le \beta e^{-\tau}\bigr\} \cdot \{a_d : |d| \le \beta\} \cdot \bigl\{u_r : |r| \le \eta\bigr\}. \tag{3.6}$$

Roughly speaking, $\mathsf{Q}^H_{\eta,\beta,\tau}$ is a *small thickening* of the (β,η)-neighborhood of the identity in AU. We write $\mathsf{Q}^H_{\beta,\tau}$ for $\mathsf{Q}^H_{\beta,\beta,\tau}$.

The following lemma will also be used in the sequel.

3.4. **Lemma** ([LM23], Lemma 2.3). *(1) Let $\tau \geq 1$, and let $0 < \eta, \beta < 0.1$. Then*

$$\Big(\big(\mathsf{Q}^H_{\eta/10\mathsf{N}^2,\beta/10\mathsf{N}^2,\tau}\big)^{\pm 1}\Big)^3 \subset \mathsf{Q}^H_{\eta,\beta,\tau}.$$

(2) For all $0 \leq \beta, \eta \leq 1$, $t, \tau > 0$, and all $|r| \leq 2$, we have

$$\big(\mathsf{Q}^H_{\eta,\beta^2,\tau}\big)^{\pm 1} \cdot a_\tau u_r \mathsf{E}_{\eta',t,\beta'} \subset a_\tau u_r \mathsf{E}_{\eta,t,\beta}, \tag{3.7}$$

where $\eta' = \eta(1 - 10\mathsf{N}^2 e^{-t})$ and $\beta' = \beta(1 - 10\mathsf{N}^2\beta)$.

Linear algebra lemma. Recall that $\dim \mathfrak{r} = 2\mathsf{m} + 1$, and that $\mathfrak{r}$ is $\mathrm{Ad}(H)$-irreducible. We have the following

3.5. **Lemma** (cf. Lemma A.1, [LMW22]). *Let $0 < \alpha \leq 1/(2\mathsf{m} + 1)$. For all $d > 0$ and all $0 \neq w \in \mathfrak{r}$, we have*

$$\int_0^1 \|a_d u_r w\|^{-\alpha}\,\mathrm{d}r \leq C e^{-\alpha \mathsf{m} d}\|w\|^{-\alpha}$$

where C is an absolute constant.

4. Avoidance principles in homogeneous spaces

In this section we will collect statements concerning avoidance principles for unipotent flows and random walks on homogeneous spaces; the reader will find this section similar to [LMW22, §4] and [LM23, §3]. The proofs are included in Appendix B for the convenience of the reader.

4.1. **Nondivergence results.** The results of this subsection are only interesting when Γ is a nonuniform lattice, i.e., when X is not compact.

4.2. **Proposition.** *There exist m_0 depending only on m and $C_3 \geq 1$ depending on X with the following property. Let $0 < \delta, \varepsilon < 1$, and let $I \subset [-10, 10]$ be an interval with $|I| \geq \delta$. For all $x \in X$, we have*

$$\big|\big\{r \in I : a_s u_r x \notin X_\varepsilon\big\}\big| < C_3 \varepsilon^{1/\mathsf{m}_0}|I|,$$

so long as $s \geq \mathsf{m}_0|\log(\delta\,\mathrm{inj}(x))| + C_3$.

Proof. There exist m', depending on m, and a function $\omega : X \to [2, \infty)$ so that for all $x \in X$

$$\mathrm{inj}(x) \geq \omega(x)^{-\mathsf{m}'},$$

moreover, for all $x \in X$ and all $s \geq \mathsf{m}'|\log\delta| + B_0'$

$$\int_I \omega(a_s u_r x)\,\mathrm{d}r \leq e^{-\star s}\omega(x) + B_0,$$

where B_0' and B_0 depends on X. See Proposition B.3 and references there.

The claim in the proposition follows from these statements and Chebyshev's inequality. $\square$

4.3. **Inheritance of the Diophantine property.** The following proposition improves the somewhat weak Diophantine property that failure of part (2) in Theorem 1.1 provides to a Diophantine property in terms of volume of periodic orbits involved. This proposition can be compared with the main theorem in [LMMS19], and will be applied as the first step in the proof of Theorem 1.1.

4.4. **Proposition.** *There exist D_0, depending on m, and C_4, s_0, depending on X, so that the following holds. Let $R, S \geq 1$. Suppose $x_0 \in X$ is so that*

$$d_X(x_0, x) \geq (\log S)^{D_0} S^{-\mathsf{m}}$$

for all x with $\mathrm{vol}(Hx) \leq R$. Then for all

$$s \geq \max\bigl\{\log S, \mathsf{m}_0 |\log(\mathrm{inj}(x_0))|\bigr\} + s_0$$

and all $0 < \eta \leq 1$, we have

$$\left|\left\{r \in [0,1] \colon \begin{array}{c}\mathrm{inj}(a_s u_r x_0) \leq \eta \textit{ or there is } x \textit{ with}\\ \mathrm{vol}(Hx) \leq R \textit{ s.t. } d_X(a_s u_r x_0, x) \leq \frac{1}{C_4 R^{D_0}}\end{array}\right\}\right| \leq C_4(\eta^{1/\mathsf{m}_0} + R^{-1}).$$

The proof of Proposition 4.4 is postponed to §B.1.

4.5. **Closing Lemma.** Let $0 < \eta \ll_X 1$ and $\beta = \eta^2$. For every $\tau \geq 0$, put

$$\mathsf{E}_\tau = \mathsf{E}_{1,\tau,\beta} = \mathsf{B}^{s,H}_\beta \cdot a_\tau \cdot \{u_r : r \in [0,1]\} \subset H. \tag{4.1}$$

where $\mathsf{B}^{s,H}_\beta = \{u^-_s : |s| \leq \beta\} \cdot \{a_d : |d| \leq \beta\}$, see (3.5).

If $y \in X$ is so that the map $\mathsf{h} \mapsto \mathsf{h}y$ is injective over E_τ, then $\mu_{\mathsf{E}_\tau . y}$ denotes the pushforward of the normalized Haar measure on E_τ to $\mathsf{E}_\tau . y \subset X$.

Let $\tau \geq 0$ and $y \in X$. For every $z \in \mathsf{E}_\tau . y$, put

$$I(z) := \bigl\{w \in \mathfrak{r} : \|w\| < \mathrm{inj}(z) \text{ and } \exp(w)z \in \mathsf{E}_\tau . y\bigr\};$$

this is a finite subset of $\mathfrak{r}$ since E_τ is bounded — we will define $I_{\mathcal{E}}(z)$ for more general sets $\mathcal{E}$ in the bootstrap phase below.

Let $0 < \alpha \leq 1$. Define the function $f : \mathsf{E}_\tau . y \to [1, \infty)$ as follows

$$f(z) = \begin{cases} \sum_{0 \neq w \in I(z)} \|w\|^{-\alpha} & \text{if } I(z) \neq \{0\} \\ \mathrm{inj}(z)^{-\alpha} & \text{otherwise} \end{cases}.$$

4.6. **Proposition.** *There exist m_1 depending on m and D_1 depending on X which satisfy the following. Let $D \geq D_1$ and $x_1 \in X$. Then for all large enough t (depending on $\mathrm{inj}(x_1)$) at least one of the following holds.*

(1) There is a subset $J(x_1) \subset [0,1]$ with $|[0,1] \setminus J(x_1)| \ll_X \eta^{1/(2\mathsf{m}_0)}$ such that for all $r \in J(x_1)$ we have the following
   - *(a) $a_{\mathsf{m}_1 t} u_r x_1 \in X_\eta$.*
   - *(b) $\mathsf{h} \mapsto \mathsf{h}.a_{\mathsf{m}_1 t} u_r x_1$ is injective over E_t.*
   - *(c) For all $z \in \mathsf{E}_t . a_{\mathsf{m}_1 t} u_r x_1$, we have*

$$f(z) \leq e^{Dt}.$$

(2) There is $x \in X$ such that Hx is periodic with

$$\mathrm{vol}(Hx) \le e^{D_1 t} \quad \text{and} \quad d_X(x, x_1) \le e^{(-D+D_1)t}.$$

The proof of Proposition 4.6 is postponed to §B.7.

5. Mixing property and equidistribution

Let us recall the following quantitative decay of correlations for the ambient space X: There exists $0 < \kappa_0 \le 1$ so that

$$\Big|\int \varphi(gx)\psi(x)\,\mathrm{d}m_X - \int \varphi\,\mathrm{d}m_X \int \psi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)\mathcal{S}(\psi)e^{-\kappa_0 d(e,g)} \tag{5.1}$$

for all $\varphi, \psi \in C_c^\infty(X)+\mathbb{C}\cdot 1$, where m_X is the G-invariant probability measure on X and d is the right G-invariant metric on G defined on p. 4. See, e.g., [KM96, §2.4] and references there for (5.1).

Here $\mathcal{S}(\cdot)$ is a certain Sobolev norm on $C_c^\infty(X)+\mathbb{C}\cdot 1$ which is assumed to dominate $\|\cdot\|_\infty$ and the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}}$. Moreover, $\mathcal{S}(g.f) \ll \|g\|^\star \mathcal{S}(f)$ where the implied constants depend only on m.

5.1. **Proposition.** *There exists $\kappa_5 \gg \kappa_0$ so that the following holds. Let $\Lambda \ge 1$, and let ν be a probability measure on B_1^G with*

$$\frac{\mathrm{d}\nu}{\mathrm{d}m_G}(g) \le \Lambda \qquad \text{for all } g \in \operatorname{supp}\nu. \tag{5.2}$$

Let $\ell_1, \ell_2 > 0$ and $0 < \eta < 1$ satisfy the following

$$\kappa_5 \ell_2 \ge \max\{\ell_1, |\log \eta|\}.$$

Then for all $x \in X_\eta$ and all $\varphi \in C_c^\infty(X)$, we have

$$\int_0^1\int_G \varphi(a_{\ell_1}u_r a_{\ell_2} gx)\,\mathrm{d}\nu(g)\,\mathrm{d}r = \int \varphi\,\mathrm{d}m_X + O\big(\mathcal{S}(\varphi)(\eta + \Lambda^{1/2}e^{-\kappa_5\ell_1})\big).$$

Proof. Put $\mathsf{B} = B_1^G$ and assume, as we may, that $\int \varphi\,\mathrm{d}m_X = 0$.

Applying Fubini's theorem and Cauchy-Schwarz inequality, we have

$$\Big|\int_0^1\int_{\mathsf{B}} \varphi(a_{\ell_1}u_r a_{\ell_2} gx)\,\mathrm{d}\nu(g)\,\mathrm{d}r\Big|^2 \le \int_{\mathsf{B}}\Big(\int_0^1 \varphi(a_{\ell_1}u_r a_{\ell_2} gx)\,\mathrm{d}r\Big)^2\,\mathrm{d}\nu(g)$$

Expanding the inner integral in the right side of the above and using (5.2), we conclude that

$$\Big|\int_0^1\int_{\mathsf{B}} \varphi(a_{\ell_1}u_r a_{\ell_2} gx)\,\mathrm{d}\nu(g)\,\mathrm{d}r\Big|^2 \le \tag{5.3}$$
$$\Lambda \int_{\mathsf{B}}\int_0^1\int_0^1 \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}gx)\varphi(a_{\ell_1}u_{r_2}a_{\ell_2}gx)\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}m_G(g).$$

For all $r_1, r_2 \in [0,1]$, let

$$\Phi_{r_1,r_2}(z) = \varphi(a_{\ell_1}u_{r_1}z)\varphi(a_{\ell_1}u_{r_2}z).$$

Then $\mathcal{S}(\Phi_{r_1,r_2}) \ll e^{M\ell_1}\mathcal{S}(\varphi)^2$ for some M depending only on the dimension. Moreover, if $|r_1 - r_2| \geq e^{-\ell_1/2}$, then (5.1) implies that

$$\Big|\int \Phi_{r_1,r_2}\,\mathrm{d}m_X\Big| = \Big|\varphi(a_{\ell_1}u_{r_1}z)\varphi(a_{\ell_1}u_{r_2}z)\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)^2 e^{-\kappa\ell_1}$$

where $\kappa = \star\kappa_0$. We will assume $\kappa \leq 1/2$.

We now estimate the second integral in (5.3). Applying Fubini's theorem,

$$\Lambda\int_{\mathsf{B}}\int_0^1\int_0^1 \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}gx)\varphi(a_{\ell_1}u_{r_2}a_{\ell_2}gx)\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}m_G(g) = \\ \Lambda\int_0^1\int_0^1\int_{\mathsf{B}} \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}gx)\varphi(a_{\ell_1}u_{r_2}a_{\ell_2}gx)\,\mathrm{d}m_G(g)\,\mathrm{d}r_1\,\mathrm{d}r_2$$

Let $\Xi = \{(r_1,r_2)\in[0,1]^2 : |r_1-r_2| \geq e^{-\ell_1/2}\}$ and $\Xi' = [0,1]^2\setminus\Xi$. Let now $(r_1,r_2)\in\Xi$, and recall that $x\in X_\eta$. Increasing M if necessary and using a partition of unity, there exist a collection $\{\psi_i\}\subset C_c^\infty(X)$ satisfying that $\mathcal{S}(\psi_i)\leq \eta^{-M}$, $\sum\int\psi_i\,\mathrm{d}m_X = 1$, and

$$\frac{1}{m(\mathsf{B})}\int_{\mathsf{B}}\Phi_{r_1,r_2}(a_{\ell_2}gy)\,\mathrm{d}m_G(g) = \\ \sum_i\int \Phi_{r_1,r_2}(a_{\ell_2}z)\psi_i(z)\,\mathrm{d}m_X(z) + O(\eta\|\Phi_{r_1,r_2}\|_\infty)$$

In view of this and (5.1), thus

$$\frac{1}{m(\mathsf{B})}\int_{\mathsf{B}}\Phi_{r_1,r_2}(a_{\ell_2}gy)\,\mathrm{d}m_G(g) = \\ \int \Phi_{r_1,r_2}\,\mathrm{d}m_X + O\big(\|\varphi\|_\infty^2\eta + \mathcal{S}(\Phi_{r_1,r_2})\eta^{-M}e^{-\kappa_0\ell_2}\big) \tag{5.4}$$

Using the above observations regarding the Sobolev norm and the integral of Φ_{r_1,r_2}, (5.4) implies that if $(r_1,r_2)\in\Xi$, then

$$\frac{1}{m(\mathsf{B})}\int_{\mathsf{B}}\Phi_{r_1,r_2}(a_{\ell_2}gy)\,\mathrm{d}m_G(g) = O\big(\mathcal{S}(\varphi)^2(\eta + e^{-\kappa\ell_1} + \eta^{-M}e^{M\ell_1}e^{-\kappa_0\ell_2})\big)$$

This, $|\Xi'| \ll e^{-\ell_1/2}$, and (5.3) imply that if $\ell_2 > M\max\{\ell_1/\kappa_0, |\log\eta|\}$, then

$$\Big|\int_0^1\int_{\mathsf{B}}\varphi(a_{\ell_1}u_r a_{\ell_2}gy)\,\mathrm{d}\nu(g)\,\mathrm{d}r\Big| = O\big(\mathcal{S}(\varphi)(\eta + \Lambda^{1/2}e^{-\kappa\ell_1})\big).$$

The proposition thus holds with $\kappa_5 = \min\{\kappa, \frac{\kappa_0}{2M}\}$. $\square$

In applying Proposition 5.1, we consider measures supported on $\mathfrak{r}$ with a finitary dimension close to $2\mathsf{m}+1$. The following lemma, based on standard arguments, establishes the connection to Proposition 5.1.

5.2. **Lemma.** *Let $0 < \delta_0 < 1$. Let $\ell_1, \ell_2 > 0$ with $\kappa_5 \ell_2 \geq \max\{\ell_1, |\log\eta|\}$ and $4\mathsf{m}\ell_2 \leq |\log\delta_0|$, and let $\varrho \leq \varrho_0$, where $0 < \varrho_0 \leq 1$ depends only on the dimension. Let μ be a probability measure on $B_{\mathfrak{r}}(0,\varrho)$ satisfying*

$$\mu(B(w,\delta)) \leq \Upsilon\delta^{2\mathsf{m}+1} \quad \text{for all } w \in \mathfrak{r} \text{ and all } \delta \geq \delta_0. \tag{5.5}$$

Then for all $\varphi \in C_c^\infty(X)$ and all $x \in X_\eta$

$$\int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)x)\,\mathrm{d}\mu(w)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \int\varphi\,\mathrm{d}m_X + O\big(\mathcal{S}(\varphi)(\varrho^\star + \eta + \Upsilon^{1/2}\varrho^{-3/2}e^{-\kappa_5\ell_1})\big)$$

Proof. We provide the details of the straightforward proof for the convenience of the reader.

As it was mentioned before, we will use Proposition 5.1 to prove the lemma. To that end we begin by convolving μ with a smooth kernel. Let

$$\hat\mu = \hat\Phi * \mu, \tag{5.6}$$

where $\hat\Phi(w) = \delta_0^{-2\mathsf{m}-1}\Phi(\delta_0^{-1}w)$ for a radially symmetric nonnegative smooth function Φ on $\mathfrak{r}$ — recall that $\dim\mathfrak{r} = 2\mathsf{m}+1$.

Then, standard computations imply that

$$\hat\mu(B_{\mathfrak{r}}(w,\delta)) \ll \Upsilon\delta^{2\mathsf{m}+1} \qquad \text{for all } w \in \mathfrak{r} \text{ and all } 0<\delta\leq 1. \tag{5.7}$$

Moreover, since $e^{\mathsf{m}(\ell_1+\ell_2)}\delta_0 \leq e^{-\ell_1}$, we have

$$\int_0^1\int_0^1\int \varphi\big(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)x)\,\mathrm{d}\mu(w)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)x)\,\mathrm{d}\hat\mu(w)\,\mathrm{d}r_2\,\mathrm{d}r_1 + O(\mathcal{S}(\varphi)e^{-\ell_1}) \tag{5.8}$$

In consequence, we now investigate the integral in the second line of (5.8). The following observation guarantees that we may replace the integral in the second line of (5.8) by a ϱ-thickening of it along H. Let $\mathsf{B}^H = B^H_\varrho$. Then

$$\int_0^1\int_0^1\int \varphi\big(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)x)\,\mathrm{d}\hat\mu(w)\,\mathrm{d}r_2\,\mathrm{d}r_1 = O\big(\mathcal{S}(\varphi)\varrho^\star\big) \; + \\ \tfrac{1}{m_H(\mathsf{B}^H)}\int_0^1\int_0^1\int_{\mathsf{B}^H}\int \varphi\big(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\mathsf{h}\exp(w)x)\,\mathrm{d}\hat\mu(w)\,\mathrm{d}\mathsf{h}\,\mathrm{d}r_2\,\mathrm{d}r_1 \tag{5.9}$$

To see (5.9), recall that for all $\mathsf{h} \in \mathsf{B}^H$, $u_{r_2}\mathsf{h} = u^-_s a_q u_r$ where $|s|, |q| \ll \varrho$ and $\mathrm{d}r = (1+O(\varrho))\,\mathrm{d}r_2$. Furthermore, $a_{\ell_2}u^-_s a_q = u^-_{e^{-\ell_2}s}a_q a_{\ell_2}$. Thus

$$a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\mathsf{h} = a_{\ell_1}u_{r_1}g''a_q a_{\ell_2}u_r = g'a_q a_{\ell_1}u_{e^{-q}r_1}a_{\ell_2}u_r,$$

where $\|g'-I\|, \|g''-I\| \ll e^{-\ell_1-\ell_2}$. This, together with $|e^{-q}-1| \ll \varrho$ and the Folner properties of $\mathrm{d}r_1$ and $\mathrm{d}r_2$, implies the claim in (5.9).

In view of (5.9), we will work with the integral on the second line of (5.9). Recall that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$. Assuming ϱ_0 is small enough, every $g \in B^G_{10\varrho_0}$

can be uniquely written as $g = \mathsf{h}\exp(w) \in H\exp(\mathfrak{r})$, moreover, the map $g \mapsto (h, w)$ is a diffeomorphism onto its image. Recall that $\varrho \le \varrho_0$ and put $\mathrm{d}\nu = \frac{1}{m_H(\mathsf{B}^H)}\,\mathrm{d}\mathsf{h}\,\mathrm{d}(\exp\hat{\mu})$. Then (5.7) and $m_H(\mathsf{B}^H) \asymp \varrho^3$ imply that

$$\nu(B^G_\delta(g)) \ll \Upsilon\varrho^{-3}\delta^{\dim G} \quad \text{for all } g \in G \text{ and all } 0 < \delta \le 1. \tag{5.10}$$

Therefore, $\frac{\mathrm{d}(u_r\nu u_{-r})}{\mathrm{d}m_G} \ll \Upsilon$ for all $r \in [0,1]$. Applying Proposition 5.1, with $u_{r_2}\nu u_{-r_2}$ and $u_{r_2}x$ for all $r_0 \in [0,1]$, thus,

$$\int_0^1\int_0^1\int_G \varphi\big(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}gx\big)\,\mathrm{d}\nu(g)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \int \varphi\,\mathrm{d}m_X + O_c\big(\mathcal{S}(\varphi)(\eta + \Upsilon^{1/2}\varrho^{-3/2}e^{-\kappa_5\ell_1})\big).$$

This, (5.9), and (5.8) complete the proof. □

6. Modified energy and projection theorems

We begin by defining a modified (and localized) α-dimensional energy for finite subsets of $\mathbb{R}^d$. Fix a norm $\|\ \|$ on $\mathbb{R}^d$ (below we will apply this with $d = 5, 7, 11$). Let $\Theta \subset B_{\mathbb{R}^d}(0,1)$ be a finite set.

For $0 \le \delta < 1$ and $0 < \alpha < d$, define $\mathcal{G}^{(\alpha)}_{\Theta,\delta} : \Theta \to (0,\infty)$ as follows:

$$\mathcal{G}^{(\alpha)}_{\Theta,\delta}(w) = \sum_{w' \in \Theta\setminus\{w\}} \max(\|w - w'\|, \delta)^{-\alpha}.$$

When $\delta = 0$, we often write $\mathcal{G}^{(\alpha)}_{\Theta}(w)$ for $\mathcal{G}^{(\alpha)}_{\Theta,0}(w)$.

This notation will also be used for finite subsets of $\mathfrak{r}$, which is an $2\mathsf{m}+1$-dimensional $\mathrm{Ad}(H)$-irreducible representation, see §3.

The following projection theorem plays a crucial role in our argument.

6.1. **Theorem.** *Let $0 < \alpha < 2\mathsf{m}+1$, there exists $\varpi > 0$ so that the following holds. Specifically, the theorem holds with ϖ as in (6.2), see also (6.3).*

Let $0 < \mathsf{c} < 10^{-4}\alpha$, and $\ell > 0$. Let $\Theta \subset B_{\mathfrak{r}}(0,1)$, and assume that $\#\Theta$ is large (depending on c). Assume further that

$$\mathcal{G}^{(\alpha)}_{\Theta,\delta}(w) \le \Upsilon \quad \textit{for every } w \in \Theta, \tag{6.1}$$

for some $0 < \delta < 1$.

There exists a subset $J \subset [0,1]$ with

$$|[0,1] \setminus J| \le \mathsf{L}_{\mathsf{c}}|\log\delta|e^{-\star\mathsf{c}^2\ell},$$

so that the following holds. Let $r \in J$, then there exists $\Theta_r \subset \Theta$ with

$$\#(\Theta \setminus \Theta_r) \le \mathsf{L}_{\mathsf{c}}|\log\delta|e^{-\star\mathsf{c}^2\ell}\cdot(\#\Theta)$$

such that for all $w \in \Theta_r$, the following is satisfied

$$\mathcal{G}^{(\alpha)}_{\Theta(w),\delta'}(\mathrm{Ad}(a_\ell u_r)w) \le \mathsf{L}_{\mathsf{c}}e^{-\varpi\ell}\delta^{-\mathsf{c}}\Upsilon$$

where $\delta' = e^{\mathsf{m}\ell}\max(\delta, (\#\Theta)^{-1/\alpha})$, L_{c} *is a constant depending on* c, *and*

$$\Theta(w) = \{\mathrm{Ad}(a_\ell u_r)w' : w' \in \Theta_r, \|w - w'\| \le e^{-\mathsf{m}\ell}\}.$$

Before proceeding with the proof of this theorem, we explicate the value of ϖ for which we will prove the theorem. Let $0 < \alpha < 2\mathsf{m} + 1$, and define

$$\varpi = \begin{cases} \alpha\mathsf{m} - \sum_{i=1}^{\lceil\alpha\rceil - 1} i & 0 < \alpha \le 2\mathsf{m} - 1 \\ \max\bigl(\mathsf{m}(1 - 2\mathsf{m} + \alpha), \mathsf{m}(1 + 2\mathsf{m} - \alpha) - 1\bigr) & 2\mathsf{m} - 1 < \alpha \le 2\mathsf{m} \\ \mathsf{m}(1 + 2\mathsf{m} - \alpha) & 2\mathsf{m} \le \alpha \le 2\mathsf{m} + 1 \end{cases} \tag{6.2}$$

Note that for any $0 < \hat\kappa < \frac{1}{10\mathsf{m}}$, and any $0 < \alpha \le 2\mathsf{m} + 1 - \hat\kappa$, we have

$$\varpi \ge \min\{\alpha\mathsf{m}, \hat\kappa\mathsf{m}\}. \tag{6.3}$$

Indeed, if $0 < \alpha \le 1$, then $\varpi = \alpha\mathsf{m}$. Let us now assume $d - 1 < \alpha \le d$ for some $2 \le d \le 2\mathsf{m} - 1$, then

$$\varpi = \alpha\mathsf{m} - \tfrac{d(d-1)}{2} \ge \tfrac{(d-1)(2\mathsf{m}-d)}{2} \ge \tfrac{1}{2}.$$

Now consider $2\mathsf{m} - 1 < \alpha \le 2\mathsf{m}$. Indeed

$$\begin{aligned} &\varpi \ge \mathsf{m}(1 + 2\mathsf{m} - \alpha) - 1 \ge \tfrac{5}{3}\mathsf{m} - 1 && \text{if } 2\mathsf{m} - 1 \le \alpha \le 2\mathsf{m} - \tfrac{2}{3} \\ &\varpi \ge \mathsf{m}(1 - 2\mathsf{m} + \alpha) \ge \tfrac{\mathsf{m}}{3} && \text{if } 2\mathsf{m} - \tfrac{2}{3} \le \alpha \le 2\mathsf{m} \end{aligned}$$

The claim in (6.3) follows.

The proof of Theorem 6.1 relies primarily on a result by Gan, Guo, and Wang [GGW22, Thm. 2.1]. More specifically, the following lemma forms the crux of the proof of Theorem 6.1. The proof of this lemma, in turn, depends on Theorem C.2 which is [GGW22, Thm. 2.1] tailored to our application.

6.2. **Lemma.** *Let* $0 < \alpha < 2\mathsf{m} + 1$, $0 < b_1 \le 1$ *and* $\ell > 0$. *Let* ρ *be the uniform measure on a finite set* $\Theta \subset B_{\mathfrak{r}}(0, 1)$ *satisfying the following*

$$\rho(B_{\mathfrak{r}}(w, b)) \le \Upsilon b^\alpha \qquad \text{for all } w \text{ and all } b \ge b_1 \tag{6.4}$$

Let $0 < \mathsf{c} < 10^{-4}$. *For every* $1 \ge b \ge e^{\mathsf{m}\ell} b_1$, *there exists a subset* $J_{\ell,b} \subset [0, 1]$ *with* $|[0, 1] \setminus J_{\ell,b}| \le C'_{\mathsf{c}} e^{-\star\mathsf{c}^2\ell}$ *so that the following holds. Let* $r \in J_{\ell,b}$, *then there exists a subset* $\Theta_{\ell,b,r} \subset \Theta$ *with*

$$\rho(\Theta \setminus \Theta_{\ell,b,r}) \le C'_{\mathsf{c}} e^{-\star\mathsf{c}^2\ell}$$

such that for all $w \in \Theta_{\ell,b,r}$ *we have*

$$\rho\bigl(\{w' \in \Theta_{\ell,b,r} : \|\mathrm{Ad}(a_\ell u_r)w - \mathrm{Ad}(a_\ell u_r)w'\| \le b\}\bigr) \le C'_{\mathsf{c}}\Upsilon e^{-\varpi\ell} b^{\alpha - \mathsf{c}}$$

where ϖ *is as in* (6.2).

Proof. We first establish the claim for $0 < \alpha \le 2\mathsf{m} - 1$, and also obtain one of the bounds in the definition $\varpi = \max\bigl(\mathsf{m}(1 - 2\mathsf{m} + \alpha), \mathsf{m}(1 + 2\mathsf{m} - \alpha) - 1\bigr)$ for $2\mathsf{m} - 1 < \alpha \le 2\mathsf{m}$, namely $\mathsf{m}(1 - 2\mathsf{m} + \alpha)$.

To that end, let $k = \lceil\alpha\rceil$ and write

$$\varpi' = \alpha\mathsf{m} - \textstyle\sum_{i=1}^{k-1} i.$$

For every $1 \le d \le 2\mathsf{m}+1$, let $\mathfrak{r}_d$ denote the space spanned by vectors with weight $\mathsf{m}, \dots, \mathsf{m}-d+1$. Let $\pi_d : \mathfrak{r} \to \mathfrak{r}_d$ denote the orthogonal projection. Apply Theorem C.2 with $b' = e^{-\mathsf{m}\ell} b \ge b_1$ and $\mathsf{c}/2$. Then for every $r \in J_{b'}$ and all $w \in \Theta_{b',r}$, we have

$$\rho\big(\{w' \in \Theta : \|\pi_k(\mathrm{Ad}(u_r)w)) - \pi_k(\mathrm{Ad}(u_r)w')\| \le b'\}\big) \le C_{\mathsf{c}} \Upsilon (b')^{\alpha - \frac{\mathsf{c}}{2}}. \tag{6.5}$$

Let $P \subset \mathfrak{r}_k$ denote the box $b' \times e^{\ell} b' \cdots \times e^{(k-1)\ell} b'$ centered at the origin, where the directions correspond to the weight spaces for a_t in decreasing order. Then $P + \pi_k(\mathrm{Ad}(a_\ell u_r)w)$ can be covered with $\ll e^{\sum_{i=1}^{k-1} i\ell}$ many boxes of size b'. Thus (6.5), applied with $2b'$, implies that

$$\begin{aligned} \pi_k(\rho|_{\Theta_{\ell,b,r}})(P + \pi_k(\mathrm{Ad}(a_\ell u_r)w)) &\le (C_{\mathsf{c}} \Upsilon (2b')^{\alpha - \frac{\mathsf{c}}{2}}) \cdot e^{\sum_{i=1}^{k-1} i\ell} \\ &\le 2^\alpha C_{\mathsf{c}} \Upsilon e^{-\varpi' \ell} b^{\alpha - \mathsf{c}}. \end{aligned} \tag{6.6}$$

Note also that

$$\begin{aligned} &\{w' \in \Theta_{b',r} : \|\mathrm{Ad}(a_\ell u_r)w) - \mathrm{Ad}(a_\ell u_r)w'\| \le b\} \subset \\ &\qquad \{w' \in \Theta_{b',r} : \|\pi_k(\mathrm{Ad}(a_\ell u_r)w)) - \pi_k(\mathrm{Ad}(a_\ell u_r)w')\| \le b\} \subset P_{\ell,r,w}. \end{aligned}$$

This and (6.6) show that

$$\begin{aligned} &\rho|_{\Theta_{\ell,b,r}}\big(\{w' \in \mathfrak{r} : \|\mathrm{Ad}(a_\ell u_r)w) - \mathrm{Ad}(a_\ell u_r)w'\| \le b\}\big) \le \\ &\qquad\qquad 2^\alpha C_{\mathsf{c}} \Upsilon e^{-\varpi' \ell} b^{\alpha - \mathsf{c}} \end{aligned} \tag{6.7}$$

which establishes the claim when $0 < \alpha \le 2\mathsf{m} - 1$, as well as when $2\mathsf{m} - 1 < \alpha \le 2\mathsf{m}$ for $\varpi' = \mathsf{m}(1 - 2\mathsf{m} + \alpha)$ (when this is positive).

We now turn to the case when $2\mathsf{m} - 1 < \alpha \le 2\mathsf{m} + 1$. For a vector $v \in \mathfrak{r}$ and $1 \le i \le 2\mathsf{m} + 1$, let v_i denote the component of v in the weight space $\mathsf{m} - i + 1$. Then for every $w \in \Theta$ and all b, we have

$$\begin{aligned} &\{w' \in \Theta : \|\mathrm{Ad}(a_\ell u_r)w) - \mathrm{Ad}(a_\ell u_r)w'\| \le b\} = \\ &\qquad\qquad \{w' \in \Theta : \forall i, |(\mathrm{Ad}(u_r)w)_i - (\mathrm{Ad}(u_r)w')_i| \le e^{(-\mathsf{m}+i-1)\ell} b\}. \end{aligned} \tag{6.8}$$

Put $b' = e^{\mathsf{m}\ell} b$, and let $P \subset \mathfrak{r}$ be the box

$$e^{-2\mathsf{m}\ell} b' \times e^{(-2\mathsf{m}+1)\ell} \times \cdots \times e^{-\ell} b' \times b'$$

centered at the origin, i.e., i-th weight space has size $e^{-\mathsf{m}-i} b'$. In view of (6.8), we will estimate the measure of sets of the form $P + w$ for $w \in \Theta$.

To that end, cover Θ with half-open disjoint boxes $\{B_q : q \in \mathcal{Q}\}$ of size b'. For all B_q, let $\rho_q = \frac{1}{\rho(B_q)} \rho|_{B_q}$, and let $\tilde\rho_q$ denote the image of ρ_q under $z \mapsto \frac{1}{b'}(z - z_q)$, where z_q is the center of B_q. Then for all $\delta \ge b_1/b'$,

$$\begin{aligned} \tilde\rho_q(B_{\mathfrak{r}}(v, \delta)) &= \tfrac{1}{\rho(B_q)} \rho(B_q \cap B_{\mathfrak{r}}(v + z_q, b'\delta)) \\ &\le \tfrac{\Upsilon}{\rho(B_q)} (b'\delta)^\alpha = \tfrac{\Upsilon e^{\alpha \mathsf{m}\ell} b^\alpha}{\rho(B_q)} \delta^\alpha \end{aligned} \tag{6.9}$$

In particular, if $b \ge e^{\mathsf{m}\ell} b_1$, then $e^{-2\mathsf{m}\ell} \ge \frac{b_1}{b'}$, thus (6.9) holds for $\delta \ge e^{-2\mathsf{m}\ell}$.

Let us first assume $2\mathsf{m}-1<\alpha\leq 2\mathsf{m}$. Put $P'=\frac{1}{b'}P$. Then using Theorem C.2, with $k=2\mathsf{m}-1$ and $\tilde\rho_q$ for all $q\in\mathcal{Q}$, there exists a subset $J_q\subset[0,1]$ with $|[0,1]\setminus J_q|\ll e^{-\star\mathsf{c}^2\ell}$ and for every $r\in J_q$ a subset $\Theta_{q,r}\subset\Theta\cap B_q$ with with $\rho_q(\Theta\setminus\Theta_{q,r})\ll e^{-\star\mathsf{c}^2\ell}$ so that for all $v\in\mathfrak{r}$,

$$\begin{aligned}\tilde\rho_q|_{\Theta_{q,r}}(\mathrm{Ad}(u_{-r})P'+v)&\leq C_{\mathsf{c}}\tfrac{\Upsilon e^{\alpha\mathsf{m}\ell}b^\alpha}{\rho(B_q)}e^{(-\sum_{i=2}^{2\mathsf{m}}i+\mathsf{c})\ell}\\&\leq C_{\mathsf{c}}\tfrac{\Upsilon}{\rho(B_q)}e^{(\alpha\mathsf{m}-\sum_{i=2}^{2\mathsf{m}}i)\ell}b^{\alpha-\mathsf{c}}\\&=C_{\mathsf{c}}\tfrac{\Upsilon}{\rho(B_q)}e^{-(\mathsf{m}(2\mathsf{m}+1-\alpha)-1)\ell}b^{\alpha-\mathsf{c}}.\end{aligned}$$

Equip $\mathcal{Q}\times[0,1]$ with $\sigma\times\mathrm{Leb}$ where $\sigma(q)=\rho(B_q)$. By Fubini's theorem, there is $J\subset[0,1]$ with $|[0,1]\setminus J|\ll e^{-\star\mathsf{c}^2\ell}$, and for all $r\in J$ a subset $\mathcal{Q}_r\subset\mathcal{Q}$ satisfying that $\sum_{q\notin\mathcal{Q}_r}\rho(B_q)\ll e^{-\star\mathsf{c}^2\ell}$ so that $r\in J_q$ for all $q\in\mathcal{Q}_r$.

For every $r\in J$, let $\Theta_{\ell,b,r}=\cup_{q\in\mathcal{Q}_r}\Theta_{q,r}$. Since $P+\mathrm{Ad}(u_r)w$ is contained in $O(1)$ many B_q's, the above and the definition of $\tilde\rho_q$ imply that

$$\rho|_{\Theta_{\ell,b,r}}(\mathrm{Ad}(u_{-r})P+w)\ll C_{\mathsf{c}}\Upsilon e^{-(\mathsf{m}(2\mathsf{m}+1-\alpha)-1)\ell}b^{\alpha-\mathsf{c}}.$$

In view of (6.8), this and (6.7) establish the claim when $2\mathsf{m}-1<\alpha\leq 2\mathsf{m}$.

The proof in the case $2\mathsf{m}\leq\alpha\leq 2\mathsf{m}+1$ is similar. Indeed, applying Theorem C.2, with $k=2\mathsf{m}$ and $\{\tilde\rho_q:q\in\mathcal{Q}\}$, we obtain $J\subset[0,1]$, for all $r\in J$ the subset $\mathcal{Q}_r\subset\mathcal{Q}$ and for each $q\in\mathcal{Q}_r$, the set $\Theta_{q,r}$ as above so that

$$\begin{aligned}\tilde\rho_q|_{\Theta_{q,r}}(P'+\mathrm{Ad}(u_r)v)&\leq C_{\mathsf{c}}\tfrac{\Upsilon e^{\alpha\mathsf{m}\ell}b^\alpha}{\rho(B_q)}e^{(-\sum_{i=1}^{2\mathsf{m}}i+\mathsf{c})\ell}\\&\leq C_{\mathsf{c}}\tfrac{\Upsilon}{\rho(B_q)}e^{(\alpha\mathsf{m}-\sum_{i=1}^{2\mathsf{m}}i)\ell}b^{\alpha-\mathsf{c}}\\&=C_{\mathsf{c}}\tfrac{\Upsilon}{\rho(B_q)}e^{-\mathsf{m}(2\mathsf{m}+1-\alpha)\ell}b^{\alpha-\mathsf{c}},\end{aligned}$$

for all $v\in\mathfrak{r}$. Put $\Theta_{\ell,b,r}=\cup_{q\in\mathcal{Q}_r}\Theta_{q,r}$. Then

$$\rho|_{\Theta_{\ell,b,r}}(P+\mathrm{Ad}(u_r)w)\ll C_{\mathsf{c}}\Upsilon e^{-\mathsf{m}(2\mathsf{m}+1-\alpha)\ell}b^{\alpha-\mathsf{c}}$$

which, thanks to (6.8), gives the claim when $2\mathsf{m}\leq\alpha<2\mathsf{m}+1$. $\square$

6.3. **Proof of Theorem 6.1.** Recall the definition of ϖ from (6.2).

Let us write $\bar\Upsilon=\frac{\Upsilon}{\#\Theta}$. To simplify the notation, $\mathrm{Ad}(a_\ell u_r)(w)$ will be denoted by $\xi_{\ell,r}(w)$ in the proof.

Let ρ denote the uniform measure on Θ. By (6.1),

$$\rho\big(B(w,b)\cap\Theta\big)\leq 2\bar\Upsilon b^\alpha\quad\text{for all } b\geq\max(\delta,(\#\Theta)^{-\frac{1}{\alpha}})=:b_1\tag{6.10}$$

Thus, Theorem C.2 and Lemma 6.2 are applicable with ρ and $b\geq b_1$.

Let $k_1=-\lceil\log b_1\rceil$ and $k_2=\mathsf{m}\ell$. Apply Lemma 6.2 with $\mathsf{c}/2$ and $b=e^{-k}$ for all $k_2\leq k\leq k_1$. Let $J=\bigcap_{k=k_2}^{k_1}J_{\ell,e^{-k}}$ where $J_{\ell,b}$ is as in Lemma 6.2. Then that lemma implies that

$$|[0,1]\setminus J|\ll_{\mathsf{c}}\sum_{k_2}^{k_1}e^{-\star\mathsf{c}^2\ell}\ll_{\mathsf{c}}|\log\delta|e^{-\star\mathsf{c}^2\ell}.$$

For every $r \in [0,1]$, let

$$\Theta_r = \bigcap_{k=k_2}^{k_1-\mathsf{m}\ell} \Theta_{\ell,e^{-k},r} \tag{6.11}$$

where $\Theta_{\ell,e^{-k},r}$ is given by Lemma 6.2. Applying that lemma, thus, we have

$$\rho(\Theta \setminus \Theta_r) \ll_{\mathsf{c}} |\log\delta| e^{-\star\mathsf{c}^2\ell}.$$

Define J and for every $r \in J$, Θ_r as above for this case. For every $w \in \Theta_r$ and every $k < k_1 - \mathsf{m}\ell$, let

$$\Theta(w,k) = \{w' \in \Theta_r : e^{-k-1} < \|\xi_{\ell,r}(w) - \xi_{\ell,r}(w')\| \le e^{-k}\};$$

define $\Theta(w, k_1 - \mathsf{m}\ell)$ similarly but without imposing a lower bound. Then by Lemma 6.2, for all $k \le k_1 - \mathsf{m}\ell$ we have

$$\rho(\Theta(w,k)) \le C'_{\mathsf{c}} \bar\Upsilon \delta^{-\mathsf{c}/2} e^{-\varpi\ell} e^{-\alpha k}. \tag{6.12}$$

Put $k'_1 = k_1 - \mathsf{m}\ell$. From the above we conclude

$$\begin{aligned}\mathcal{G}^{(\alpha)}_{\Theta(w),\delta'}(w) &= \sum_{w'\in\Theta(w)} \max(\|\xi_{\ell,r}(w) - \xi_{\ell,r}(w')\|, \delta')^{-\alpha}\\ &\le \rho(\Theta(w,k'_1)) \cdot e^{k'_1\alpha} + \sum_{k=k_2}^{k'_1-1} \sum_{\Theta(w,k)} \|\xi_{\ell,r}(w) - \xi_{\ell,r}(w')\|^{-\alpha}\\ &\ll_{\mathsf{c}} \sum_{k_2}^{k'_1} \bar\Upsilon \delta^{-\frac{\mathsf{c}}{2}} e^{-\varpi\ell} e^{-\alpha k} e^{\alpha k} \cdot (\#\Theta)\\ &\ll_{\mathsf{c}} |\log\delta| e^{-\varpi\ell}\delta^{-\frac{\mathsf{c}}{2}}\Upsilon \ll_{\mathsf{c}} e^{-\varpi\ell}\delta^{-\mathsf{c}}\Upsilon,\end{aligned}$$

where we used (6.12) in the third line, $\bar\Upsilon = \frac{\Upsilon}{\#\Theta}$ and $k_1 \le |\log\delta|$ in the second to last inequality, and $|\log\delta|\delta^{-\mathsf{c}/2} \le \delta^{-\mathsf{c}}$ in the last inequality.

The proof is complete. □

6.4. **Linear algebra lemma and the energy.** In this section, we detail how Lemma 3.5 can be employed to improve the initial estimate provided by Proposition 4.6. While the formulation of Lemma 6.5 below bears similarities to Theorem 6.1 albeit with small α, a key observation is that, unlike that theorem, this step avoids any loss of scales.

In the proof of Proposition 8.1, we will use Lemma 6.5 to refine the initial estimate provided by Proposition 4.6, achieving a positive initial dimension. Subsequently, Theorem 6.1 will be applied to further improve this dimension, bringing it close to full dimension.

6.5. **Lemma.** *Let $0 < \alpha \le \frac{1}{2\mathsf{m}+1}$ and $\ell > 0$. Let $\Theta \subset B_{\mathfrak{r}}(0,1)$ be a finite set satisfying that*

$$\mathcal{G}^{(\alpha)}_{\Theta}(w) \le \Upsilon \quad \textit{for every } w \in \Theta, \tag{6.13}$$

for some $\Upsilon \ge 1$.

There exists a subset $J \subset [0,1]$ with $|[0,1] \setminus J| \le \mathsf{L}_\alpha e^{-\star\alpha\ell}$, so that the following holds. Let $r \in J$, then there exists a subset $\Theta_r \subset \Theta$ with

$$\#(\Theta \setminus \Theta_r) \le \mathsf{L}_\alpha e^{-\star\alpha\ell} \cdot (\#\Theta)$$

such that for all $w \in \Theta_r$ we have

$$\mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)\Theta}(\mathrm{Ad}(a_\ell u_r)w) \le \mathsf{L}_\alpha e^{-\frac{3\alpha}{4}\ell}\Upsilon$$

where L_α is a constant depending on α.

Proof. Recall from Lemma 3.5 that for all $0 \ne v \in \mathfrak{r}$, we have

$$\int_0^1 \|a_\ell u_r v\|^{-\alpha}\,\mathrm{d}r \le C_\alpha e^{-\alpha\ell}\|v\|^{-\alpha}$$

where C_α depends on α. This and the definition of $\mathcal{G}^{(\alpha)}_\Theta$ imply that

$$\begin{aligned}\int_0^1 \mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)\Theta}(\mathrm{Ad}(a_\ell u_r)w)\,\mathrm{d}r &= \int_0^1 \sum_{w\ne w'} \|\mathrm{Ad}(a_\ell u_r)(w-w')\|^{-\alpha}\,\mathrm{d}r\\ &\le C_\alpha e^{-\alpha\ell}\sum_{w\ne w'}\|w-w'\|^{-\alpha}\\ &= C_\alpha e^{-\alpha\ell}\mathcal{G}^{(\alpha)}_\Theta(w) \le C_\alpha e^{-\alpha\ell}\Upsilon,\end{aligned}$$

for all $w \in \Theta$.

Applying Chebyshev's inequality, we conclude that if for $w \in \Theta$ we put

$$J(w) = \left\{r \in [0,1] : \mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)\Theta}(\mathrm{Ad}(a_\ell u_r)w) > C_\alpha e^{-3\alpha\ell/4}\Upsilon\right\},$$

then $|J(w)| \le e^{-\alpha\ell/4}$. Let

$$\begin{aligned}\Xi &= \left\{(w,r) \in \Theta \times [0,1] : \mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)\Theta}(\mathrm{Ad}(a_\ell u_r)w) \le C_\alpha e^{-3\alpha\ell/4}\Upsilon\right\}\\ &= \{(w,r) : w \in \Theta, r \in [0,1] \setminus J(w)\}.\end{aligned}$$

The above discussion implies that $\rho \times \mathrm{Leb}\,(\Xi) > 1 - O(e^{-\alpha\ell/4})$, where ρ denotes the uniform measure on Θ. This and Fubini's theorem imply that there exists $J \subset [0,1]$ with $|[0,1] \setminus J| \ll e^{-\star\alpha\ell}$, so that for every $r \in J$

$$\rho(\{w : (w,r) \in \Xi\}) > 1 - O(e^{-\star\alpha\ell}).$$

For every $r \in J$, let

$$\Theta_r = \{w : (w,r) \in \Xi\} \tag{6.14}$$

The claim in the lemma thus follows with J and Θ_r as above. $\square$

7. Margulis functions and projection theorems

Let us put

$$\mathsf{E} = \{u_s^- : |s| \le \beta\} \cdot \{a_d : |d| \le \beta\} \cdot \{u_r : |r| \le \eta\},$$

see (3.5). Let $F \subset B_{\mathfrak{r}}(0,\beta)$ be a finite set. Let $y \in X_\eta$ be so that $(h,w) \mapsto h\exp(w)y$ is injective over $\mathsf{E} \times F$ and

$$\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\} \subset X_\eta \tag{7.1}$$

For every $(h,z) \in H \times \mathcal{E}$, define

$$I_{\mathcal{E}}(h,z) := \big\{w \in \mathfrak{r} : \|w\| < \mathrm{inj}(hz),\ \exp(w)hz \in h\mathcal{E}\big\}. \tag{7.2}$$

Note that $I_{\mathcal{E}}(h,z)$ contains 0 for all $z \in \mathcal{E}$, moreover, since E is bounded, $I_{\mathcal{E}}(h,z)$ is a finite set for all $(h,z) \in H \times \mathcal{E}$.

Fix some $0 < \alpha \le 2\mathsf{m}+1$. For every $0 \le \delta < 1$, define a modified Margulis function $f^{(\alpha)}_{\mathcal{E},\delta} : H \times \mathcal{E} \to [1,\infty)$ as follows.

$$f^{(\alpha)}_{\mathcal{E},\delta}(h,z) = \sum_{0 \ne w \in I_{\mathcal{E}}(h,z)} \max(\|w\|,\delta)^{-\alpha}$$

In this paper, we will primarily use the above objects with $h = e$. Hence, and in order to simplify the notation, we denote $I_{\mathcal{E}}(e,z)$ and $f^{(\alpha)}_{\mathcal{E},\delta}(e,z)$ by $I_{\mathcal{E}}(z)$ and $f^{(\alpha)}_{\mathcal{E},\delta}(z)$, respectively.

The modified Margulis function and the modified energy discussed in §6 are closely related. Specifically, the following lemma [LMW22, Lemma 9.2] establishes, in a general sense, that bounding one of these quantities implies a bound on the other, with the exception of potential *edge effects*. When $\mathfrak{r}$ is a Lie subalgebra, this connection becomes more straightforward. However, in the general case, understanding the relationship between $I(z)$ and $I(\exp(w)z)$ demands further elaboration; see [LMW22, Lemma 9.2] for details.

7.1. **Lemma.** *Let the notation be as above, and assume that*

$$f^{(\alpha)}_{\mathcal{E},\delta}(z) \le \Upsilon \qquad \textit{for all } z \in \mathcal{E}.$$

Then for every $z \in \big(\overline{\mathsf{E} \setminus \partial_{5\beta^2}\mathsf{E}}\big).\{\exp(w)y : w \in F\}$ *and all* $w \in I_{\mathcal{E}}(z)$,

$$\mathcal{G}^{(\alpha)}_{I_{\mathcal{E}}(z),\delta}(w) \ll \Upsilon,$$

where $\mathcal{G}$ *is defined as in §6.*

An iterative dimension improvement lemma. The following lemma outlines a general iterative process for improving the dimension under suitable projection theorems. Readers may compare it to [LMW22, Lemmas 9.1 and 10.7] and [LM23, Lemma 7.10]. In the next section, we apply Lemma 7.2 to establish Proposition 8.1, a central component of our argument.

Let $0 < \varepsilon < 10^{-10}$ be a small parameter, and $t > 0$ a large parameter. We also fix some $0 < \varepsilon' < 10^{-6}\varepsilon$, and put

$$\ell = \tfrac{\varepsilon t}{10\mathsf{m}}, \quad \beta = e^{-\varepsilon' t}, \quad \text{and} \quad \eta^2 = \beta.$$

Throughout this section, we fix a large parameter d_0. The results of this section will be applied with $\mathsf{d}_0 = 3t + 3d_{\mathrm{fn}}\ell$, where d_{fn} is the number of steps we need to apply Lemma 7.2 in the proof of Proposition 8.1, and is explicated in the next section.

Recall that $F \subset B_{\mathfrak{r}}(0, \beta)$, and

$$\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\} \subset X_\eta.$$

We assume $\mathcal{E}$ is equipped with an admissible measure $\mu_{\mathcal{E}}$, see §D.3.

7.2. **Lemma.** *Let $\frac{1}{2\mathsf{m}+1} \le \alpha < 2\mathsf{m}+1$, and define ϖ as in (6.2) (ϖ is some explicit function of α that goes to zero as $\alpha \to 2\mathsf{m}+1$). Let $\Upsilon \le e^{\mathsf{D}t}$ for some $\mathsf{D} > 0$, and assume that*

$$f^{(\alpha)}_{\mathcal{E},\delta}(z) \le \Upsilon \qquad \text{for all } z \in \mathcal{E}$$

where $\delta = 0$ if $\alpha = \frac{1}{2\mathsf{m}+1}$ and $e^{-\mathsf{D}t} \le \delta < 1$ otherwise.

The following holds so long as t is large enough, depending on X, D, and $2\mathsf{m}+1-\alpha$, see (7.6). There exists a collection $\{\mathcal{E}_i : 1 \le i \le N\}$ of sets

$$\mathcal{E}_i = \mathsf{E}.\{\exp(w)y_i : w \in F_i\} \subset X_\eta$$

where for every $1 \le i \le N$, $F_i \subset B_{\mathfrak{r}}(0,\beta)$ with

$$\beta^{4\mathsf{m}+6} \cdot (\#F) \le \#F_i \le \beta^{4\mathsf{m}+4}\#F,$$

and for every i, an admissible measure $\mu_{\mathcal{E}_i}$, so that both of the following hold

(1) For all $1 \le i \le N$ and all $z \in \mathsf{E}.\{\exp(w)y_i : w \in F_i\}$,

$$f^{(\alpha)}_{\mathcal{E}_i,\delta'}(z) \le e^{-\frac{\varpi}{2}\ell} \cdot \Upsilon + e^{\varepsilon t} \cdot (\#F_i) \tag{7.3}$$

where $\delta' = 0$ if $\alpha = \frac{1}{2\mathsf{m}+1}$ and $\delta' = e^{\mathsf{m}\ell}\max(\delta, (\#F)^{-\frac{1}{\alpha}})$ otherwise.

(2) For all $0 < \mathsf{d} \le \mathsf{d}_0 - \ell$, all $|s| \le 2$, and every $\varphi \in C_c^\infty(X)$, we have

$$\int_0^1 \int \varphi(a_{\mathsf{d}} u_s a_\ell u_r z)\, \mathrm{d}\mu_{\mathcal{E}}(z)\mathrm{d}r = \sum_i c_i \int \varphi(a_{\mathsf{d}} u_s z)\, \mathrm{d}\mu_{\mathcal{E}_i}(z) + O(\mathrm{Lip}(\varphi)\beta^{\kappa_6})$$

where $0 \le c_i \le 1$ and $\sum_i c_i = 1 - O(\beta^{\kappa_6})$, $\mathrm{Lip}(\varphi)$ is the Lipschitz norm of φ, and κ_6 and the implied constants depend on X.

The proof of Lemma 7.2 relies on Lemma 6.5 (for $\alpha = 1/(2\mathsf{m}+1)$) and Theorem 6.1 (for $\alpha > 1/(2\mathsf{m}+1)$) and will be completed in several steps. The basic idea is straightforward: By applying these results to the sets $I_{\mathcal{E}}(z)$ for all $z \in \mathcal{E}$ and a few applications of Fubini's theorem, we obtain a *large* subset $L \subset [0,1]$ and for each $r \in L$ a *large* subset $\mathcal{E}(r) \subset \mathcal{E}$ where the upper bound on the modified Margulis function improves under the application of $a_\ell u_r$. We then trim and smear $a_\ell u_r \mathcal{E}(r)$ using the results in [LMW22, §6–8], see also §D.3, to establish the lemma. The details follow.

Proof of Lemma 7.2. Let us write

$$\mathsf{E}' = \overline{\mathsf{E} \setminus \partial_{5\beta^2}\mathsf{E}} \quad \text{and} \quad \mathcal{E}' = \mathsf{E}'.\{\exp(w)y : w \in F\}.$$

Recall from Lemma 7.1 that our condition

$$f^{(\alpha)}_{\mathcal{E},\delta}(\bar{z}) \leq \Upsilon$$

implies that for every $\bar{z} \in \mathcal{E}'$ and all $v \in I_{\mathcal{E}}(\bar{z})$, we have $\mathcal{G}^{(\alpha)}_{I_{\mathcal{E}}(\bar{z}),\delta}(v) \ll \Upsilon$.

Let $\bar{z} \in \mathcal{E}'$, we will estimate

$$\mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)I,\delta'}(\mathrm{Ad}(a_\ell u_r)v)$$

for a certain subset $I \subset I_{\mathcal{E}}(\bar{z})$ which will be explicated below.

First let us assume $\alpha = \frac{1}{2\mathsf{m}+1}$, then $\varpi = \alpha$ and $\delta' = 0$. In this case, Lemma 6.5 applied with $\Theta = I_{\mathcal{E}}(\bar{z})$ implies that there is $\theta > 0$ (depending on m) so that for all large enough ℓ the following holds. Denote by $J_{\bar{z}}$ the set $J \subset [0, 1]$ in Lemma 6.5, and for all $r \in J_{\bar{z}}$, let $I_{\bar{z},r}$ be the set Θ_r in Lemma 6.5 applied with $\Theta = I_{\mathcal{E}}(\bar{z})$, see (6.14). Then

$$\#(I_{\mathcal{E}}(\bar{z}) \setminus I_{\bar{z},r}) \leq e^{-\theta\ell} \cdot (\#I_{\mathcal{E}}(\bar{z})), \tag{7.4}$$

and for every $v \in I_{\bar{z},r}$, we have

$$\begin{aligned}\mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)I^{\mathrm{int}}_{\bar{z},r}(v),\delta'}(\mathrm{Ad}(a_\ell u_r)v) &\leq \mathcal{G}^{(\alpha)}_{\mathrm{Ad}(a_\ell u_r)I_{\mathcal{E}}(\bar{z})}(\mathrm{Ad}(a_\ell u_r)v) \\ &\ll e^{-3\alpha\ell/4}\Upsilon\end{aligned} \tag{7.5}$$

where $I^{\mathrm{int}}_{\bar{z},r}(v) = \{v' \in I_{\mathcal{E}}(\bar{z}) : \|v - v'\| \leq e^{-\mathsf{m}\ell}\}$ and we used $\varpi = \alpha$ and $\delta' = 0$ in the case at hand.

We now turn to the case $\alpha > \frac{1}{2\mathsf{m}+1}$. The goal is to establish a similar estimate, albeit using Theorem 6.1 in this case. To that end, first let c be small enough so that $e^{\mathsf{cD}t} < e^{\varpi\ell/4}$.

$$e^{-\varpi\ell}\delta^{-\mathsf{c}}\Upsilon \leq e^{-3\varpi\ell/4}\Upsilon, \tag{7.6}$$

where we used $\delta \geq e^{-\mathsf{D}t}$. Now apply Theorem 6.1 with $\Theta = I_{\mathcal{E}}(\bar{z})$ and this c. There is $\theta > 0$ so that for all large enough ℓ the following holds. Denote by $J_{\bar{z}}$ the set $J \subset [0, 1]$ in Theorem 6.1, and for all $r \in J_{\bar{z}}$, let $I_{\bar{z},r} \subset I_{\mathcal{E}}(\bar{z})$ be the set Θ_r in Theorem 6.1 applied with $\Theta = I_{\mathcal{E}}(\bar{z})$, see (6.11). Then

$$\#(I_{\mathcal{E}}(\bar{z}) \setminus I_{\bar{z},r}) \leq e^{-\theta\mathsf{c}^2\ell} \cdot (\#I_{\mathcal{E}}(\bar{z})), \tag{7.7}$$

and for every $v \in I_{\bar{z},r}$, we have

$$\mathcal{G}_{\mathrm{Ad}(a_\ell u_r)I^{\mathrm{int}}_{\bar{z},r}(v),\delta'}(\mathrm{Ad}(a_\ell u_r)v) \ll e^{-3\varpi\ell/4}\Upsilon \tag{7.8}$$

where $\delta' = e^{\mathsf{m}\ell}\max(\delta, (\#F)^{-1/\alpha})$,

$$I^{\mathrm{int}}_{\bar{z},r}(v) = \{v' \in I_{\bar{z},r} : \|v - v'\| \leq e^{-\mathsf{m}\ell}\},$$

and we used (7.6).

The set $L_{\mu_{\mathcal{E}}}$. Equip $\mathcal{E}\times[0,1]$ with $\sigma:=\mu_{\mathcal{E}}\times\mathrm{Leb}$, where Leb denotes the normalized Lebesgue measure on $[0,1]$. Extend the definition of $I_{\bar z,r}$ to all $r\in[0,1]$ by letting $I_{\bar z,r}=\emptyset$ if $r\notin J_{\bar z}$, and put

$$Z=\bigl\{(\bar z,r)\in\hat{\mathcal{E}}\times[0,1]:\#(I_{\mathcal{E}}(\bar z)\setminus I_{\bar z,r})\leq e^{-\theta\mathsf{c}^2\ell}\cdot(\#I_{\mathcal{E}}(\bar z))\bigr\},\tag{7.9}$$

where $\hat{\mathcal{E}}=\hat{\mathsf{E}}.\{\exp(w)y:w\in F\}$ and $\hat{\mathsf{E}}=\overline{\mathsf{E}\setminus\partial_{200\beta^2}\mathsf{E}}$.

Then, using (7.4) if $\alpha=\frac{1}{2\mathsf{m}+1}$ or (7.7) when $\alpha>\frac{1}{2\mathsf{m}+1}$, we have that

$$\{(\bar z,r):r\in J_{\bar z}\}\subset Z\qquad\text{for all }\bar z\in\hat{\mathcal{E}}.$$

Recall moreover that $\mu_{\mathcal{E}}(\mathcal{E}\setminus\hat{\mathcal{E}})\ll\beta$. We conclude that

$$\sigma(\mathcal{E}\times[0,1]\setminus Z)\ll\beta+e^{-\star\mathsf{c}^2\ell}\ll e^{-\star\mathsf{c}^2\ell},$$

where we assumed $\mathsf{c}\leq\kappa$. This and Fubini's theorem imply that there is a subset $L=L_{\mu_{\mathcal{E}}}\subset[0,1]$ with $|[0,1]\setminus L|\ll e^{-\star\mathsf{c}^2\ell}$ so that

$$\mu_{\mathcal{E}}\bigl(\mathcal{E}\setminus Z_r\bigr)\ll e^{-\star\mathsf{c}^2\ell}\qquad\text{for all }r\in L,\tag{7.10}$$

where $Z_r=\{\bar z\in\hat{\mathcal{E}}:(\bar z,r)\in Z\}$.

Decomposing $a_\ell u_r\mu_{\mathcal{E}}$ as a convex combination. Fix a maximal $e^{-3\mathsf{d}_0}$ separated subset $\mathcal{L}$ of L, and d and s as in the statement of Lemma 7.2. Applying Lemma D.4, see also [LMW22, Lemma 8.9], for every $r\in\mathcal{L}$ we can write

$$\int\varphi(a_{\mathsf{d}}u_sa_\ell u_rz)\,\mathrm{d}\mu_{\mathcal{E}}(z)=\sum_i c'_{i,r}\int\varphi(a_{\mathsf{d}}u_sz)\,\mathrm{d}\mu_{\mathcal{E}'_{i,r}}(z)+O(\beta^\star\,\mathrm{Lip}(\varphi)),\tag{7.11}$$

where the $\mathcal{E}'_{i,r}$s are sets of the form

$$\mathcal{E}'_{i,r}=\mathsf{E}.\{\exp(w)y_{i,r}:w\in F'_{i,r}\}\subset X_\eta$$

with $F'_{i,r}\subset B_{\mathfrak{r}}(0,\beta)$ that satisfies

$$\beta^{4\mathsf{m}+5}\cdot(\#F)\leq F'_{i,r}\leq\beta^{4\mathsf{m}+4}\cdot(\#F).$$

We will now further divide the sets $\mathcal{E}'_{i,r}$.

Let $\mathcal{B}$ denote the cubes of size 2^{-n} with $2^{-n}\geq e^{(-2\mathsf{m}-1)\ell}>2^{-n-1}$ which are obtained by scaling some fixed partition of $\mathfrak{r}$ into unit size cubes by 2^{-n}. Applying Lemma D.1, with the collection of cubes $\mathcal{B}$, we can write $F'_{i,r}=F''_{i,r}\sqcup\bigl(\sqcup_\varsigma\hat F'_{i,r,\varsigma}\bigr)$ where the union is disjoint,

$$\#F''_{i,r}\ll\beta^{1/2}(\#F'_{i,r}),$$

and both of the following hold for all ς

a. $\#\hat F'_{i,r,\varsigma}\geq\beta^{0.6}\cdot(\#F'_{i,r})$, and

b. There exists $w_{i,\varsigma} \in \mathfrak{r}$ so that if $\mathsf{C}_1 \neq \mathsf{C}_1' \in \mathcal{B}$ intersect $\hat{F}'_{i,r,\varsigma} + w_{i,\varsigma}$ non-trivially, the distance between $\mathsf{C}_1 \cap (\hat{F}'_{i,r,\varsigma} + w_{i,\varsigma})$ and $\mathsf{C}_1' \cap (\hat{F}'_{i,r,\varsigma} + w_{i,\varsigma})$ is $\gg e^{(-2\mathsf{m}-1)\ell}\beta$.

Replacing $\mu_{\mathcal{E}'_{i,r}}$ by $\sum c_{i,r,\varsigma}\mu_{\mathcal{E}'_{i,r,\varsigma}}$, where

$$\mathcal{E}'_{i,r,\varsigma} = \mathsf{E}.\{\exp(w)y_{i,r} : w \in F'_{i,r,\varsigma}\} \qquad \text{for all } \varsigma,$$

we have that

$$\int \varphi(a_{\mathsf{d}}u_s a_\ell u_r z)\,\mathrm{d}\mu_{\mathcal{E}}(z) = \sum_{i,\varsigma} c'_{i,r,\varsigma}\int \varphi(a_{\mathsf{d}}u_s z)\,\mathrm{d}\mu_{\mathcal{E}'_{i,r,\varsigma}}(z) + O(\beta^\star\,\mathrm{Lip}(\varphi)).$$

By reindexing, may assume that b. above holds already for $F'_{i,r}$. In other words, we may assume that there exists a disjoint collection $\mathcal{B}'_i$ of cubes of size $2^{-n} \geq e^{(-2\mathsf{m}-1)\ell}\beta \geq 2^{-n-1}$ which cover $B_{\mathfrak{r}}(0,\beta)$ so that if $\mathsf{C}_1 \neq \mathsf{C}_1' \in \mathcal{B}'_i$ intersect $\hat{F}'_{i,r}$ non-trivially, then

$$\text{the distance between } \mathsf{C}_1 \cap \hat{F}'_{i,r} \text{ and } \mathsf{C}_1' \cap \hat{F}'_{i,r} \text{ is } \gg e^{(-2\mathsf{m}-1)\ell}\beta. \tag{7.12}$$

After this additional refinement, we have

$$\beta^{4\mathsf{m}+6}\cdot(\#F) \leq F'_{i,r} \leq \beta^{4\mathsf{m}+4}\cdot(\#F).$$

Incremental improvements using Z_r. Let $r \in L$, and let $\bar{z} \in Z_r$, see (7.10). Fix $\bar{w} \in I_{\bar{z},r}$; by definition of $I_{\bar{z},r}$ we have that $z = \exp(\bar{w})\bar{z} \in \hat{\mathcal{E}}$. We will estimate

$$\sum_{v \in I_z^{\mathrm{int}}} \max(\|a_\ell u_r v\|, \delta')^{-\alpha},$$

where I_z^{int} is explicated in (7.17).

For any small $w \in \mathfrak{r}$, we have

$$\exp(w)z = \exp(w)\exp(\bar{w})\bar{z} = \mathsf{h}\exp(\mathsf{v}_w)\bar{z} \tag{7.13}$$

with $\mathsf{h} \in H$, $\mathsf{v}_w \in \mathfrak{r}$, $\|\mathsf{h} - I\| \ll \beta\|w\|$ and

$$\|\mathsf{v}_w - (w + \bar{w})\| \ll \|\bar{w}\|\|w\|, \tag{7.14}$$

see Lemma 3.2. On the other hand, if $z \in \hat{\mathcal{E}}$ and $w \in I_{\mathcal{E}}(z)$, we have

$$\exp(w)z = \exp(w)h\exp(w_z)y = h\mathsf{h}_z\exp(w_{z,w})y$$

for some $w_z, w_{z,w} \in F$, $h \in \hat{\mathsf{E}}$, and $\mathsf{h}_z \in H$ with $\|\mathsf{h}_z - I\| \ll \beta\|w\|$. Thus (7.13) implies that

$$\exp(\mathsf{v}_w)\bar{z} = \mathsf{h}^{-1}\exp(w)z = \mathsf{h}^{-1}h\mathsf{h}_z\exp(w_{z,w})y \in \mathcal{E}.$$

That is, $\mathsf{v}_w \in I_{\mathcal{E}}(\bar{z})$. Hence, $w \mapsto \mathsf{v}_w$ is one-to-one from $I_{\mathcal{E}}(z)$ into $I_{\mathcal{E}}(\bar{z})$.

Moreover, since $e^{-\mathsf{m}\ell}\|v\| \ll \|a_\ell u_r v\| \ll e^{\mathsf{m}\ell}\|v\|$ for all $v \in \mathfrak{r}$, we conclude from (7.13) and (7.14) that if $\|\bar{w}\| \leq e^{(-2\mathsf{m}-1)\ell}$, then

$$\tfrac{1}{2}\|a_\ell u_r(\mathsf{v}_w - \bar{w})\| \leq \|a_\ell u_r w\| \leq 2\|a_\ell u_r(\mathsf{v}_w - \bar{w})\|. \tag{7.15}$$

Recall that $\bar z\in Z_r$, thus using (7.5) if $\alpha=\frac{1}{2\mathsf{m}+1}$ and (7.8) if $\alpha>\frac{1}{2\mathsf{m}+1}$, we conclude that for every $v\in I_{\bar z,r}$,

$$\sum_{v\neq v'\in I^{\rm int}_{\bar z,r}(v)}\max(\|a_\ell u_r(v-v')\|,\delta')^{-\alpha}\ll e^{-3\varpi\ell/4}\Upsilon \tag{7.16}$$

where $I^{\rm int}_{\bar z,r}(v)=\{v'\in I_{\bar z,r}:\|v-v'\|\leq e^{-\mathsf{m}\ell}\}$.

Let us now put

$$I^{\rm int}_z=\big\{w\in I_{\mathcal{E}}(z):\|w\|\leq e^{(-\mathsf{m}-1)\ell},\mathsf{v}_w\in I_{\bar z,r}\big\}. \tag{7.17}$$

Note that for any $w\in I^{\rm int}_z$, we have $\|\bar w-\mathsf{v}_w\|\ll\beta\|w\|$, hence $\mathsf{v}_w\in I^{\rm int}_{\bar z,r}(\bar w)$. Thus, (7.15) and (7.16) (applied with $v=\bar w$) imply that

$$\begin{aligned}\sum_{0\neq w\in I^{\rm int}_z}&\max(\|a_\ell u_r w\|,\delta')^{-\alpha}\\ &\leq 2^{2\mathsf{m}+1}\sum_{\bar w\neq v'\in I^{\rm int}_{\bar z,r}(\bar w)}\max(\|a_\ell u_r(\bar w-v')\|,\delta')^{-\alpha}\ll e^{-3\varpi\ell/4}\Upsilon.\end{aligned} \tag{7.18}$$

The set $\mathcal{E}(r)$. Recall that $\mathcal{B}$ denotes the cubes of size 2^{-n} with $2^{-n}\geq e^{(-2\mathsf{m}-1)\ell}>2^{-n-1}$ which are obtained by scaling some fixed partition of $\mathfrak{r}$ into unit size cubes by 2^{-n}. Let $\mathcal{C}\subset\mathcal{B}$ denote the collection of those cubes $\mathsf{C}\in\mathcal{B}$ with the following property: there exists $w_{\mathsf{C}}\in F\cap\mathsf{C}$ satisfying that

$$\begin{aligned}\mu_{w_{\mathsf{C}}}\big\{\bar z\in\hat{\mathsf{E}}\exp(w_{\mathsf{C}})y\cap Z_r:\tfrac{\#I_{\bar z,r}\cap\mathsf{B}}{\#I_{\mathcal{E}}(\bar z)\cap\mathsf{B}}\geq(1-O(e^{-\star\mathsf{c}^2\ell})\big\}\geq&\\ \big(1-O(e^{-\star\mathsf{c}^2\ell})\big)\mu_{w_{\mathsf{C}}}(\hat{\mathsf{E}}\exp(w_{\mathsf{C}})y)&\end{aligned}$$

where $\mathsf{B}=B_{\mathfrak{r}}(0,e^{(-2\mathsf{m}-1)\ell})$ and $\mu_{w_{\mathsf{C}}}$ is the restriction of the measure $\mu_{\hat{\mathcal{E}}}$ to $\hat{\mathsf{E}}\exp(w_{\mathsf{C}})y$, see D.3. For every $\mathsf{C}\in\mathcal{C}$, fix one such w_{C}, and let

$$Z_r(w_{\mathsf{C}})=\big\{\bar z\in\hat{\mathsf{E}}\exp(w_{\mathsf{C}})y\cap Z_r:\tfrac{\#I_{\bar z,r}\cap\mathsf{B}}{\#I_{\mathcal{E}}(\bar z)\cap\mathsf{B}}\geq(1-O(e^{-\star\mathsf{c}^2\ell})\big\}.$$

Fix a covering $\{\mathsf{P}h'_j\}_j$ of $\hat{\mathsf{E}}$ with multiplicity bounded by an absolute constant, where

$$\mathsf{P}=\big\{u^-_s:|s|\leq 10\beta^2\big\}\cdot\{a_\tau:|\tau|\leq 10\beta^2\}\cdot\{u_r:|r|\leq 10e^{-\ell}\eta\};$$

cf. e.g. [LM23, Lemma 7.9]. Note that by our choice of constants, β^2 is much larger than $e^{-\ell}\eta$.

For every $\mathsf{C}\in\mathcal{C}$, let $\mathcal{J}_{\mathsf{C}}$ denote the collection of j so that $\mathsf{P}h'_j\exp(w_{\mathsf{C}})\cap Z_r(w_{\mathsf{C}})\neq\emptyset$. For any $j\in\mathcal{J}_{\mathsf{C}}$, fix some $\mathsf{h}'_j\in\mathsf{P}$ so that

$$\bar z_{\mathsf{C},j}:=\mathsf{h}'_jh'_j\exp(w_{\mathsf{C}})\in Z_r(w_{\mathsf{C}}).$$

With this notation, set $I_{\mathsf{C},j}=I_{\bar z_{\mathsf{C},j},r}\cap B_{\mathfrak{r}}(0,e^{(-2\mathsf{m}-1)\ell})$, and let

$$\mathcal{E}^{\mathsf{C},j}(r)=\{\mathsf{h}\exp(\bar w)\bar z_{\mathsf{C},j}:\mathsf{h}\in\mathsf{P},\bar w\in I_{\mathsf{C},j}\}\cap\hat{\mathcal{E}}.$$

Define $\mathcal{E}(r) := \bigcup_{\mathsf{C}\in\mathcal{C}}\bigcup_{\mathcal{J}_{\mathsf{C}}}\mathcal{E}^{\mathsf{C},j}(r)$; this covering has multiplicity $\leq \mathsf{K}$, where K is absolute. It follows from (7.9), (7.10), and the above definitions that

$$\mu_{\mathcal{E}}(\mathcal{E}\setminus\mathcal{E}(r)) \ll e^{-\star\mathsf{c}^2\ell}. \tag{7.19}$$

Trimming $\mathcal{E}'_{i,r}$ using $\mathcal{E}(r)$. Recall from part (2) in Lemma D.4 the following property of $F'_{i,r}$

$$\mathsf{Q}^H_\ell.\exp(w)y_{i,r} \subset a_\ell u_r\mathcal{E} \qquad \text{for all } w\in F'_{i,r},$$

where $\mathsf{Q}^H_\ell = \{u^-_s : |s|\leq \beta^2 e^{-\ell}\}\cdot\{a_\tau : |\tau|\leq\beta^2\}\cdot\{u_r : |r|\leq\eta\}$.

For all i, all $\mathsf{C}_1\in\mathcal{B}'_i$, $\mathsf{C}_2\in\mathcal{C}$, and all $j\in\mathcal{J}_{\mathsf{C}_2}$, let

$$F'_{i,\mathsf{C}_1}(\bar z_{\mathsf{C}_2,j}) = \{w\in\mathsf{C}_1 : \exists\bar w\in I_{\mathsf{C}_2,j},\ a_\ell u_r\exp(\bar w)\bar z_{\mathsf{C}_2,j}\in\mathsf{Q}^H_\ell.\exp(w)y_{i,r}\},$$

roughly speaking, this set captured points in $F'_{i,r}\cap\mathsf{C}_1$ which may be *traced back* to $I_{\mathsf{C}_2,j} = I_{\bar z_{\mathsf{C}_2,j},r}\cap B_{\mathfrak{r}}(0,e^{(-2\mathsf{m}-1)\ell})$.

For every i, let

$$\mathcal{B}_i := \{\mathsf{C}_1\in\mathcal{B}'_i : \max_{\mathsf{C}_2,j}\#F'_{i,\mathsf{C}_1}(\bar z_{\mathsf{C}_2,j}) \geq (1-10^{-6})\cdot(\#F'_{i,r}\cap\mathsf{C}_1)\},$$

and put $\mathcal{I}_r = \{i : \bigcup_{\mathsf{C}_1\in\mathcal{B}_i}(\mathsf{C}_1\cap F'_{i,r}) \geq \beta^{1/2}\cdot\#(F'_{i,r})\}$.

For every $i\in\mathcal{I}_r$ and all $\mathsf{C}_1\in\mathcal{B}_i$, choose $\bar z_{i,\mathsf{C}_1}\in\hat{\mathsf{E}}\exp(w_{\mathsf{C}_2,j})y)\cap Z_r$ so that $\#F'_{i,\mathsf{C}_1}(\bar z_{i,\mathsf{C}_1})$ realizes the above maximum. Put

$$F_{i,r} = \bigcup\nolimits_{\mathsf{C}_1\in\mathcal{B}_i} F'_{i,\mathsf{C}_1}(\bar z_{i,\mathsf{C}_1}) \quad\text{and}\quad \mathcal{E}_{i,r} = \mathsf{E}.\{\exp(w)y_{i,r} : w\in F_{i,r}\}. \tag{7.20}$$

For every $i\in\mathcal{I}_r$, let $\mu_{i,r}$ denote the restriction of $\mu_{\mathcal{E}'_{i,r}}$ to $\mathcal{E}_{i,r}$ normalized to be a probability measure.

7.3. **Lemma.** *With the above notation, we have*

$$f^{(\alpha)}_{\mathcal{E}_{i,r},\delta'}(z) \leq e^{-\varpi\ell/2}\Upsilon + e^{\varepsilon t}\cdot(\#F_{i,r})$$

for all $i\in\mathcal{I}_r$ and all $z\in\mathcal{E}_{i,r}$.

Moreover, we have

$$\int\varphi(a_{\mathsf{d}}u_s z)\,\mathrm{d}(a_\ell u_r\mu_{\mathcal{E}})(z) = \sum_{\mathcal{I}_r} c_{i,r}\int\varphi(a_{\mathsf{d}}u_s z)\,\mathrm{d}\mu_{\mathcal{E}_{i,r}}(z) + O(\beta^\star\,\mathrm{Lip}(\varphi)). \tag{7.21}$$

We postpone the proof of Lemma 7.3 to after the completion of the proof of Lemma 7.2; a task which we now undertake.

Conclusion of the proof of Lemma 7.2. First recall that

$$\iint_0^1 \varphi(a_{\mathsf{d}} u_s a_\ell u_r z)\,\mathrm{d}r\,\mathrm{d}\mu_{\mathcal{E}}(z) = \iint \varphi(a_{\mathsf{d}+\ell} u_{r+se^{-\ell}} z)\,\mathrm{d}r\,\mathrm{d}\mu_{\mathcal{E}}(z).$$

Since $|[0,1] \setminus L_{\mu_{\mathcal{E}}}| \ll e^{-\star \mathsf{c}^2 \ell} = \beta^\star$ and $\mathcal{L} \subset L_{\mu_{\mathcal{E}}}$ is a maximal $e^{-3\mathsf{d}_0}$ separated subset, we have

$$\begin{aligned}\iint \varphi(a_{\mathsf{d}+\ell} u_{r+se^{-\ell}} z)\,\mathrm{d}r\,\mathrm{d}\mu_{\mathcal{E}}(z) &= \\ \sum_{r\in\mathcal{L}} \int \varphi(a_{\mathsf{d}+\ell} u_{r+se^{-\ell}} z)\,\mathrm{d}\mu_{\mathcal{E}}(z) + O\big(\beta^\star \operatorname{Lip}(\varphi)\big) &= \\ \sum_{r\in\mathcal{L}} \int \varphi(a_{\mathsf{d}} u_s a_\ell u_r z)\,\mathrm{d}\mu_{\mathcal{E}}(z) + O\big(\beta^\star \operatorname{Lip}(\varphi)\big).&\end{aligned}$$

Combining these and the second assertion in Lemma 7.3, we conclude that

$$\iint \varphi(a_{\mathsf{d}} u_s a_\ell u_r z)\,\mathrm{d}r\,\mathrm{d}\mu_{\mathcal{E}} = \sum_{\mathcal{I}} c_{i,r} \int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_{i,r}}(z) + O(\beta^\star \operatorname{Lip}(\varphi));$$

moreover, by the first claim in Lemma 7.3, we have

$$f_{\mathcal{E}_{i,r},\delta'}(z) \le e^{-\varpi\ell/2}\Upsilon + e^{\varepsilon t}\cdot(\#F_{i,r}).$$

Finally, since $\beta^{4\mathsf{m}+5}\cdot(\#F) \le \#F'_{i,r} \le \#F$ and $\#F_{i,r} \ge \beta^{1/2}\cdot(\#F'_{i,r})$,

$$\beta^{4\mathsf{m}+6}\cdot(\#F) \le \#F_{i,r} \le \beta^{4\mathsf{m}+4}\cdot(\#F).$$

The proof of Lemma 7.2 is complete. □

Proof of Lemma 7.3. We begin by establishing the first claim in the lemma. Fix some i, and denote $F_{i,r}$, $y_{i,r}$, and $\mathcal{E}_{i,r}$ by F_{nw}, y_{nw}, and $\mathcal{E}_{\mathrm{nw}}$, respectively.

First note that

$$f^{(\alpha)}_{\mathcal{E}_{\mathrm{nw}},\delta'}(z) \le \sum_{0\ne w\in I^{\mathrm{int}}_{\mathcal{E}_{\mathrm{nw}}}(z)} \max(\|w\|,\delta')^{-\alpha} + e^{\varepsilon t}\cdot(\#F_{\mathrm{nw}}) \tag{7.22}$$

where $I^{\mathrm{int}}_{\mathcal{E}_{\mathrm{nw}}}(z) = \{w \in I_{\mathcal{E}_{\mathrm{nw}}}(z) : \|w\| \le e^{(-2\mathsf{m}-2)\ell}\}$.

We also recall the definition of F_{nw} and $\mathcal{E}_{\mathrm{nw}}$ from (7.20). In particular, for every $\mathsf{C}_1 \in \mathcal{B}_i$, there exists $\mathsf{C}_2 \in \mathcal{C}$ and $\bar z = \bar z_{\mathsf{C}_2,j}$ so that for all $w \in F_{\mathrm{nw}} \cap \mathsf{C}_1$, there is some $\bar w \in I_{\mathsf{C}_2,j} = I_{\bar z,r} \cap B_{\mathfrak{r}}(0, e^{(-2\mathsf{m}-1)\ell})$ with

$$a_\ell u_r \exp(\bar w)\bar z \in \mathsf{Q}^H_\ell . \exp(w) y_{\mathrm{nw}}.$$

For $k = 1, 2$, let $z_k = a_\ell u_r \exp(\bar w_k)\bar z$, and write

$$z_k = \mathsf{h}_k \exp(w_k) y_{\mathrm{nw}},$$

where $w_k \in F_{\mathrm{nw}}$ and $\mathsf{h}_k \in \mathsf{Q}^H_\ell$. Then we have

$$\begin{aligned} z_2 &= \mathsf{h}_2 \exp(w_2) y_{\mathrm{nw}} = \mathsf{h}_2 \exp(w_2)\exp(-w_1)\mathsf{h}_1^{-1} z_1 \\ &= \mathsf{h}_2\mathsf{h}_1^{-1} \exp(\operatorname{Ad}(\mathsf{h}_1) w_2)\exp(-\operatorname{Ad}(\mathsf{h}_1) w_1) z_1 \\ &= \mathsf{h}_2\mathsf{h}_1^{-1}\hat{\mathsf{h}} \exp(\hat w) z_1 \end{aligned} \tag{7.23}$$

where $\hat{\mathsf{h}} \in H$ and $\hat{w} \in \mathfrak{r}$, moreover, by Lemma 3.2, we have

$$\|\hat{\mathsf{h}} - I\| \le C_2 \beta \|\hat{w}\| \qquad \text{and} \tag{7.24a}$$

$$\tfrac{1}{2}\|\operatorname{Ad}(\mathsf{h}_1)(w_2 - w_1)\| \le \|\hat{w}\| \le 2\|\operatorname{Ad}(\mathsf{h}_1)(w_2 - w_1)\|. \tag{7.24b}$$

Now let $v \in I_{\mathcal{E}_{\rm nw}}(z_1)$. Then there exist $w_v \in F_{\rm nw}$ and $h'_v \in \mathsf{E}$ so that

$$\exp(v)z_1 = h'_v \exp(w_v) y_{\rm nw}.$$

Moreover, if $\|v\| \le e^{(-2\mathsf{m}-2)\ell}$, then by (7.12) $w_v \in \mathsf{C}_1$, thus there exist $\bar{w}_v \in I_{\bar{z},r}$ and $\mathsf{h}_v \in \mathsf{Q}^H_\ell$ so that

$$a_\ell u_r \exp(\bar{w}_v)\bar{z} = \mathsf{h}_v \exp(w_v) y_{\rm nw}.$$

Altogether, if $\|v\| \le e^{(-2\mathsf{m}-2)\ell}$, then

$$a_\ell u_r \exp(\bar{w}_v)\bar{z} = h_v \exp(v) z_1 \tag{7.25}$$

where $h_v = \mathsf{h}_v h'^{-1}_v \in \mathsf{B}^H_{1.1\eta}$.

Applying (7.23) with $w_2 = w_v$, we get that

$$a_\ell u_r \exp(\bar{w}_v)\bar{z} = \mathsf{h}_v \mathsf{h}_1^{-1} \hat{\mathsf{h}}_v \exp(\hat{w}_v) z_1 \tag{7.26}$$

where $\hat{\mathsf{h}}_v$ and $\hat{w}_v$ satisfy (7.24a) and (7.24b), and $\mathsf{h}_1, \mathsf{h}_v \in \mathsf{Q}^H_\ell$.

Recall now that $(\hat{h}, \hat{w}) \mapsto \hat{h}\exp(\hat{w})z_1$ is injective over $\mathsf{B}^H_{10\eta} \times B_{\mathfrak{r}}(0, 10\eta)$. Thus, we conclude from (7.26) and (7.25) that

$$\hat{w}_v = v. \tag{7.27}$$

Since $\|v\| \le e^{(-2\mathsf{m}-2)\ell}$, (7.24a) and (7.27) imply that

$$\|\hat{\mathsf{h}}_v - I\| \le C_2\beta\|v\| \le e^{(-2\mathsf{m}-2)\ell}\beta \le \beta^2 e^{-\ell};$$

recall that $\beta \ge e^{-\ell}$ and $\mathsf{m} \ge 2$. Moreover, $\mathsf{h}_1, \mathsf{h}_v \in \mathsf{Q}^H_\ell$. Therefore,

$$\hat{\mathsf{h}}_v^{-1}\mathsf{h}_1\mathsf{h}_v^{-1} a_\ell u_r \exp(\bar{w}_v)\bar{z} \in a_\ell u_r \mathcal{E},$$

see (3.7). This, (7.26), and (7.27) yield

$$\begin{aligned} \exp(v) a_\ell u_r \exp(\bar{w}_1)\bar{z} &= \exp(v) z_1 \\ &= \hat{\mathsf{h}}^{-1}\mathsf{h}_1\mathsf{h}_v^{-1} a_\ell u_r \exp(\bar{w}_v)\bar{z} \in a_\ell u_r \mathcal{E}. \end{aligned}$$

Recall also that $\bar{w}_v \in I_{\bar{z},r}$, and note that that since $\|v\| \le e^{(-2\mathsf{m}-2)}$, we have $\|\operatorname{Ad}((a_\ell u_r)^{-1})v\| \le e^{(-\mathsf{m}-1)\ell}$. Altogether, we conclude that

$$\operatorname{Ad}((a_\ell u_r)^{-1})v \in I^{\rm int}_{\exp(\bar{w}_1)\bar{z}}, \tag{7.28}$$

see (7.17). Let

$$I^{\rm int}_{\mathcal{E}_{\rm nw}}(z_1) = \{v \in I_{\mathcal{E}_{\rm nw}}(z_1) : \|v\| \le e^{(-2\mathsf{m}-2)\ell}\}$$

Since $\|\bar{w}_1\| \leq e^{(-2\mathsf{m}-1)\ell}$, we conclude from (7.28) and (7.18), applied with $\exp(\bar{w}_1)\bar{z} =: \hat{z}$, that

$$\begin{aligned} f^{(\alpha)}_{\mathcal{E}_{\rm nw},\delta'}(z_1) &\leq \sum_{0\neq v\in I^{\rm int}_{\mathcal{E}_{\rm nw}}(z_1)} \max(\|v\|,\delta')^{-\alpha} + e^{\varepsilon t}\cdot(\#F_{\rm nw}) \\ &\leq \sum_{0\neq w\in I^{\rm int}_{\hat{z}}} \max(\|a_\ell u_r w\|,\delta')^{-\alpha} + e^{\varepsilon t}\cdot(\#F_{\rm nw}) \\ &\leq Ce^{-3\varpi\ell/4}\Upsilon + e^{\varepsilon t}\cdot(\#F_{\rm nw}), \end{aligned} \tag{7.29}$$

where C depends on X, see also (7.22) for the first inequality.

Let now $z = h\exp(w)y_{\rm nw} \in \mathcal{E}_{\rm nw}$ be arbitrary. Then there exists some $z_1 = a_\ell u_r \exp(\bar{w}_1)\bar{z}$ so that $z_1 = \mathsf{h}\exp(w)y_{\rm nw}$, and we have

$$f^{(\alpha)}_{\mathcal{E}_{\rm nw},\delta'}(z) \leq 2f^{(\alpha)}_{\mathcal{E}_{\rm nw},\delta'}(z_1).$$

This and (7.29) complete the proof of the first claim, assuming ℓ is large enough so that $2Ce^{-3\varpi\ell/4} \leq e^{-\varpi\ell/2}$.

The second claim in the lemma follows from (7.19) and (7.11) as we now explicate. Fix some i, and suppose that $\mathsf{C}_1 \notin \mathcal{B}_i$. Then for all $\mathsf{C}_2 \in \mathcal{C}$ and all $j \in \mathcal{J}_{\mathsf{C}_2}$, we have

$$\begin{aligned} \#\{w \in \mathsf{C}_1 \cap F'_{i,r} : \exp(w)y_{i,r} \notin (\mathsf{Q}^H_\ell)^{-1}.a_\ell u_r \exp(I_{\mathsf{C}_2,j})\bar{z}_{\mathsf{C}_2,j}\} \\ \geq 10^{-6}\cdot(\#(\mathsf{C}_1 \cap F'_{i,r})). \end{aligned}$$

Note now that $(a_\ell u_r)^{-1}(\mathsf{Q}^H_\ell)^{-1}.a_\ell u_r \subset \mathsf{P}$ and

$$m_H((a_\ell u_r)^{-1}(\mathsf{Q}^H_\ell)^{-1}.a_\ell u_r \asymp m_H(\mathsf{P}).$$

Moreover, note that the multiplicity of the covering $\{\mathcal{E}^{j,\mathsf{C}}_r\}$ is $\leq \mathsf{K}$. We thus conclude that the contribution of each $\mathsf{C}_1 \notin \mathcal{B}_i$ to $\mu_{\mathcal{E}}(\mathcal{E}\setminus\mathcal{E}_r)$ is

$$\asymp (\#(\mathsf{C}_1 \cap F'_{i,r}))m_H(\mathsf{P}).$$

Therefore, the total number of $w \in \bigcup_i F'_{i,r}$ that we have omitted is $\ll \beta^\star(\#\bigcup_i F'_{i,r})$, which implies the claim. $\square$

8. The main proposition

We begin by fixing several parameters that will remain constant throughout this section. Roughly speaking, κ is a small parameter representing the incremental improvements in dimension. These improvements are achieved by inductively applying Lemma 7.2 with various choices of α. The constants $d.$ indicate the number of times the lemma is applied for a fixed α, while $p_{\rm fn}$ represents the number of times α is adjusted in increments of κ. See the discussion proceeding the statement of the theorem for more details.

Let $0 < \kappa < 1$ be a small parameter — in our application, we will let $\kappa = (\frac{\kappa_5}{40\mathsf{m}})^2$, see Proposition 5.1 and Lemma 5.2. Set

$$p_{\rm fn} := \lceil \tfrac{4\mathsf{m}^2+4\mathsf{m}}{\kappa(2\mathsf{m}+1)} \rceil - 10,$$

and let $\kappa_{\rm fn} = \frac{99}{100} \times (\frac{3}{4})^{p_{\rm fn}}$ and $\varepsilon = \kappa \cdot \kappa_{\rm fn}$.

Let $t > 0$ be a large parameter, and let $D \geq D_1$, see Proposition 4.6. Let

$$\begin{aligned} &d_0 = \lceil \tfrac{(10D-9)(4\mathsf{m}^2+2\mathsf{m})}{\varepsilon} \rceil, \\ &d_1 = \lceil \tfrac{495}{200\varepsilon} \rceil \quad \text{and} \quad d_{p+1} = \lceil \tfrac{3}{4} d_p \rceil \quad \text{for } 0 < p < p_{\rm fn} \end{aligned}$$

Set $d_{\rm fn} = \sum_{p=0}^{p_{\rm fn}} d_p$, and let

$$\varepsilon' = 10^{-6} \mathsf{m}^{-1} d_{\rm fn}^{-2}, \quad \beta = e^{-\varepsilon' t}, \quad \text{and} \quad \eta^2 = \beta.$$

Put $\ell = \frac{\varepsilon t}{10\mathsf{m}}$. Let ν_d denote the probability measure on H defined by

$$\int \varphi \, \mathrm{d}\nu_d = \int_0^1 \varphi(a_d u_r) \, \mathrm{d}r,$$

for any $\varphi \in C_c(H)$, and put

$$\mu_{t,\ell,n} = \nu_\ell * \cdots * \nu_\ell * \nu_t$$

where ν_ℓ appears $n \geq 0$ times in the above expression.

The following proposition, whose proof is based on Lemma 7.2, is a crucial tool in our argument.

8.1. **Proposition.** *Let $x_1 \in X$, and assume that Proposition 4.6(2) does not hold for x_1 with parameters $D \geq D_1$ and $t > 0$. Let $r_1 \in I(x_1)$ and put $x_2 = a_{\mathsf{m}_1 t} u_{r_1} x_1$, see Proposition 4.6(1).*

There exists a collection $\Xi = \{\mathcal{E}_i : 1 \leq i \leq N\}$ of sets

$$\mathcal{E}_i = \mathsf{E}.\{\exp(w) y_i : w \in F_i\} \subset X_\eta$$

satisfying that for all $1 \leq i \leq N$, we have $F_i \subset B_{\mathfrak{r}}(0, \beta)$ and

$$e^{0.98t} \leq \# F_i \leq e^t$$

so that both of the following hold

(1) For all $1 \leq i \leq N$ and all $z \in (\overline{\mathsf{E} \setminus \partial_{10\beta}\mathsf{E}}).\{\exp(w) y_i : w \in F_i\}$,

$$f_{\mathcal{E}_i,\delta}^{(2\mathsf{m}+1-20\kappa)}(z) \leq e^{2\varepsilon t} \cdot (\# F_i) \qquad \text{where } \delta = e^{-\kappa_{\rm fn} t}. \tag{8.1}$$

(2) For all $\tau \leq \ell d_{\rm fn}$, all $|s| \leq 2$, and for every $\varphi \in C_c^\infty(X)$,

$$\int \varphi(a_\tau u_s h x_2) \, \mathrm{d}\mu_{t,\ell,d_{\rm fn}}(h) = \sum_i c_i \int \varphi(a_\tau u_s z) \, \mathrm{d}\mu_{\mathcal{E}_i}(z) + O(\mathrm{Lip}(\varphi) \beta^{\kappa_7})$$

where

- *$0 \leq c_i \leq 1$ and $\sum_i c_i = 1 - O(\beta^{\kappa_7})$,*
- *$\mu_{\mathcal{E}_i}$ is an admissible measure on $\mathcal{E}_i$ with parameter depending only on D and X, see §D.3,*
- *$\mathrm{Lip}(\varphi)$ is the Lipschitz norm of φ, and*
- *κ_7 and the implied constants depend on X.*

The reader may compare this proposition to [LMW22, Prop. 10.1]. Indeed in [LMW22, Prop. 10.1], we established similar bound concerning modified Margulis functions, but for $\alpha < 1$ (in the setting considered in [LMW22], $\dim \mathfrak{r}^+ = 1$) and localized to balls of size $e^{-\sqrt{\varepsilon} t}$. Requiring this localized estimate, albeit for smaller α, was responsible for *different* stopping times — in [LMW22, Prop. 10.1] we could not specify the stopping time, only guarantee that there will be at least one good stopping time n in an interval of length $O(1/\sqrt{\varepsilon})$. This resulted in a less transparent endgame analysis. The estimate established in (8.1) concerns α dimensional energy for α very close to $2\mathsf{m}+1 = \dim \mathfrak{r}$. This is made possible thanks to stronger projection theorems, see Theorem 6.1.

Proof of Proposition 8.1. The proof of Proposition 8.1 relies on Lemma 7.2. Indeed with Lemma 7.2 in place, the general strategy is straightforward: Let us put $\rho = \tau + t + d_{\rm fn}\ell$. Using results in [LMW22, §6–8] and the fact that x_2 satisfies conditions in part (1) of Proposition 4.6, we can write

$$\int \varphi(a_{\rho-t} u_s h x_2)\,\mathrm{d}\mu_{t,\ell,0}(h) = \sum_i c_i \int \varphi(a_\tau u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z) + O(\mathrm{Lip}(\varphi)\beta^\star)$$

where the decomposition is similar to the one claimed in the proposition, but with $\beta^{4\mathsf{m}+5} e^t \le \# F_i \le e^t$, and where we only have the initial estimate

$$f^{(\alpha)}_{\mathcal{E}_i,0}(z) \le e^{Dt} \qquad \text{for all } 0 < \alpha \le 1.$$

Then we apply Lemma 7.2 with $\alpha_0 = \frac{1}{2\mathsf{m}+1}$ (hence $\varpi = \alpha_0$ and $\delta' = 0$) for $d_0 = \lceil \frac{(10D-9)(4\mathsf{m}^2+2\mathsf{m})}{\varepsilon} \rceil$ many steps. For every $|s| \le 2$, thus

$$\int \varphi(a_{\rho-t-d_0\ell} u_s h x_2)\,\mathrm{d}\mu_{t,\ell,d_1}(h) = \sum_i c_i \int \varphi(a_{\tau'} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i'}(z) + O(\mathrm{Lip}(\varphi)\beta^\star)$$

where the decomposition is as in the proposition, but now we have $e^{0.99t} \le \# F_i' \le e^t$ for all i, and the improved estimate

$$f^{(\alpha_0)}_{\mathcal{E}_i',0}(z) \le 2e^{\varepsilon t} \cdot (\# F_i').$$

In the next phase, we improve the above estimate for $f^{(\alpha_0)}$ inductively to obtain similar estimate for $\alpha_p = \alpha_0 + p\kappa$ for all $0 \le p < p_{\rm fn}$: Assume

$$f^{(\alpha_p)}_{\mathcal{E}_i,\delta_p}(z) \le 2e^{\varepsilon t} \cdot (\# F_i),$$

where $\delta_1 = e^{-0.99t}$ and $\delta_{p+1} = \delta_p^{3/4}$ for all $p \ge 1$. We conclude that

$$f^{(\alpha_{p+1})}_{\mathcal{E}_i,\delta_p}(z) \le 2e^{\varepsilon t} \cdot \delta_p^{-\kappa} \cdot (\# F_i)$$

We now apply Lemma 7.2 with $\alpha = \alpha_{p+1}$ for d_{p+1} many steps. Since $\alpha_0 \le \alpha_{p+1} \le 2\mathsf{m}+1-10\kappa\mathsf{m}$ we have $\varpi \ge 9\kappa\mathsf{m}$, and each application of Lemma 7.2 improves the bound on f by $e^{-4\kappa\mathsf{m}\ell}$ while the scale δ is replaced by $\delta' = e^{\mathsf{m}\ell}\delta$.

Applying this, $p_{\rm fn}-1$ many times, we conclude that

$$\int \varphi(a_\tau u_s h x_2)\,\mathrm{d}\mu_{t,\ell,d_{\rm fn}}(h) = \sum_i c_i \int \varphi(a_\tau u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z) + O(\mathrm{Lip}(\varphi)\beta^\star)$$

where the decomposition is as in the proposition, and

$$f^{(\alpha_{p_{\rm fn}})}_{\mathcal{E}_i,\delta_{p_{\rm fn}}}(z) \leq 2e^{\varepsilon t}\cdot(\#F_i),$$

and the proposition follows.

Let us now turn to the more detailed argument.

Smearing and Folner property. The following equality which is a simple consequence of commutation relations, will be used throughout the proof

$$a_{t_1}u_{r_1}a_{t_2}u_{r_2} = a_{t_1+t_2}u_{r_2+e^{-t_2}r_1} \tag{8.2}$$

Let λ denote the uniform measure on $\mathsf{B}^{s,H}_{\beta+100\beta^2}$, where as before for all $\delta>0$, we put

$$\mathsf{B}^{s,H}_\delta = \{u^-_s : |s|\leq\delta\}\cdot\{a_\tau : |\tau|\leq\delta\}$$

In view of [LMW22, Lemma 7.4], for all $x\in X$ and all $\varphi\in C^\infty_c(X)$,

$$\int \varphi(hx)\,\mathrm{d}(\nu_{t_1+t_2})(h) = \int \varphi(hx)\,\mathrm{d}(\lambda*\nu_{t_2}*\lambda*\nu_{t_1})(h) + O(\beta\,\mathrm{Lip}(\varphi)) \tag{8.3}$$

so long as $e^{-t_i}\leq\beta^2$.

Closing lemma and initial separation. Recall that $x_2 = a_{\mathsf{m}_1 t}u_{r_1}x_1$ where $r_1\in I(x_1)$. In particular, the map $h\mapsto hx_2$ is injective over $\mathsf{B}^{s,H}_\beta\cdot a_t\cdot U_1$, see part (1) in Proposition 4.6. Put

$$\rho = \tau + d_{\rm fn}\ell + t = \tau + \textstyle\sum_0^{p_{\rm fn}} d_p\ell + t.$$

Recall also that $\mu_{t,\ell,d_{\rm fn}} = \nu_\ell^{(d_{\rm fn})}*\nu_t$. An inductive application of (8.2) implies that for any $h_0\in\mathrm{supp}(\nu_\ell^{d_{\rm fn}})$ and any $|s|\leq 2$, there exists $|s'|\leq 2$ so that $a_\tau u_s h_0 = a_{\rho-t}u_{s'}$. Thus

$$\begin{aligned}\int\varphi(a_\tau u_s h x_2)\,\mathrm{d}\mu_{t,\ell,d_{\rm fn}}(h) &= \int\int_0^1 \varphi(a_\tau u_s h_0 a_t u_r x_2)\,\mathrm{d}\nu_\ell^{(d_{\rm fn})}(h_0)\,\mathrm{d}r\\ &= \int\int_0^1 \varphi(a_{\rho-t}u_{s'}a_t u_r x_2)\,\mathrm{d}\nu_\ell^{(d_{\rm fn})}(h_0)\,\mathrm{d}r\end{aligned} \tag{8.4}$$

Now applying [LMW22, Lemma 8.4], see also Lemma D.4, with x_2 we get the following: for every $\varphi\in C^\infty_c(X)$, and all $|s'|\leq 2$,

$$\left|\int_0^1\varphi(a_{\rho-t}u_{s'}a_t u_r x_2)\,\mathrm{d}r - \sum_i c_i\int\varphi(a_{\rho-t}u_{s'}z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\right| \ll \mathrm{Lip}(\varphi)\beta^\star \tag{8.5}$$

where $\sum c_i = 1-O(\beta^\star)$ and the implied constants depend only on X.

Combining (8.4) and (8.5), and using $a_\tau u_s h_0 = a_{\rho-t}u_{s'}$, we have

$$\int \varphi(a_\tau u_s h x_2)\,\mathrm{d}\mu_{t,\ell,d_{\rm fn}}(h) = \sum_i c_i \int \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}\nu_\ell^{(d_{\rm fn})}(h) + O(\mathrm{Lip}(\varphi)\beta^\star) \tag{8.6}$$

Moreover, $\mathcal{E}_i = \mathsf{E}.\{\exp(w)y_i : w \in F_i\}$ where $y_i \in X_\eta$ and $\mathcal{E}_i \subset X_\eta$. Since $\dim \mathfrak{r} = 2\mathsf{m}+1$, [LMW22, Lemma 8.1 and 8.2], see also Lemma D.2 and recall that $\mathsf{m} = 1$ in [LMW22], we have

$$\beta^{4\mathsf{m}+7}e^t \le \#F_i \le e^t \tag{8.7}$$

— in the references above, the upper bound $\beta^{-3}e^t$ is asserted, further subdividing F_i, we may replace that by e^t as it is claimed here.

Furthermore, in view of the definition of $\mathcal{E}_i$, see [LMW22, Lemma 8.4], and the fact that x_2 satisfies properties in part (1) of Proposition 4.6,

$$f^{(\alpha)}_{\mathcal{E}_i,0}(z) \le e^{Dt} \qquad \text{for all } 0 < \alpha \le 1 \tag{8.8}$$

for all i and all $z \in \mathcal{E}_i$.

Applying Lemma 7.2 with $\alpha = \frac{1}{2\mathsf{m}+1}$. The following lemma will be used to carry out the second phase in the above outline. Before stating the lemma, let us recall that $\ell = \frac{\varepsilon t}{10\mathsf{m}}$, and

$$d_0 = \lceil \tfrac{(10D-9)(4\mathsf{m}^2+2\mathsf{m})}{\varepsilon} \rceil.$$

Also recall that we set $\varepsilon' = 10^{-6}\mathsf{m}^{-1}d_{\rm fn}^{-2}$, $\beta = e^{-\varepsilon' t}$, and $\eta^2 = \beta$.

8.2. **Lemma.** *Fix some $\mathcal{E}_i$ as in (8.6) and some $0 \le k \le d_0$. For every $\varphi \in C_c(X)$ and all $|s| \le 2$ we have*

$$\int \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}\nu_\ell^{(d_{\rm fn})}(h) = \sum_j c_{ij} \iint \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(z)\,\mathrm{d}\nu_\ell^{(d_{\rm fn}-k)}(h) + O(\mathrm{Lip}(\varphi)\beta^\star)$$

and all the following hold

(1) $0 \le c_{ij} \le 1$ and $\sum c_{ij} = 1 + O(\beta^\star)$.
(2) $\mathcal{E}_{ij} = \mathsf{E}.\{\exp(w)y_{ij} : w \in F_{ij}\} \subset X_\eta$ and

$$\beta^{(4k+4)\mathsf{m}+6k+7}e^t \le \#F_{ij} \le e^t \qquad \text{for all } j.$$

(3) Let $\alpha_0 = \frac{1}{2\mathsf{m}+1}$. For all j and all $z \in \mathcal{E}_{ij}$

$$f^{(\alpha_0)}_{\mathcal{E}_{ij},0}(z) \le e^{Dt - \frac{k\alpha_0}{2}\ell} + e^{\varepsilon t}\cdot(\#F_{ij}).$$

Proof. We prove the lemma using induction on k. The base case, $k = 0$, follows from (8.8) applied with $\alpha = \alpha_0$. That is

$$f^{(\alpha_0)}_{\mathcal{E}_i,0}(z) \le e^{Dt} \qquad \text{for all } z \in \mathcal{E}_i. \tag{8.9}$$

Assume now the statement for some $0 \le k < d_0$:

$$\int \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}\nu_\ell^{(d_{\mathrm{fn}})}(h) = \sum_j c_{ij} \iint \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(s)\,\mathrm{d}\nu_\ell^{(d_{\mathrm{fn}}-k)}(h) + O(\mathrm{Lip}(\varphi)\beta^\star)$$

and properties (1), (2), and (3) above hold.

To obtain desired assertions for $k+1$, first note that (8.3) implies

$$\iint \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(s)\,\mathrm{d}\nu_\ell^{(d_{\mathrm{fn}}-k)}(h) = \iint_0^1\int \varphi(a_\tau u_s h a_\ell u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}\mathrm{d}r\,\mathrm{d}\nu_\ell^{(d_{\mathrm{fn}}-k-1)}(h) + O(\beta^\star\,\mathrm{Lip}(\varphi)) \tag{8.10}$$

Note again that for every $h \in \mathrm{supp}(\nu_\ell^{(d_{\mathrm{fn}}-k-1)})$ we have

$$a_\tau u_s h = a_{\tau+(d_{\mathrm{fn}}-k-1)\ell} u_{s'}, \qquad \text{for some } |s'| \le 2.$$

Thus, for all $|s| \le 2$ and all $h \in \mathrm{supp}(\nu_\ell^{(d_{\mathrm{fn}}-k-1)})$, we apply Lemma 7.2 with $\mathcal{E} = \mathcal{E}_{ij}$ for $\alpha = \alpha_0$ and $\mathsf{d} = (d_{\mathrm{fn}} - k - 1)\ell$ and s' as above, and conclude

$$\int_0^1\int \varphi(a_\tau u_s h a_\ell u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(z)\,\mathrm{d}r = \sum_\varsigma c'_\varsigma \int \varphi(a_\tau u_s h z)\,\mathrm{d}\mu_{\mathcal{E}'_\varsigma}(z) + O(\mathrm{Lip}(\varphi)\beta^\star), \tag{8.11}$$

where $0 \le c'_\varsigma \le 1$, $\sum c'_\varsigma = 1 + O(\beta^\star)$, and both of the following hold

- For all ς, $F'_\varsigma \subset B_{\mathfrak{r}}(0,\beta)$ and $\mathcal{E}'_\varsigma = \mathsf{E}.\{\exp(w)y'_\varsigma : w \in F'_\varsigma\} \subset X_\eta$, moreover
$$\beta^{4\mathsf{m}+6} \cdot (\#F_{ij}) \le \#F'_\varsigma \le \#F_{ij}. \tag{8.12}$$
- For all ς and all $z \in \mathsf{E}.\{\exp(w)y_i : w \in F'_\varsigma\}$, we have
$$f^{(\alpha_0)}_{\mathcal{E}'_\varsigma,0}(z) \le e^{-\alpha_0\ell/2} \cdot \left(e^{(Dt - \frac{k\alpha_0}{2}\ell)} + e^{\varepsilon t} \cdot (\#F_{ij})\right) + e^{\varepsilon t} \cdot (\#F'_\varsigma)$$
where we used $\delta_0 = 0$ and $\varpi = \alpha_0$ when $\alpha = \alpha_0$.

Using (2) in the inductive hypothesis (8.12), we have

$$\begin{aligned}\beta^{(4(k+1)+4)\mathsf{m}+6(k+1)+7}e^t &= \beta^{4\mathsf{m}+6}\beta^{(4k+4)\mathsf{m}+6k+7}e^t\\ &\le \beta^{4\mathsf{m}+6} \cdot (\#F_{ij}) \le \#F'_\varsigma \le \#F_{ij} \le e^t.\end{aligned}$$

Moreover, recall that $\varepsilon' = 10^{-6}\mathsf{m}^{-1}d_{\mathrm{fin}}^{-2}$ and $\beta = e^{-\varepsilon' t}$. Thus $e^{-\alpha_0\ell/2}\beta^{-4\mathsf{m}-7} \le 1$, and using (8.20) and $\#F_{ij} \le e^t$, we conclude

$$e^{-\alpha_0\ell/2}(e^{\varepsilon t} \cdot (\#F_{ij})) \le e^{-\alpha_0\ell/2}\beta^{-4\mathsf{m}-6}e^{\varepsilon t} \cdot (\#F'_\varsigma) \le e^{\varepsilon t} \cdot (\#F'_\varsigma),$$

Thus one obtains

$$\begin{aligned} e^{-\alpha_0\ell/2}\cdot\left(e^{(Dt-\frac{k\alpha_0}{2}\ell)}+e^{\varepsilon t}\cdot(\#F_{ij})\right) &\le e^{-\alpha_0\ell/2}e^{Dt-\frac{k\alpha_0}{2}\ell}+e^{\varepsilon t}\cdot(\#F'_\varsigma)\\ &\le e^{Dt-\frac{(k+1)\alpha_0}{2}\ell}+e^{\varepsilon t}\cdot(\#F'_\varsigma). \end{aligned}$$

Therefore, integrating (8.11) over $h\in\mathrm{supp}(\nu_\ell^{(d_{\mathrm{fn}}-k-1)})$, the proof of the inductive step and of the lemma is complete. □

Lemma 8.2 and initial dimension. Recall now that ε is small, $\ell=\frac{\varepsilon t}{10\mathsf{m}}$, $d_0=\lceil\frac{(10D-9)(4\mathsf{m}^2+2\mathsf{m})}{\varepsilon}\rceil$, $\varepsilon'=10^{-6}\mathsf{m}^{-1}d_{\mathrm{fn}}^{-2}$, and $\beta=e^{-\varepsilon' t}$. Thus

$$\begin{aligned} &e^{0.99t}\le\beta^{(4d_1+4)\mathsf{m}+6d_1+7}e^t\le\#F_{ij}\le e^t,\quad\text{and}\\ &e^{Dt-\frac{d_0\alpha_0}{2}\ell}\le e^{0.9t},\quad\text{where }\alpha_0=\tfrac{1}{2\mathsf{m}+1}. \end{aligned}$$

Applying Lemma 8.2 with $k=d_0$, hence, we conclude

$$\begin{aligned} \int\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}\nu_\ell^{(d_{\mathrm{fn}})}(h) &=\\ \sum_j c_{ij}\iint\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(z)\,\mathrm{d}\nu_\ell^{(d_1)}(h)&+O(\mathrm{Lip}(\varphi)\beta^\star) \end{aligned}\tag{8.13}$$

where $\mathcal{E}_i$ is any set appearing in (8.6), and all the following are satisfied

(D-1) $0\le c_{ij}\le 1$ and $\sum c_{ij}=1+O(\beta^\star)$.

(D-2) $\mathcal{E}_{ij}=\mathsf{E}.\{\exp(w)y_{ij}:w\in F_{ij}\}\subset X_\eta$ and

$$e^{0.99t}\le\#F_{ij}\le e^t\qquad\text{for all }j.$$

(D-3) For all j and all $z\in\mathcal{E}_{ij}$

$$f^{(\alpha_0)}_{\mathcal{E}_{ij},\delta}(z)\le 2e^{\varepsilon t}\cdot(\#F_{ij}).\tag{8.14}$$

Lemma 7.2 and incremental dimension improvement. Recall that $p_{\mathrm{fn}}=\lceil\frac{4\mathsf{m}^2+4\mathsf{m}}{\kappa(2\mathsf{m}+1)}\rceil-10$ and $\ell=\frac{\varepsilon t}{10\mathsf{m}}$. We let $\delta_1=e^{-0.99t}$ and let $d_1=\frac{495}{200\varepsilon}$. For all $0<p\le p_{\mathrm{fn}}$, let

$$\delta_{p+1}=\delta_p^{3/4}\quad\text{and}\quad d_{p+1}=\tfrac34 d_p.$$

Also, for all $0<p\le p_{\mathrm{fn}}$, and all $0<k\le d_p$, let

$$\delta_{p,0}=\delta_p\quad\text{and}\quad\delta_{p,k}=e^{\mathsf{m}\ell}\delta_{p,k-1};$$

note that $\delta_{p,d_p}=\delta_{p+1}$, and

$$\delta_p^{-\kappa}e^{-4\mathsf{m}\kappa d_p\ell}\le 1\tag{8.15}$$

Finally, we let $\alpha_p=\alpha_0+p\kappa$, for every $0\le p\le p_{\mathrm{fn}}$, and let ϖ_p be ϖ defined as in (6.2) with $\alpha=\alpha_p$. In particular,

$$\varpi_p\ge 9\mathsf{m}\kappa\quad\text{for all }0<p\le p_{\mathrm{fn}}.\tag{8.16}$$

The following lemma, which is an analogue of Lemma 8.2, will be used to inductively improve the dimension from $\frac{1}{2\mathsf{m}+1}$ to $2\mathsf{m}+1-10\mathsf{m}\kappa$.

8.3. **Lemma.** *Fix some $\mathcal{E}_i$ as in (8.6). Let $0<p<p_{\rm fn}$ and let $0\le k\le d_{p+1}$. For every $\varphi\in C_c(X)$ and all $|s|\le 2$, we have*

$$\int\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}\nu_\ell^{(d_{\rm fn})}(h)=\\ \sum_j c_{ij}\iint\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(z)\,\mathrm{d}\nu_\ell^{(d'(p)-k)}(h)+O(\mathrm{Lip}(\varphi)\beta^\star)$$

where $d'(p)=d_{\rm fn}-d_0-d(p)$, $d(p)=\sum_{q=1}^{p-1}d_q$ for $p>1$, and $d(1)=0$.

Moreover, all the following hold

(1) $0\le c_{ij}\le 1$ and $\sum c_{ij}=1+O(\beta^\star)$.
(2) $\mathcal{E}_{ij}=\mathsf{E}.\{\exp(w)y_{ij}:w\in F_{ij}\}\subset X_\eta$ and

$$\beta^{(4\mathsf{m}+6)(d(p)+k)}e^{0.99t}\le\#F_{ij}\le e^t\qquad\text{for all } j.$$

(3) For all j and all $z\in\mathcal{E}_{ij}$, we have

$$f^{(\alpha_p)}_{\mathcal{E}_{ij},\delta_{p,k}}(z)\le 2e^{\varepsilon t}\cdot(\delta_p^{-\kappa}e^{-4\mathsf{m}\kappa k\ell}+1)\cdot(\#F_{ij})$$

Proof. The proof is similar to the proof of Lemma 8.2, and is completed by induction on p and k. Indeed the case $p=1$ and $k=0$ follows from (8.13) and properties (D-1), (D-2), (D-3). Indeed, the assertions in (1) and (2) in this lemma (D-1) and (D-2) are the same. To see (3) in the lemma follows from (D-3), note that (8.14) implies

$$f^{(\alpha_0)}_{\mathcal{E}_{ij},0}(z)\le 2e^{\varepsilon t}\cdot(\#F_{ij})$$

Recall that $\alpha_1=\alpha_0+\kappa$ and $\delta_1=\delta_{1,0}=e^{-0.99t}$. Thus

$$\begin{aligned}f^{(\alpha_1)}_{\mathcal{E}_{ij},\delta_{1,0}}(z)&\le\delta_1^{-\kappa}\sum\max(\|w\|,\delta_1)^{-\alpha_0}\\&\le 2e^{\varepsilon t}\cdot\delta_1^{-\kappa}\cdot(\#F_{ij})\end{aligned}\tag{8.17}$$

as it is claimed in part (3) for $p=1$ and $k=0$.

Fix some p and assume now the statement for some $0\le k<d_{p+1}$:

$$\int\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}\nu_\ell^{(d_{\rm fn})}(h)=\\ \sum_j c_{ij}\iint\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(z)\,\mathrm{d}\nu_\ell^{(d'(p)-k)}(h)+O(\mathrm{Lip}(\varphi)\beta^\star)$$

and properties (1), (2), and (3) above hold.

To obtain desired assertions for $k+1$, first note that (8.3) implies

$$\iint\varphi(a_\tau u_s hz)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}(s)\,\mathrm{d}\nu_\ell^{(d'(p)-k)}(h)=\\ \iint_0^1\int\varphi(a_\tau u_s ha_\ell u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{ij}}\mathrm{d}r\,\mathrm{d}\nu_\ell^{(d'(p)-k-1)}(h)+O(\beta^\star\,\mathrm{Lip}(\varphi))\tag{8.18}$$

For every $h \in \operatorname{supp}\big(\nu_\ell^{(d'(p)-k)}\big)$ we have

$$a_\tau u_s h = a_{\tau+(d'(p)-k-1)\ell} u_{s'}, \qquad \text{for some } |s'| \le 2.$$

Thus, for all $|s| \le 2$ and $h \in \operatorname{supp}\big(\nu_\ell^{(d'(p)-k-1)}\big)$, Lemma 7.2 applied with $\mathcal{E} = \mathcal{E}_{ij}$ for $\alpha = \alpha_p$ and $\mathsf{d} = (d'(p) - k - 1)\ell$ and s' as above, gives

$$\int_0^1 \int \varphi(a_\tau u_s h a_\ell u_r z)\, \mathrm{d}\mu_{\mathcal{E}_{ij}}(z)\, \mathrm{d}r = \sum_\varsigma c'_\varsigma \int \varphi(a_\tau u_s h z)\, \mathrm{d}\mu_{\mathcal{E}'_\varsigma}(z) + O(\operatorname{Lip}(\varphi)\beta^\star), \tag{8.19}$$

where $0 \le c'_\varsigma \le 1$, $\sum c'_\varsigma = 1 + O(\beta^\star)$, and both of the following hold

- For all ς, $F'_\varsigma \subset B_{\mathfrak{r}}(0, \beta)$ and $\mathcal{E}'_\varsigma = \mathsf{E}.\{\exp(w) y'_\varsigma : w \in F'_\varsigma\} \subset X_\eta$, moreover

$$\beta^{4\mathsf{m}+6} \cdot (\# F_{ij}) \le \# F'_\varsigma \le \# F_{ij}. \tag{8.20}$$

- For all ς and all $z \in \mathsf{E}.\{\exp(w) y_i : w \in F'_\varsigma\}$, we have

$$f^{(\alpha_p)}_{\mathcal{E}'_\varsigma, \delta_{p,k+1}}(z) \le e^{-\varpi_p \ell/2} \cdot \big(2e^{\varepsilon t} \cdot (\delta_p^{-\kappa} e^{-4\mathsf{m}\kappa k \ell} + 1) \cdot (\# F_{ij})\big) + e^{\varepsilon t} \cdot (\# F'_\varsigma)$$

  where, $\delta_{p,k+1} = e^{\mathsf{m}\ell} \delta_{p,k}$.

Using (2) in the inductive hypothesis (8.20), we have

$$\begin{aligned} \beta^{(4\mathsf{m}+6)(d(p)+k+1)} e^{0.99t} &= \beta^{4\mathsf{m}+6} \beta^{(4\mathsf{m}+6)(d(p)+k)} e^{0.99t} \\ &\le \beta^{4\mathsf{m}+6} \cdot (\# F_{ij}) \le \# F'_\varsigma \le \# F_{ij} \le e^t. \end{aligned}$$

Moreover, recall from (8.16) that $\alpha'_p > 9\mathsf{m}\kappa$, also recall that $\beta = e^{-\varepsilon' t}$ where $\varepsilon' = 10^{-6} \mathsf{m}^{-1} d_{\text{fin}}^{-2}$. Using (8.20), thus

$$\begin{aligned} e^{-\varpi_p \ell/2} \cdot &\big(2e^{\varepsilon t} \cdot (\delta_p^{-\kappa} e^{-4\mathsf{m}\kappa k\ell} + 1) \cdot (\# F_{ij})\big) \\ &\le e^{-\varpi_p \ell/2} \big(2e^{\varepsilon t} \cdot (\delta_p^{-\kappa} e^{-4\mathsf{m}\kappa k\ell} + 1) \cdot \beta^{-4\mathsf{m}-6} \cdot (\# F'_\varsigma)\big) \\ &\le \delta_p^{-\kappa} e^{-4\mathsf{m}\kappa(k+1)\ell} \cdot (\# F'_\varsigma) \end{aligned}$$

This and the above estimates imply that

$$f^{(\alpha_p)}_{\mathcal{E}'_\varsigma, \delta_{p,k+1}}(z) \le (\delta_p^{-\kappa} e^{-4\mathsf{m}\kappa(k+1)\ell} + e^{\varepsilon t}) \cdot (\# F'_\varsigma)$$

where, $\delta_{p,k+1} = e^{\mathsf{m}\ell} \delta_{p,k}$.

Therefore, integrating (8.19) over $h \in \operatorname{supp}(\nu_\ell^{(d'(p)-k-1)})$, the proof of the inductive step.

After d_p steps, thus, one obtains

$$f^{(\alpha_p)}_{\mathcal{E}'_\varsigma, \delta_{p,d_p}}(z) \le (\delta_p^{-\kappa} e^{-4\mathsf{m}\kappa d_p \ell} + e^{\varepsilon t}) \cdot (\# F'_\varsigma).$$

Since $\delta_{p,d_p} = \delta_{p+1}$ and $\delta_p^{-\kappa} e^{-4\mathsf{m}\kappa d_p \ell} \le 1$, see (8.15), we conclude that

$$f^{(\alpha_p)}_{\mathcal{E}'_\varsigma, \delta_{p+1}}(z) \le 2e^{\varepsilon t} \cdot (\# F'_\varsigma). \tag{8.21}$$

Using $\alpha_{p+1} = \alpha_p + \kappa$, the above yields,

$$f^{(\alpha_{p+1})}_{\mathcal{E}'_\varsigma, \delta_{p+1}}(z) \le 2e^{\varepsilon t} \cdot \delta_{p+1}^{-\kappa} \cdot (\# F'_\varsigma).$$

Therefore, we may repeat the above for $0 \le k < d_{p+1}$.

This completes the proof. □

Conclusion of the proof of Proposition 8.1. Recall that

$$\kappa_{\rm fn} = \tfrac{99}{100} \times (\tfrac{3}{4})^{p_{\rm fn}} \quad \text{and} \quad \varepsilon = \kappa \cdot \kappa_{\rm fn}.$$

Also note that it follows from $\delta_{p,d_p} = \delta_p^{3/4} = \delta_{p+1}$ and $\delta_1 = e^{-0.99t}$ that

$$\delta_{p_{\rm fn}, d_{p_{\rm fn}}} = \delta_{p_{\rm fn}}^{3/4} = e^{-\kappa_{\rm fn} t}$$

Applying Lemma 8.2 with $\alpha_{p_{\rm fn}}$ and $k = d_{p_{\rm fn}}$, hence, we conclude

$$\int \varphi(a_\tau u_s h z)\, \mathrm{d}\mu_{\mathcal{E}_i}(z)\, \mathrm{d}\nu_\ell^{(d_{\rm fn})}(h) = \sum_j c_{ij} \iint \varphi(a_\tau u_s h z)\, \mathrm{d}\mu_{\mathcal{E}_{ij}}(z) + O(\mathrm{Lip}(\varphi)\beta^\star) \tag{8.22}$$

where $\mathcal{E}_i$ is any set appearing in (8.6), and all the following are satisfied

(1) $0 \le c_{ij} \le 1$ and $\sum c_{ij} = 1 + O(\beta^\star)$.

(2) $\mathcal{E}_{ij} = \mathsf{E}.\{\exp(w) y_{ij} : w \in F_{ij}\} \subset X_\eta$ and

$$e^{0.98t} \le \# F_{ij} \le e^t \qquad \text{for all } j,$$

where we used $e^{0.98t} \le \beta^{(4\mathsf{m}+6)d(p_{\rm fn})} e^{0.99t} \le \# F_{ij} \le e^t$.

(3) For all j and all $z \in \mathcal{E}_{ij}$

$$f^{(2\mathsf{m}+1-20\kappa)}_{\mathcal{E}_{ij}, \delta}(z) \le 2e^{\varepsilon t} \cdot (\# F_{ij}) \qquad \text{where } \delta = e^{-\kappa_{\rm fn} t},$$

see (8.21) and note that $\alpha_{p_{\rm fn}} \ge 2\mathsf{m} + 1 - 20\kappa$

In view of (1), (2), and (3), Proposition 8.1 follows from (8.22) and (8.6). □

9. From large dimension to equidistribution

The main result of this section is Proposition 9.1 which will be used in the final step of our proof of Theorem 1.1.

Let $0 < \kappa_5 \le 1$ be the constant given by Proposition 5.1 — this constant is closely related to the spectral gap (or mixing rate) in G/Γ, c.f. (5.1). Throughout this section, let $\kappa = (\frac{\kappa_5}{40\mathsf{m}})^2$ and $p_{\rm fn} = \lceil \frac{4\mathsf{m}^2 + 4\mathsf{m}}{\kappa(2\mathsf{m}+1)} \rceil - 10$. Let

$$\kappa_{\rm fn} = \tfrac{99}{100} \times (\tfrac{3}{4})^{p_{\rm fn}} \quad \text{and} \quad \varepsilon = \kappa \cdot \kappa_{\rm fn}$$

We also recall that $\beta = e^{-\varepsilon' t}$ and $\eta^2 = \beta$ where $0 < \varepsilon' < 10^{-6}\varepsilon^4$. Let

$$\alpha = 2\mathsf{m} + 1 - 20\kappa.$$

9.1. **Proposition.** *The following holds for all large enough t. Let $F \subset B_{\mathfrak{r}}(0, \beta)$ be a finite set with $\#F \geq e^{0.9t}$. Let*

$$\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\} \subset X_\eta$$

be equipped with an admissible measure $\mu_{\mathcal{E}}$, see §D.3. Assume further that the following is satisfied: For all $z = h\exp(w)y$ with $h \in \overline{\mathsf{E} \setminus \partial_{10\beta}\mathsf{E}}$,

$$f^{(\alpha)}_{\mathcal{E},\delta_0}(e, z) \leq 2e^{\varepsilon t} \cdot (\#F) \qquad \text{where } \delta_0 = e^{-\kappa_{\rm fn} t}. \tag{9.1}$$

Let τ be a parameter in the range $\frac{\kappa_{\rm fn}t}{8\mathsf{m}} \leq \tau \leq \frac{\kappa_{\rm fn}t}{4\mathsf{m}}$. Then

$$\Big| \int_0^1 \int \varphi(a_\tau u_r z) \,\mathrm{d}\mu_{\mathcal{E}}(z) \,\mathrm{d}r - \int \varphi \,\mathrm{d}m_X \Big| \ll \mathcal{S}(\varphi)\beta^\star$$

for all $\varphi \in C_c^\infty(X)$.

Proof. The proof of Proposition 9.1, which will be completed in a few steps, is based on Lemma 5.2, and in turn on Proposition 5.1.

Folner property. Let τ be as in the statement, and write $\tau = \ell_1 + \ell_2$ where

$$\ell_2 = \tau/(1+\kappa_5) \quad \text{and} \quad \ell_1 = \kappa_5 \ell_2; \tag{9.2}$$

In particular, $4\mathsf{m}\ell_2 \leq 4\mathsf{m}\tau \leq \kappa_{\rm fn}t = |\log\delta_0|$ and $\ell_2 \geq \tau/2 \geq |\log\eta|/\kappa_5$, where the parameter η is fixed above, see Lemma 5.2 for these choices.

For any $\varphi \in C_c^\infty(X)$, we have

$$\begin{aligned}\int_0^1 \int \varphi(a_\tau u_r z) \,\mathrm{d}\mu_{\mathcal{E}}(z) \,\mathrm{d}r = \\ \int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z) \,\mathrm{d}\mu_{\mathcal{E}}(z) \,\mathrm{d}r_2 \,\mathrm{d}r_1 + O(e^{-\ell_2}\operatorname{Lip}(\varphi))\end{aligned} \tag{9.3}$$

where the implied constant depends on X.

In view of (9.3), the proof of Proposition 9.1 is reduced to investigating the following

$$\int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z) \,\mathrm{d}\mu_{\mathcal{E}}(z) \,\mathrm{d}r_2 \,\mathrm{d}r_1.$$

9.2. **Conditional measures of $\mu_{\mathcal{E}}$.** Recall that $\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\}$. Fix some $v \in F$ and let $z = \exp(v)y$. Then

$$\begin{aligned}\mathsf{h}\exp(w)y &= \mathsf{h}\exp(w)\exp(-v)\exp(v)y \\ &= \mathsf{h}\mathsf{h}_w\exp(v_w)z\end{aligned} \tag{9.4}$$

where $\|\mathsf{h}_w - I\| \ll \beta^2$ and $\frac{1}{2}\|w - v\| \leq \|v_w\| \leq 2\|w - v\|$, see Lemma 3.2.

For our application here, it will be more convenient to *recenter* $\mathcal{E}$ from y to z. To that end, note that $w \mapsto v_w$, see (9.4), is a one-to-one map. Let $F_v = \{v_w : w \in F\}$, and let $\hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{20\beta}\mathsf{E}}$. Set

$$\hat{\mathcal{E}} := \hat{\mathsf{E}}.\{\exp(w)z : w \in F_v\}.$$

Then by (9.4) and since $\|\mathsf{h}_w - I\| \ll \beta^2$, we have $\hat{\mathcal{E}} \subset \mathcal{E}$; moreover,

$$\bar{\mu}_{\mathcal{E}}(\mathcal{E} \setminus \hat{\mathcal{E}}) \ll \beta.$$

Thus it suffices to show the claim in the lemma with $\bar{\mu}_{\mathcal{E}}$ replaced by

$$\hat{\mu} := \frac{1}{\bar{\mu}_{\mathcal{E}}(\hat{\mathcal{E}})} \bar{\mu}_{\mathcal{E}}|_{\hat{\mathcal{E}}}.$$

For later reference, let us also record that $\|\mathsf{h}_w - I\| \ll \beta^2$ and (9.4) imply that indeed

$$\hat{\mathcal{E}} \subset \mathcal{E}' := \mathsf{E}'.\{\exp(w)y : w \in F\} \tag{9.5}$$

where $\mathsf{E}' = \overline{\mathsf{E} \setminus \partial_{10\beta}\mathsf{E}}$. In particular, (9.1) holds for all $z \in \hat{\mathcal{E}}$.

Recall that $\hat{\mu}$ is the probability measure proportional to $\sum_w \hat{\mu}_w$ where $\mathrm{d}\hat{\mu}_w = \hat{\rho}_w \,\mathrm{d}m_H$ and $\hat{\rho}_w \asymp 1$. As it was mentioned earlier the proof of Proposition 9.1 relies on Proposition 5.1. To set the stage for the latter to be applicable, we will use Fubini's theorem to change the order of disintegration of $\hat{\mu}$ as follows. Let $z \in \hat{\mathcal{E}}$, then

$$z = \mathsf{h}\exp(v)z = \exp(\mathrm{Ad}(\mathsf{h})v)\mathsf{h}z \in \hat{\mathcal{E}}.$$

Moreover, $\mathrm{Ad}(\mathsf{h})v \in B_{\mathfrak{r}}(0, 8\bar{\eta}b)$. Since $\bar{\eta}/2 \le \mathrm{inj}(z') \le 2\bar{\eta}$ for every $z' \in \mathcal{E}$, we conclude that

$$\mathrm{Ad}(\mathsf{h})v \in I_{\mathcal{E}}(\mathsf{h}z).$$

Let $\pi : \hat{\mathcal{E}} \to \mathsf{E}.z$ denote the projection $z' = \mathsf{h}\exp(w)z \mapsto \mathsf{h}z$. Using Fubini's theorem, we have

$$\hat{\mu} = \int \hat{\mu}^{\mathsf{h}} \,\mathrm{d}\pi_*\hat{\mu}(\mathsf{h}.z),$$

where $\hat{\mu}^{\mathsf{h}}$ denotes the conditional measure of $\hat{\mu}$ for the factor map π.

Note that $\hat{\mu}^{\mathsf{h}}$ is supported on $I_{\mathcal{E}}(\mathsf{h}z)$. In view of the above discussion, $\mathrm{d}\pi_*\hat{\mu}$ is proportional to $\hat{\rho}\,\mathrm{d}m_H$ restricted to the support of $\pi_*\hat{\mu}$ where $1 \ll \hat{\rho} \ll 1$, moreover, for every $w \in \mathrm{supp}(\hat{\mu}^{\mathsf{h}})$,

$$\hat{\mu}^{\mathsf{h}}(w) \asymp (\#F)^{-1} \tag{9.6}$$

where the implied constant depends on X.

Now, using Fubini's theorem, we have

$$\int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\hat{\mu}(z)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \int_{\hat{\mathsf{E}}.z}\int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z)\,\mathrm{d}\hat{\mu}^{\mathsf{h}}(w)\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}\pi_*\hat{\mu}(\mathsf{h}.z).$$

Fix some $\mathsf{h} \in \hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{20b}\mathsf{E}}$. The proof of the proposition is thus reduced to investigating the following

$$\int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z)\,\mathrm{d}\hat{\mu}^{\mathsf{h}}(w)\,\mathrm{d}r_2\,\mathrm{d}r_1. \tag{9.7}$$

Discretized dimension of $\hat\mu^{\mathsf{h}}$ and Lemma 5.2. Set

$$F^{\mathsf{h}} := \operatorname{supp}(\hat\mu^{\mathsf{h}}) = \{\mathrm{Ad}(\mathsf{h})w : w \in F\}.$$

Note that (9.5) implies

$$\exp(\mathrm{Ad}(\mathsf{h})w)\mathsf{h}z = \mathsf{h}\exp(w)z \in \hat{\mathcal{E}}_i \subset \mathcal{E}'$$

Moreover, (9.1) and Lemma 7.1 imply that for every $w \in F^{\mathsf{h}}$,

$$\mathcal{G}^{(\alpha)}_{F^{\mathsf{h}},\delta}(w) \ll 2e^{\varepsilon t} \cdot (\#F),$$

where $\mathcal{G}$ is defined as in §6. This and (9.6) yield

$$\hat\mu^{\mathsf{h}}(B(w,\delta)) \ll 2e^{\varepsilon t}\delta^{\alpha} \qquad \text{for all } \delta \ge \delta_0 \tag{9.8}$$

where the implied constant depends only on X.

Therefore, Lemma 5.2 applies with $\mu = \hat\mu^{\mathsf{h}}$, $\varrho = \beta$, $\delta_0 = e^{-\kappa_{\rm fn} t}$,

$$\Upsilon = 2\delta_0^{-20\kappa}e^{\varepsilon t} = 2e^{20\kappa\kappa_{\rm fn}t}e^{\varepsilon t},$$

and ℓ_1 and ℓ_2 as above, see (9.2). By the conclusion of that lemma, thus,

$$\begin{aligned}\int_0^1\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z)\,\mathrm{d}\hat\mu^{\mathsf{h}}(w)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \int\varphi\,\mathrm{d}m_X + O\big(\mathcal{S}(\varphi)(\beta^\star + \eta + \Upsilon^{1/2}\beta^{-3/2}e^{-\kappa_5\ell_1})\big)\end{aligned} \tag{9.9}$$

Recall now that $\tau \ge \frac{\kappa_{\rm fn}t}{8\mathsf{m}}$ and $\ell_1 = \kappa_5\ell_2 = \frac{\kappa_5\tau}{1+\kappa_5}$. Therefore, $\ell_1 \ge \frac{\kappa_{\rm fn}\kappa_5 t}{10\mathsf{m}}$, and

$$\Upsilon^{1/2}\beta^{-3/2}e^{-\kappa_5\ell_1} \le e^{10\kappa_{\rm fn}\kappa t}e^{\varepsilon t}e^{-\frac{\kappa_{\rm fn}\kappa_5^2}{10\mathsf{m}}t} \le e^{-\frac{\kappa_{\rm fn}\kappa_5^2}{20\mathsf{m}}t},$$

where in the second to last inequality we used $2e^{\varepsilon t/2}\beta^{-3/2} \le e^{\varepsilon t}$, and in last inequality, we used $\kappa = (\frac{\kappa_5}{40\mathsf{m}})^2$ and $\varepsilon = \kappa\cdot\kappa_{\rm fn}$.

In view of (9.9), thus, the proof of Proposition 9.1 is complete. □

10. Proof of Theorem 1.1

The proof will be completed in some steps and is based on various propositions which were discussed so far.

Fixing the parameters. Let $0 < \kappa_5 \le 1$ be the constant given by Proposition 5.1. Let $\kappa = (\frac{\kappa_5}{40\mathsf{m}})^2$ and $p_{\rm fn} = \lceil\frac{4\mathsf{m}^2+4\mathsf{m}}{\kappa(2\mathsf{m}+1)}\rceil - 10$. Put

$$\kappa_{\rm fn} = \tfrac{99}{100}\times(\tfrac{3}{4})^{p_{\rm fn}} \quad \text{and} \quad \varepsilon = \kappa\cdot\kappa_{\rm fn} \tag{10.1}$$

Let $D = D_0D_1 + 2D_1$ where D_0 is as in Proposition 4.4 and D_1 is as in Proposition 4.6; we will always assume $D_1, D_0 \ge 10\mathsf{m}$. We will show the claim holds with

$$A_1 = 2D_0(2\mathsf{m}+1) + \mathsf{m}_0 + \mathsf{m}_1 + 4, \tag{10.2}$$

where m_0 is as in Proposition 4.2 and m_1 is as in Proposition 4.6.

Let us also assume (as we may) that

$$R \ge \max\{(10C_3)^3\,\mathrm{inj}(x_0)^{-\mathsf{m}_0}, e^{C_3}, e^{s_0}, C_4\}, \tag{10.3}$$

see Proposition 4.2 and Proposition 4.4.

Let $T \geq R^{A_1}$, and suppose that Theorem 1.1(2) does not hold. That is, for every $x \in X$ with Hx periodic and $\mathrm{vol}(Hx) \leq R$,

$$d_X(x, x_0) > R^{A_1}(\log T)^{A_1} T^{-\mathsf{m}} \geq (\log S)^{D_0} S^{-\mathsf{m}} \tag{10.4}$$

where $S := R^{-A_1}T$.

Folner property and random walks. We put $d_{\mathrm{fn}} = \sum_{p=0}^{p_{\mathrm{fn}}} d_p$, where

$$\begin{aligned} d_0 &= \lceil \tfrac{(10D-9)(4\mathsf{m}^2+2\mathsf{m})}{\varepsilon} \rceil, \\ d_1 &= \lceil \tfrac{495}{200\varepsilon} \rceil \quad \text{and} \quad d_{p+1} = \lceil \tfrac{3}{4} d_p \rceil \quad \text{for } 0 < p < p_{\mathrm{fn}} \end{aligned}$$

Let $t = \frac{1}{D_1}\log R$, and let $\ell = \frac{\varepsilon t}{10\mathsf{m}}$. Then

$$d_{\mathrm{fn}}\ell \leq \big(\tfrac{(10D-9)(2\mathsf{m}+1)}{5} + \tfrac{99}{100\mathsf{m}}\big)t + \varepsilon t \tag{10.5}$$

We now write $\log T = t_3 + t_2 + t_1 + t_0$ where

$$t_1 = \mathsf{m}_1 t, \quad t_2 = t + d_{\mathrm{fn}}\ell, \quad \text{and} \quad t_3 = \tfrac{\kappa_{\mathrm{fn}}}{8\mathsf{m}} t \tag{10.6}$$

The choice of t_2 is motivated by Proposition 8.1, the choices of t_1 and t_3 are motivated by Proposition 4.6 and Proposition 9.1, respectively. Thus

$$\begin{aligned} t_0 &= \log T - (t_1 + t_2 + t_3) \\ &\geq \log T - \big(\mathsf{m}_1 + 1 + \big(\tfrac{(10D-9)(2\mathsf{m}+1)}{5} + \tfrac{99}{100\mathsf{m}}\big) + \varepsilon + \tfrac{\kappa_{\mathrm{fn}}}{8\mathsf{m}}\big)t \\ &\geq \log T - \tfrac{\mathsf{m}_1 + 2D(2\mathsf{m}+1)+2}{D_1}\log R \end{aligned}$$

we used (10.6) and (10.5) in the second line, and used $t = \frac{1}{D_1}\log R$ in the third line. Using this and (10.2), we conclude that

$$\begin{aligned} t_0 &\geq \log T - A_1 \log R + (\mathsf{m}_0 + 2)\log R \\ &\geq \log S + \mathsf{m}_0 |\log \mathrm{inj}(x_0)| + 2\log R. \end{aligned} \tag{10.7}$$

we used $R \geq \mathrm{inj}(x_0)^{-\mathsf{m}_0}$ and $\log S = \log T - A_1 \log R$ in the last inequality.

Since $a_{\rho_1} u_r a_{\rho_2} = a_{\rho_1+\rho_2} u_{e^{-\rho_2}r}$, for any $\varphi \in C_c^\infty(X)$, we have

$$\begin{aligned} \int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r = O(\|\varphi\|_\infty e^{-t}) + \\ \int_0^1\int_0^1\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}a_{t_1}u_{r_1}a_{t_0}u_{r_0}x_0)\,\mathrm{d}r_3\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}r_0 \end{aligned} \tag{10.8}$$

where the implied constant is absolute and we used $t_0, t_1, t_2 \geq t$.

Finally, recall that $0 < \varepsilon' < 10^{-6}\varepsilon^4$ and we put $\beta = e^{-\varepsilon' t}$ and $\eta^2 = \beta$.

Improving the Diophantine condition. Apply Proposition 4.4 with $S = R^{-A_1}T$, then for all

$$s \geq \max\{\log S, \mathsf{m}_0|\log \mathrm{inj}(x_0)|\} + s_0,$$

we have the following

$$\left|\left\{r \in [0,1] : \begin{array}{c} a_s u_r x_0 \notin X_\eta \text{ or } \exists\, x \text{ with } \mathrm{vol}(Hx) \leq R \\ \text{so that } d_X(x, a_s u_r x_0) \leq R^{-D_0-1} \end{array}\right\}\right| \ll \eta^{1/\mathsf{m}_0}, \tag{10.9}$$

where we also used $\eta^{1/\mathsf{m}_0} \geq R^{-1}$ and $R \geq C_4$.

Let $J_0 \subset [0,1]$ be the set of those $r_0 \in [0,1]$ so that $a_{t_0} u_{r_0} x_0 \in X_\eta$ and

$$d_X(x, a_{t_0} u_{r_0} x_0) > R^{-D_0-1} = e^{-D_1(D_0+1)t}$$

for all x with $\mathrm{vol}(Hx) \leq R = e^{D_1 t}$. Then since by (10.7) and (10.3) we have

$$t_0 \geq \log S + \mathsf{m}_0|\log \mathrm{inj}(x_0)| + 2\log R \geq \max(\log S, \mathsf{m}_0|\log \mathrm{inj}(x_0)|) + s_0,$$

the assertion in (10.9) implies that $|[0,1] \setminus J_0| \ll \eta^{1/\mathsf{m}_0}$. In consequence,

$$\begin{aligned}\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r = O(\|\varphi\|_\infty \eta^{1/\mathsf{m}_0}) &+ \\ \int_{J_0}\int_0^1\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}a_{t_1}u_{r_1}x(r_0))\,\mathrm{d}r_3\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}r_0&\end{aligned} \tag{10.10}$$

where $x(r_0) = a_{t_0} u_{r_0} x_0$ and the implied constant depends on X.

Applying the closing lemma. For every $r_0 \in J_0$, we now apply Proposition 4.6 with $x(r_0)$, $D = D_0 D_1 + 2D_1$ and the parameter t. For any such r_0, we have

$$d_X(x, x(r_0)) > e^{-D_1(D_0+1)t} = e^{(-D+D_1)t}$$

for all x with $\mathrm{vol}(Hx) \leq e^{D_1 t}$. Thus Proposition 4.6(1) holds. Let

$$J_1(r_0) = I(x(r_0)) = I(a_{t_0} u_{r_0} x_0)$$

Then

$$\begin{aligned}\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r = O(\|\varphi\|_\infty \eta^{\frac{1}{2\mathsf{m}_0}}) &+ \\ \int_{J_0}\int_{J_1(r_0)}\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}x(r_0,r_1))\,\mathrm{d}r_3\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}r_0&\end{aligned} \tag{10.11}$$

where $x(r_0, r_1) = a_{t_1} u_{r_1} a_{t_0} u_{r_0} x_0$ and the implied constant is absolute.

Improving the dimension phase. Fix some $r_0 \in J_0$, and let $r_1 \in J(r_0)$. Put $x_1 = x(r_0, r_1)$. Recall that

$$\mu_{t,\ell,d_{\mathrm{fn}}} = \nu_\ell * \cdots * \nu_\ell * \nu_t$$

where ν_ℓ appears d_{fn} times in the above expression. Then

$$\begin{aligned}(10.12)\quad & \int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}x_1)\,\mathrm{d}r_3\,\mathrm{d}r_2 = \\ & \qquad \iint_0^1 \varphi(a_{t_3}u_{r_3}hx_1)\,\mathrm{d}r_3\,\mathrm{d}\mu_{t,\ell,d_{\mathrm{fn}}}(h) + O(\mathrm{Lip}(\varphi)e^{-\ell}),\end{aligned}$$

where we used $t_2 = d_{\mathrm{fn}}\ell$ and (8.2), see also [LMW22, Lemma 7.4].

We now apply Proposition 8.1 with x_1 and $\tau = t_3$ and $s = r_3$. Then

$$\begin{aligned}(10.13)\quad & \int \varphi(a_{t_3}u_{r_3}hx_1)\,\mathrm{d}\mu_{t,\ell,d_{\mathrm{fn}}}(h) = \\ & \qquad \sum_i c_i\int \varphi(a_{t_3}u_{r_3}z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z) + O(\mathrm{Lip}(\varphi)\beta^\star)\end{aligned}$$

where $0 \le c_i \le 1$ and $\sum_i c_i = 1 - O(\beta^\star)$ and the implied constants depend on X. Moreover, for all i we have

$$\mathcal{E}_i = \mathsf{E}.\{\exp(w)y_i : w \in F_i\} \subset X_\eta$$

with $F_i \subset B_{\mathfrak{r}}(0,\beta)$, $e^{0.98t} \le \#F_i \le e^t$, and

$$(10.14)\quad f^{(\alpha)}_{\mathcal{E}_i,\delta}(z) \le e^{2\varepsilon t}\cdot(\#F_i) \quad \text{where } \delta = e^{-\kappa_{\mathrm{fn}}t} \text{ and } \alpha = 2\mathsf{m} + 1 - 20\kappa$$

for all $z \in (\overline{\mathsf{E} \setminus \partial_{10\beta}\mathsf{E}}).\{\exp(w)y_i : w \in F_i\}$.

From large dimension to equidistribution. We now apply Proposition 9.1 with $\mathcal{E}_i$ (see (10.14) and recall that $t_3 = \frac{\kappa_{\mathrm{fn}}t}{8\mathsf{m}}$). Hence,

$$(10.15)\qquad \Big|\int_0^1\int \varphi(a_{t_3}u_{r_3}z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}r_3 - \int \varphi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)\beta^\star$$

where the implied constant depends on X.

Recall now that $\beta = R^{-\star}$. Thus, (10.15), (10.13), (10.12), (10.11), (10.10), and (10.8), imply

$$\Big|\int_0^1 \varphi(a_{\log T}u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)R^{-\star},$$

where the implied constant depends on X. The proof is complete. □

11. Proof of Theorem 1.2

In this section, we use an argument analogous to [LMW22, §16] to establish Theorem 1.2. As in [LMW22, §16] and previously noted, the proof relies on Theorem 1.1 and linearization techniques of Dani and Margulis, albeit in their quantitative form, which were developed in [LMMS19].

11.1. **Lemma** (cf. [LMW22], Lemma 16.1)**.** *There exist A_5, A_6, and C_5 (depending on X) so that the following holds. Let $S, M > 0$, and $0 < \eta < 1/2$ satisfy*

$$S \geq M^{A_5} \quad \text{and} \quad M \geq C_5 \eta^{-A_5}.$$

Let $x_1 \in X_\eta$, and suppose there exists $\mathrm{Exc} \subset \{r \in [-S, S] : u_r x_1 \in X_\eta\}$ with

$$|\mathrm{Exc}| > C_5 \eta^{1/A_6} S$$

so that for every $r \in \mathrm{Exc}$, there exists $y_r \in X$ with

$$\mathrm{vol}(H.y_r) \leq M \quad \text{and} \quad d(u_r x_1, y_r) \leq M^{-A_5}.$$

Then one of the following holds

(1) There exists $x \in G/\Gamma$ with $\mathrm{vol}(H.x) \leq M^{A_5}$, and for every $r \in [-S, S]$ there exists $g \in G$ with $\|g\| \leq M^{A_5}$ so that

$$d_X(u_s x_1, gH.x) \leq M^{A_5} \left(\frac{|s - r|}{S}\right)^{1/A_6} \quad \text{for all } s \in [-S, S].$$

(2) There is a parabolic subgroup $P \subset G$ and some $x \in G/\Gamma$ satisfying that $\mathrm{vol}(R_u(P).x) \leq M^{A_5}$, and for every $r \in [-S, S]$ there exists $g \in G$ with $\|g\| \leq M^{A_5}$ so that

$$d_X(u_s x_1, gR_u(P).x) \leq M^{A_5} \left(\frac{|s - r|}{S}\right)^{1/A_6} \quad \text{for all } s \in [-S, S].$$

In particular, X is not compact.

Arithmetic groups. Recall that $G = \mathbf{G}(\mathbb{R})$ where $\mathbf{G}$ is an absolutely almost simple $\mathbb{R}$-group, and $H = \mathbf{H}(\mathbb{R})^\circ$ where $\mathbf{H} \simeq \mathrm{SL}_2$ or PGL_2 is an $\mathbb{R}$-subgroup of $\mathbf{G}$ and the connected component is considered as a Lie group.

Since Γ is an arithmetic lattice, there exists a semisimple simply connected $\mathbb{Q}$-almost simple $\mathbb{Q}$-group $\tilde{\mathbf{G}} \subset \mathrm{SL}_{\mathsf{D}}$, for some D, and an epimorphism

$$\rho : \tilde{\mathbf{G}}(\mathbb{R}) \to \mathbf{G}(\mathbb{R}) = G$$

of $\mathbb{R}$-groups with compact kernel so that Γ is commensurable with $\rho(\tilde{\mathbf{G}}(\mathbb{Z}))$. Moreover, since $\tilde{\mathbf{G}}$ is simply connected, we can identify $\tilde{\mathbf{G}}(\mathbb{R})$ with $G \times G'$ where $G' = \ker(\rho)$ is compact.

We are allowed to choose the parameter M in the lemma to be large depending on Γ, therefore, by passing to a finite index subgroup, we will assume that $\Gamma \subset \tilde{\Gamma} := \rho(\tilde{\mathbf{G}}(\mathbb{Z}))$, where $\tilde{\mathbf{G}}(\mathbb{Z}) = \tilde{\mathbf{G}}(\mathbb{R}) \cap \mathrm{SL}_{\mathsf{D}}(\mathbb{Z})$.

Thus, every $\gamma \in \Gamma$ lifts uniquely to $(\gamma, \sigma(\gamma)) \in \tilde{\Gamma}$, where σ is (a collection of) Galois automorphisms. For every $g \in G$, put

$$\hat{g} = (g, 1) \in G \times G'.$$

If $g \in G$ is so that $Hg\Gamma$ is periodic, let $\Delta_g = \Gamma \cap g^{-1}Hg$ and let $\tilde{\Delta}_g = \rho^{-1}(\Delta_g) \cap \tilde{\Gamma}$. Let $\tilde{\mathbf{H}}_g$ be the Zariski closure of $\tilde{\Delta}_g$. Then $\tilde{\mathbf{H}}_g$ is a semisimple $\mathbb{Q}$-subgroup, and the restriction of ρ to $\tilde{\mathbf{H}}_g$ surjects onto $g^{-1}\mathbf{H}g$.

Let $\tilde{H}_g = \tilde{\mathbf{H}}_g(\mathbb{R})^\circ$, as a Lie group, then

$$\overline{\hat{g}^{-1}\hat{H}\hat{g}\tilde{\Gamma}} = \tilde{H}_g\tilde{\Gamma}$$

Lie algebras and the adjoint representation. Recall that $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(H) = \mathfrak{h}$, which are considered as $\mathbb{R}$-vector spaces. Let v_H be a unit vector on the line $\wedge^3\mathfrak{h}$. Then

$$N_G(H) = \{g \in G : gv_H = v_H\}$$

which contains H as a subgroup of finite index.

Let $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{\mathbf{G}}(\mathbb{R}))$ and $\tilde{\mathfrak{g}}_{\mathbb{Z}} := \tilde{\mathfrak{g}} \cap \mathfrak{sl}_N(\mathbb{Z})$. Then $\tilde{\mathfrak{g}}$ has a natural $\mathbb{Q}$-structure, and $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ is a $\tilde{\mathbf{G}}(\mathbb{Z})$-stable lattice in $\tilde{\mathfrak{g}}$.

Assuming H has periodic orbits in $X = G/\Gamma$, we fix $\mathsf{g}_1, \ldots, \mathsf{g}_k$ so that $\mathrm{vol}(H\mathsf{g}_i\Gamma) \ll 1$ (the implied constant and k depend on Γ) and that every $\tilde{H}_g$ is conjugate to some $\tilde{H}_i = \tilde{H}_{\mathsf{g}_i}$ in $\tilde{G}$. Let

$$\mathbf{v}_i \in \wedge^{\dim \tilde{H}_i}(\mathrm{Lie}(\tilde{H}_i)) \subset \wedge^{\dim \tilde{H}_i}\tilde{\mathfrak{g}}$$

be a primitive integral vector. Then $N_{\tilde{G}}(\tilde{H}_i) = \{g \in \tilde{G} : gv_i = v_i\}$, and $\tilde{H}_i \subset N_{\tilde{G}}(\tilde{H}_i)$ has finite index. For all i,

$$\mathbf{v}_i = c_i \cdot \big((g_i^{-1}v_H) \wedge v_i'\big) \quad \text{where } v_i' \in \wedge\mathrm{Lie}(G') \text{ and } |c_i| \asymp 1.$$

More generally, if $\mathbf{L} \subset \tilde{\mathbf{G}}$ is a $\mathbb{Q}$-subgroup, we let $\mathbf{v}_L$ be a primitive integral vector on the line $\wedge^{\dim L}\mathrm{Lie}(L) \subset \wedge^{\dim L}\tilde{\mathfrak{g}}$ where $L = \mathbf{L}(\mathbb{R})^\circ$. Recall from [LMMS19] the definition of the height of $\mathbf{L}$

$$\mathrm{ht}(\mathbf{L}) = \|\mathbf{v}_L\|. \tag{11.1}$$

Fix a right invariant metric on $\tilde{G}$ defined using the killing form and the maximal compact subgroup $\tilde{K} = K \times G'$; this metric induces the right invariant metric on G which we fixed on p. 4.

11.2. **Lemma.** *Let $Hg\Gamma$ be a periodic orbit, and let $\tilde{\mathbf{H}}_g$ be as above. Both of the following properties hold:*

$$\mathrm{ht}(\tilde{\mathbf{H}}_g)^\star \ll \mathrm{vol}(\tilde{H}_g\tilde{\Gamma}/\tilde{\Gamma}) \ll \mathrm{ht}(\tilde{\mathbf{H}}_g)^\star$$

$$\|g\|^{-\star}\mathrm{vol}(Hg\Gamma) \ll \mathrm{vol}(\tilde{H}_g\tilde{\Gamma}/\tilde{\Gamma}) \ll \|g\|^\star\mathrm{vol}(Hg\Gamma)$$

Proof. We refer the reader to [LMW22, Lemma 16.2] and references there for the proof of this lemma. □

We now proceed to prove Lemma 11.1. This proof follows an adaptation of the argument presented in [LMW22, §16], which we recount here for the reader's convenience.

Proof of Lemma 11.1. Given our assumption in the lemma, periodic H orbits exist. Let $\tilde{H}_1, \ldots, \tilde{H}_k$ be defined as above. We introduce the constants A_5 and A_6 which will be chosen as sufficiently large values to be explicated later. Specifically, we will require that $A_5 > \max(A, D_2)$, $A_6 > D$ and

$C_5 > \max\{kE_1, C_8\}$ where A, D, and E_1 are as in [LMMS19, Thm. 1.4] applied with $\{\hat{u}_r\} \subset \tilde{G}$, and D_2 and C_8 are as described in Lemma B.10.

Write $x_1 = g_1\Gamma$, where $\|g_1\| \leq C_8\eta^{-D_2} \leq M$, see Lemma B.10 and our assumption in this lemma. For every $r \in \mathrm{Exc}$, write $y_r = g(r)\Gamma$ where $\|g(r)\| \leq M$, indeed, for every such r there exists $\gamma_r \in \Gamma$ so that

$$u_r g_1 \gamma_r = \epsilon(r) g(r) \quad \text{and} \quad \|u_r g_1 \gamma_r\| \leq M + 1, \tag{11.2}$$

where $\|\epsilon(r)\| \ll M^{-A_5}$.

For every $1 \leq i \leq k$, let

$$\mathrm{Exc}_i = \{r \in \mathrm{Exc} : \tilde{H}^r := \tilde{H}_{g(r)} \text{ is a conjugate of } \tilde{H}_i\}.$$

There is some i so that $|\mathrm{Exc}_i| \geq |\mathrm{Exc}|/k$. Replacing Exc by Exc_i, we assume that $\tilde{H}^r$ is a conjugate of $\tilde{H}_i$ for all $r \in \mathrm{Exc}$. Put $\tilde{H}^r = \tilde{g}(r)^{-1}\tilde{H}_i\tilde{g}(r)$. Then

$$\tilde{g}(r) = (\mathrm{g}_i^{-1} g(r), \tilde{g}'(r)) \in G \times G',$$

and $\mathbf{v}^r := \frac{\|\mathbf{v}_{\tilde{H}^r}\|}{\|\tilde{g}(r)^{-1}\mathbf{v}_i\|}\tilde{g}(r)^{-1}\mathbf{v}_i = \pm\mathbf{v}_{\tilde{H}^r}$. Moreover, we have

$$\mathbf{v}^r = c_r \cdot \big((g(r)^{-1}v_H) \wedge (\tilde{g}'(r)^{-1}v_i')\big) \quad \text{where } |c_r| \ll \mathrm{ht}(\tilde{\mathbf{H}}_g)M \ll M^\star \tag{11.3}$$

where we used Lemma 11.2 to conclude $\mathrm{ht}(\tilde{\mathbf{H}}_g)M \ll M^\star$.

In view of (11.2), we have

$$\hat{u}_r\hat{g}_1(\gamma_r, \sigma(\gamma_r)).\mathbf{v}^r = c_r \cdot \Big((\epsilon(r)v_H) \wedge \big((\sigma(\gamma_r)\tilde{g}'(r)^{-1})v_i'\big)\Big), \tag{11.4}$$

where $\hat{g} = (g, 1)$ for all $g \in G$. Since G' is compact, (11.4) implies

$$\|\hat{u}_r\hat{g}_1(\gamma_r, \sigma(\gamma_r)).\mathbf{v}^r\| \leq M^{\hat{A}}, \quad \text{for some } \hat{A}. \tag{11.5}$$

Let $z \in \mathfrak{g}$ be a vector so that $u_r = \exp(rz)$. Using (11.4) and associativity of the exterior algebra, we have

$$\begin{aligned}\|\hat{z} \wedge \big(\hat{u}_r\hat{g}_1(\gamma_r, \sigma(\gamma_r)).\mathbf{v}^r\big)\| &= |c_r|\big\|\big(z \wedge \epsilon(r)v_H\big) \wedge \big((\sigma(\gamma_r)\tilde{g}'(r)^{-1})v_i'\big)\big\| \\ &\ll M^\star M^{-A_5} < \eta^A M^{-A\hat{A}}/E_1. \end{aligned} \tag{11.6}$$

where we used $\|\epsilon(r)\| \ll M^{-A_5}$ in the second to last inequality, A and E_1 are as in [LMMS19, Thm. 1.4], and we choose A_5 large enough so that the last estimate holds.

In view of (11.5) and (11.6), conditions in [LMMS19, Cor. 7.2] are satisfied. Hence, there exist $r \in \mathrm{Exc}$, $\tilde{\gamma} = (\gamma, \sigma(\gamma)) \in \tilde{\Gamma}$, and a subgroup

$$\tilde{\mathbf{H}}' \subset \tilde{\gamma}^{-1}\tilde{\mathbf{H}}^r\tilde{\gamma} \cap \tilde{\mathbf{H}}^r$$

satisfying that $\tilde{\mathbf{H}}'(\mathbb{C})$ is generated by unipotent subgroups (see [LMMS19, p. 3]) so that for all $r \in [-S, S]$ both of the following hold

$$\|\hat{u}_r\hat{g}_1\mathbf{v}_{\tilde{H}'}\| \ll M^\star \tag{11.7a}$$

$$\|\hat{z} \wedge (\hat{u}_r\hat{g}_1\mathbf{v}_{\tilde{H}'})\| \ll S^{-1/D}M^\star. \tag{11.7b}$$

Let $\tilde{H}' = \tilde{\mathbf{H}}'(\mathbb{R})^\circ$. Since $\|g_1\| \leq M$, applying (11.7a) with $r = 0$, we get

$$\|\mathbf{v}_{\tilde{H}'}\| \ll M^\star. \tag{11.8}$$

There are two possibilities for $\tilde{H}'$:

Case 1. $\rho(\tilde{H}')$ is a conjugate of H.

This in particular implies that

$$\rho(\tilde{H}') = g(r_0)^{-1}Hg(r_0) \quad \text{where } r_0 \in \mathrm{Exc} \text{ is as above.}$$

Put $g' = g(r_0)$. Then $\|g'\| \le M$ and we have

$$\begin{aligned}\mathrm{vol}(Hg'\Gamma/\Gamma) &\ll \|g'\|^\star \mathrm{vol}(g'^{-1}Hg'\Gamma/\Gamma)\\ &\ll M^\star \mathrm{ht}(\tilde{\mathbf{H}}') \ll M^\star\end{aligned} \tag{11.9}$$

where we used Lemma 11.2 in the second and (11.8) in the last inequality.

Recall that if we choose a maximal compact $K' \subset G$ associated to a Cartan involution which stabilizes H, then $G = K\exp(\mathfrak{r}')H$, where $\mathfrak{r}' = \mathfrak{r} \cap (\mathrm{Lie}(K'))^\perp$, see e.g. [EMS96, Thm. A.1]. This and (11.7a) imply that for every $r \in [-S, S]$, we may write

$$u_r g_1 g' v_H = g'_r g' v_H, \quad \text{where } \|g'_r\| \ll M^\star.$$

Since the map $s \mapsto u_s g_1 g' v_H$ is a polynomial with coefficients $\ll M^\star$,

$$u_s g_1 g' v_H = \epsilon'(s, r) g'_r g' v_H \quad \text{where } \|\epsilon'(s, r)\| \ll M^\star \big(|s - r|/S\big)^\star.$$

Using the fact that d is right invariant, the above implies

$$d(u_s g_1, g'_r g'^{-1} H g') \ll M^\star \big(|s - r|/S\big)^\star;$$

hence part (1) in the lemma holds if for every $r \in [-S, S]$ we let $g = g'_r g'^{-1}$.

Case 2. $\rho(\tilde{H}') = g'^{-1}Ug'$ where $U = \{u_r\}$.

First note that if this holds, then Γ is a non-uniform lattice and we may identify $\tilde{\mathbf{G}}$ and $\mathbf{G}$ as $\mathbb{R}$-groups. Thus $\mathbf{v}_{\tilde{H}'} \in \mathrm{Lie}(G)$, and we have

$$\exp(\mathbf{v}_{\tilde{H}'}) \in \tilde{H}' \cap \Gamma.$$

Recall from §3 that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ where $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{r}$ are 3 and $2\mathsf{m} + 1$ irreducible representation $\mathrm{Ad}(H)$, respectively. Let us write

$$g_1 \mathbf{v}_{\tilde{H}'} = w + w', \quad \text{where } w_1 \in \mathfrak{h} \text{ and } w_2 \in \mathfrak{r}.$$

Now (11.7a) implies that for every $r \in [-S, S]$ we have

$$\|\,\mathrm{Ad}(u_r)w\|, \|\,\mathrm{Ad}(u_r)w'\| \ll M^\star$$

Using standard representation theory of $\mathrm{SL}_2(\mathbb{R})$ (recall that $H \simeq \mathrm{SL}_2(\mathbb{R})$ or $H \simeq \mathrm{PSL}_2(\mathbb{R})$), we may write $\mathrm{Ad}(u_r)\bullet$ in the basis consisting of weight spaces for a_t and conclude that for all $t \in [\log M, \log S]$,

$$\begin{aligned}&\|(\mathrm{Ad}(a_{-t}u_r)w)_i\| \ll M^\star e^{-t} \quad \text{for } i = -1, 0, 1,\\ &\|(\mathrm{Ad}(a_{-t}u_r)w')_j\| \ll M^\star e^{-\mathsf{m}t} \quad \text{for } j = -\mathsf{m}, \ldots, \mathsf{m}\end{aligned}$$

We apply this with $t = \star \log M$ large enough so that the above implies

$$\|u_r a_{-t} g_1 \mathbf{v}_{\tilde{H}'}\| \le M^{-1} \qquad \text{for all } r \in [-e^{-t}S, e^t S] \tag{11.10}$$

Now (11.10) implies that

$$u_r a_{-t} g_1 \Gamma \notin X_{1/M} \qquad \text{for all } r \in [-e^{-t}S, e^t S].$$

Thus by Theorem A.1, there exists a $\mathbb{Q}$-parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ satisfying that $\mathbf{v}_{\tilde{H}'} \in \mathrm{Lie}(R_u(\mathbf{P}))$ so that

$$\|u_r a_{-t} g_1 \mathbf{v}_{R_u(P)}\| \ll M^{-\star} \qquad \text{for all } r \in [-e^{-t}S, e^t S].$$

Conjugating back, and using $t = \star \log M$, we get

$$\|u_r g_1 \mathbf{v}_{R_u(P)}\| \ll M^{\star} \qquad \text{for all } r \in [-S, S]. \tag{11.11}$$

Arguing as in the previous case, recall also that $G = KP = KLR_u(P)$ where L is a Levi subgroup of P, we get that part (2) in the lemma holds. $\square$

11.3. **Proof of Theorem 1.2.** Let A_1 be as Theorem 1.1, and let A_5, A_6 and C_5 be as in Lemma 11.1. By increasing A_5 and A_6 if necessary, we may assume $A_5, A_6 \geq 10A_1$. Let $E' \geq A_{10}$ and $F' \geq A_9$, see Theorem A.1. Increasing F' if necessary, we will assume $F' \geq \mathsf{m}_0$ in Proposition 4.2. We will now demonstrate that the theorem holds with

$$A_2 = \star A_5 \quad \text{and} \quad A_3 = A_6$$

Let $C = \max\{E', (10C_3)^3, e^{C_3}, e^{s_0}, C_4, C_5\}$, see (10.3). Let $R \geq C^2$, and

$$d = A_5 \log R \quad \text{and} \quad \eta = (C/R)^{\frac{1}{F' A_5}}.$$

Let $T > R^{A_2}$, and put $T_1 = e^{-d}T \geq R^{A_5}$. Then

$$\begin{aligned} \frac{1}{T}\int_0^T \varphi(u_r x_0)\,\mathrm{d}r &= \frac{1}{T_1}\int_0^{T_1} \varphi(a_d u_{r_1} a_{-d} x_0)\,\mathrm{d}r_1 \\ &= \frac{1}{T_1}\int_0^{T_1}\int_0^1 \varphi(a_d u_r u_{r_1} a_{-d} x_0)\,\mathrm{d}r\,\mathrm{d}r_1 + O(\|\varphi\|_\infty T_1^{-1}) \end{aligned} \tag{11.12}$$

where the implied constant is absolute.

Put $x_1 = a_{-d} x_0$, and define

$$\mathrm{Exc}_1 = \{r_1 \in [0, T_1] : u_{r_1} x_1 \notin X_\eta\} \tag{11.13a}$$

$$\mathrm{Exc}_2 = \left\{ r_1 \in [0, T_1] : \begin{array}{c} \text{there exists } x \text{ with } \mathrm{vol}(Hx) \leq R \\ \text{and } d(u_{r_1} x_1, x) \leq R^{A_1} d^{A_1} e^{-d} \end{array} \right\}. \tag{11.13b}$$

Let us first assume that

$$|\mathrm{Exc}_1| \leq C\eta^{1/F'} T_1 \quad \text{and} \quad |\mathrm{Exc}_2| \leq 2C^2 R^{-\kappa} T_1, \tag{11.14}$$

where $\kappa = \min\{1/(F' A_5 A_6)\}$.

For every

$$r_1 \in [0, T_1] \setminus \big(\mathrm{Exc}_1 \cup \mathrm{Exc}_2\big),$$

put $x(r_1) = u_{r_1} x_1$. Then

$$R = C\eta^{-F' A_5} \geq C\,\mathrm{inj}(x(r_1))^{-\mathsf{m}_0},$$

see (10.3); moreover, $e^d = R^{A_5} > R^{A_1}$. Thus, conditions of Theorem 1.1 are satisfied with the parameters e^d, R, and $x(r_1)$. However, by the definition of Exc_2, part (2) in Theorem 1.1 does not hold with these choices. Consequently, we conclude that for every r_1 as described above,

$$\Big|\int_0^1 \varphi(a_d u_r x(r_1))\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \le \mathcal{S}(\varphi)R^{-\kappa_1}.$$

Together with (11.14) and (11.12), this implies that

$$\Big|\frac{1}{T}\int_0^T \varphi(u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \le (R^{-\kappa_1} + 3C^2R^{-\kappa} + 2T_1^{-1})\mathcal{S}(\varphi),$$

where we used $C\eta^{1/F'} \le C^2R^{-\kappa}$.

Hence, part (1) in Theorem 1.2 holds with $\kappa_2 = \min(\kappa_1, \kappa)/2$ if we assume R is large enough.

We now assume to the contrary that (11.14) fails:

Assume that $|\mathrm{Exc}_1| > C\eta^{1/F'}T_1$. We will show that part (3) in the theorem holds under this condition.

First note that under this condition Γ is non-uniform, thus, $\tilde{\mathbf{G}}$ may be identified with $\mathbf{G}$ as $\mathbb{R}$ groups. Let us write $x_0 = g_0\Gamma$. Since $|\mathrm{Exc}_1| > C\eta^{1/F'}T_1$, then in view of our choices of F' and C, we conclude from Theorem A.1 that there exists a $\mathbb{Q}$-parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ so that

$$\|u_r a_{-d} g_0 \mathbf{v}_{R_u(P)}\| \le A_{10}\eta^{1/A_9} \qquad \text{for all } r \in [-T_1, T_1].$$

Conjugating back with a_d and using $T = e^dT_1$ and $e^d = R^{A_5}$,

$$\|u_r g_0 \mathbf{v}_{R_u(P)}\| \le A_{10}R^{\star A_5} \qquad \text{for all } r \in [-T, T].$$

Arguing as in Case 1 (or Case 2) of the proof of Lemma 11.1, we get that part (3) holds with $A_2 = \star A_5$ and $A_3 = A_6$.

Assume that $|\mathrm{Exc}_2| > 2C^2R^{-\kappa}T_1$. If $|\mathrm{Exc}_1| > C\eta^{1/F'}T_1$, then part (3) in the theorem holds as we just discussed. Thus, we may assume that

$$|\mathrm{Exc}_2| > 2C^2R^{-\kappa}T_1 \quad \text{and} \quad |\mathrm{Exc}_1| \le C\eta^{1/F'}T_1.$$

Put $\mathrm{Exc}' := \mathrm{Exc}_2 \setminus \mathrm{Exc}_1$. Then

$$\mathrm{Exc}' = \left\{ r_1 \in [0, T_1] : \begin{array}{c} u_{r_1}x_1 \in X_\eta \text{ and there exists } x \text{ with} \\ \mathrm{vol}(Hx) \le R \text{ and } d(u_{r_1}x_1, x) \le R^{A_1}d^{A_1}e^{-d} \end{array} \right\},$$

and $|\mathrm{Exc}'| \ge C^2R^{-\kappa}T_1 \ge C_5\eta^{1/A_6}T_1$. Moreover, for R large enough,

$$R^{A_1}d^{A_1}e^{-d} = R^{A_1}(A_5\log R)^{A_1}R^{-2A_5} \le R^{-A_5}.$$

Fix some $r_1 \in \mathrm{Exc}'$ for the rest of the argument. Put

$$x_2 = u_{r_1}x_1 = u_{r_0}a_{-d}x_0 \quad \text{and} \quad \mathrm{Exc} = \mathrm{Exc}' - r_1 \subset [-T_1, T_1].$$

Then the conditions in Lemma 11.1 are satisfied with x_2, Exc, η, $M = R$, and $S = T_1 = R^{-A_5}T$.

First, assume that part (1) of Lemma 11.1 holds. Then there exists $x \in G/\Gamma$ with $\mathrm{vol}(H.x) \le R^{A_5}$. Moreover, for every $r \in [-T_1, T_1]$ there exists $g \in G$ with $\|g\| \le R^{A_5}$ so that

$$d_X(u_s x_2, gHx) \le R^{A_5} \left(\frac{|s-r|}{T_1}\right)^{1/A_6} \qquad \text{for all } s \in [-T_1, T_1].$$

Since $s - r_1, r - r_1 \in [-T_1, T_1]$ for all $s, r \in [0, T_1]$, the above implies

$$\begin{aligned} d_X(u_{e^d s} x_0, a_d gHx) &= d_X(a_d u_s a_{-d} x_0, a_d gHx) \\ &= d_X(a_d u_{s-r_1} u_{r_1} a_{-d} x_0, a_d gHx) \\ &= d_X(a_d u_{s-r_1} x_2, a_d gHx) \\ &\ll e^{\star d} d_X(u_{s-r_1} x_2, gHx) \le R^{\star A_5} \Big(\frac{|e^d s - e^d r|}{T}\Big)^{1/A_6}. \end{aligned}$$

That is part (2) in the theorem holds with $A_2 = \star A_5$ and $A_3 = A_6$ for all large enough R.

Assume now that part (2) in Lemma 11.1 holds. Then arguing as above, we conclude that part (3) of the theorem with $A_2 = \star A_5$ and $A_3 = A_6$. $\square$

12. Equidistribution of expanding circles

In this section, we record the following theorem concerning equidistribution of large circle; this theorem is a corollary of Theorem 1.1 as it was shown in [LMW23, §5].

We keep the notation from Theorem 1.1. In particular, G is any of the following groups

$$\mathrm{SL}_3(\mathbb{R}), \quad \mathrm{SU}(2,1), \quad \mathrm{Sp}_4(\mathbb{R}), \quad \mathbf{G}_2(\mathbb{R}),$$

and $H \subset G$ is the image of the principal $\mathrm{SL}_2(\mathbb{R})$ in G. For all $t, r \in \mathbb{R}$ and all $\theta \in [0, 2\pi]$, let a_t, u_r and k_θ denote the images of

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

in H, respectively.

We let $\Gamma \subset G$ be an arithmetic lattice, and let m_X denote the probability Haar measure on $X = G/\Gamma$.

12.1. **Theorem.** *For every $x_0 \in X$, and large enough R (depending explicitly on x_0), for any $T \ge R^{A_7}$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$ and 2π-periodic smooth function ξ on $\mathbb{R}$, we have

$$\Big| \int_0^1 \varphi(a_{\log T} k_\theta x_0) \xi(\theta) \,\mathrm{d}\theta - \int_0^{2\pi} \xi(\theta) \,\mathrm{d}\theta \int \varphi \,\mathrm{d}m_X \Big| \le \mathcal{S}(\varphi) \mathcal{S}(\xi) R^{-\kappa_8}$$

where $\mathcal{S}(\cdot)$ denotes appropriate Sobolev norms on X and $\mathbb{R}$, respectively.

(2) There exists $x \in X$ such that Hx is periodic with $\mathrm{vol}(Hx) \le R$, and

$$d_X(x, x_0) \le R^{A_7} (\log T)^{A_7} T^{-\mathsf{m}}.$$

The constants A_7 and κ_8 are positive, and depend on X but not on x_0.

Proof. The proof of [LMW23, Thm.1.4] remains valid without modification when [LMW22, Thm.1.1] is replaced by Theorem 1.1 from this paper. □

Part 2. Application to quadratic forms

The second part of this article concerns applications to quantitative version of Oppenheim conjecture. In particular, the proof of Theorem 2.3 and Theorem 2.1 will be completed in this part. We employ the strategy pioneered by Eskin, Margulis and Mozes in [EMM98]. This result covered all indefinite quadratic forms in $d \geq 3$ variables except forms of signature $(2,2)$, or $(2,1)$ where a Diophantine condition is needed for the requisite nondivergence results to be true.

The case of $(2,2)$ forms was treated by Eskin, Margulis and Mozes [EMM05], and required significant additional ideas. A more precise version, suitable for a quantitative counting result, for the special case of $\Gamma \subset \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ was given by the first three authors of this paper in [LMW23].

More recently Wooyeon Kim succeeded in giving a treatment of the case of forms of signature $(2,1)$ in [Kim24]. Since the actual results we need are only implicit in [Kim24], we give a full treatment here; but it is possible to isolate what is needed already from [Kim24].

13. Upper bound estimates

In this section we will state Proposition 13.3 which yields the upper bound estimates needed for the proof of Theorem 2.3.

Let

$$Q_0(\mathsf{x},\mathsf{y},\mathsf{z}) = 2\mathsf{x}\mathsf{z} - \mathsf{y}^2.$$

If Q is an indefinite ternary quadratic form of determinant 1, and Then there exists some $g_Q \in \mathrm{SL}_3(\mathbb{R})$ so that $Q(v) = Q_0(g_Q v)$ for all $v \in \mathbb{R}^3$.

Let $H = \mathrm{SO}(Q_0)^\circ \subset \mathrm{SL}_3(\mathbb{R}) = G$; note that $H \simeq \mathrm{PSL}_2(\mathbb{R})$. In the case at hand, the groups a_t and u_r featuring in Theorem 1.1 can be more explicitly described as follows

$$a_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad u_r = \begin{pmatrix} 1 & r & \frac{r^2}{2} \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}.$$

Also let $K = H \cap \mathrm{SO}(3) \simeq \mathrm{SO}(2)$. Indeed, $K = \{k_\theta : \theta \in [0, 2\pi]\}$ where

$$k_\theta = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}. \tag{13.1}$$

Definition 13.1. A rational line $L \subset \mathbb{R}^3$ will be said to be (ρ, A, t)-*exceptional* for Q if the vector v defined up to sign by $L \cap \mathbb{Z}^3 = \mathrm{Span}\{v\}$ satisfies

$$\|v\| \leq e^{\rho t} \quad \text{and} \quad |Q(v)| \leq e^{-A\rho t}.$$

Existence of 5 exceptional lines yields an approximation of Q with a rational form with low complexity, see Lemma 13.4 below.

We identify $\wedge^2\mathbb{R}^3$ with the dual of $\mathbb{R}^3$, which will be identified with $\mathbb{R}^3$ using the standard Euclidean inner product. Then for every $g \in G$

$$gv_1 \wedge gv_2 = g^*(v_1 \wedge v_2) \tag{13.2}$$

where $g^* = (g^t)^{-1}$. We put $Q^*(v) := Q_0(g^*v)$ for any $v \in \mathbb{R}^3$.

Similar to Definition 13.1, we have the following:

Definition 13.2. Let $P \subset \mathbb{R}^3$ be a rational plane and suppose $P \cap \mathbb{Z}^3 = \mathrm{Span}\{w_1, w_2\}$. Then P will be said to be (ρ, A, t)-*exceptional* if the line spanned by $w_1 \wedge w_2$ is (ρ, A, t)-exceptional for Q^*.

This definition can be equivalently stated in terms of the height of the plane P and the determinant of the rational form induced by Q on P.

Let $f \in C_c(\mathbb{R}^3)$. For every $h \in H$, define

$$\tilde{f}_{\rho,A,t}(h; g_Q) = \sum_{v \in \mathcal{N}_{Q,t}} f(hv) \tag{13.3}$$

where $\mathcal{N}_{Q,t}$ denotes the set of vectors in $\mathbb{Z}^3$ which are not contained in any (ρ, A, t)-exceptional line or plane.

13.3. **Proposition.** *For all large enough A, for all small enough ρ (depending on A), and all large enough t (depending on A and ρ), at least one of the following holds:*

(1) Let $\tilde{\mathcal{C}}_t = \{k \in K : \tilde{f}_{\rho,A,t}(a_t k; g_Q\Gamma) \geq Ae^{A\rho t}\}$. Then

$$\int_{\tilde{\mathcal{C}}_t} \tilde{f}_{\rho,A,t}(a_t k; g_Q\Gamma)\, \mathrm{d}k \ll e^{-\rho^2 t/2}.$$

(2) There exists $P \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|P\| \leq e^{D_0\rho t}$ so that

$$\|Q - \lambda P\| \leq \|P\|^{-A/2} \quad \textit{where } \lambda = (\det P)^{-1/3}$$

The implied constant depends on $\|g\|$ and D_0 is absolute.

The reader may compare Proposition 13.3 and its proof with [LMW23, Prop. 6.1]. It is also worth noting that Proposition 13.3 is equivalent to showing that $\tilde{f}$ belongs to $L^{1+\epsilon}$. The proof of this proposition will occupy the rest of this section.

Let A and t be large parameters (A is absolute and $t > t_0(Q)$), and let $\rho > 0$ a small parameter; these will be explicated later. In what follows η represents a small number.

Linear algebra lemmas. We begin with the following lemmas from linear algebra.

13.4. **Lemma.** *Let $A, D \geq 1$ be two parameters. Let $v_1, \ldots, v_5 \in \mathbb{Z}^3$ be five vectors so that no three of them are co-planar and $|Q(v_i)| \leq \eta^A$ for all i. Then the following hold:*

(L-1) *Let $\gamma = (v_1, v_2, v_3)$, and assume that*
$$|\det \gamma| \leq \eta^{-D} \quad \textit{and} \quad \|\gamma^{-1} v_i\| \leq \eta^{-D} \ \textit{ for } i = 4, 5.$$
Then there is a matrix $P \in \mathrm{Mat}_3(\mathbb{Z})$ with $\det(P) \neq 0$ and $\|P\| \ll \eta^{-\star D}$ so that
$$\left\|\gamma^t Q \gamma - P\right\| \ll \eta^{A-\star D}.$$

(L-2) *If $\|v_i\| \leq \eta^{-D}$ for all i, then there is a matrix P as in (L-1) so that*
$$\|Q - \lambda P\| \ll \eta^{A-\star D} \quad \textit{where } \lambda = (\det P)^{-1/3}.$$

Proof. Writing Q in the basis $\{v_1, v_2, v_3\}$ implies that
$$\gamma^t Q \gamma = B = (b_{ij})$$
where $|b_{ii}| \ll \eta^A$. Write
$$\gamma^{-1} v_4 = (a_1, a_2, a_3)^t$$
$$\gamma^{-1} v_5 = (a_1', a_2', a_3')^t.$$
Then $a_i, a_i' \in \mathbb{Q}$ have height $\leq \eta^{-D}$.

Note that since no three of the v_i's are co-planar, a_i, a_i' are all nonzero. Suppose (a_2a_3, a_1a_3, a_1a_2) and $(a_2'a_3', a_1'a_3', a_1'a_2')$ are colinear. Then $\frac{a_2a_3}{a_2'a_3'} = \frac{a_1a_2}{a_1'a_2'}$ so $\frac{a_3}{a_3'} = \frac{a_1}{a_1'}$ and similarly for the other pairs. But this means v_4 and v_5 are colinear, in contradiction.

Since for $i = 4, 5$, $|Q(v_i)| \leq \eta^A$, we have
$$\sum_{0<i<j\leq 3} b_{ij} a_i a_j \ll \eta^{A-2D} \quad \text{and} \quad \sum_{0<i<j\leq 3} b_{ij} a_i' a_j' \ll \eta^{A-2D}$$
Since $a_i, a_i' \in \mathbb{Q}$ have height $\leq \eta^{-D}$, the above imply (L-1).

The second claim (L-2) follows from (L-1) and $\|\gamma\| \ll \eta^{-D}$. $\square$

13.5. **Lemma.** *Let $0 < \sigma < 1$, and let $0 < \delta < 1 - \sigma$. For every $v \in \mathbb{R}^3$ which satisfies $|Q_0(v)| \geq e^{-2\sigma t}$, we have*
$$\int_K \|a_t k v\|^{-1-\delta} \, \mathrm{d}k \ll \delta^{-1} e^{(-1+\delta+\sigma)t} \, \|v\|^{-1-\delta},$$
where the implied constant is absolute.

Proof. Recall that K acts transitively on the level sets
$$\{v \in \mathbb{R}^3 : \|v\| = c_1, Q_0(v) = c_2\}.$$
Thus we may assume $v = (0, \varepsilon, 1)$ where $\varepsilon = |Q_0(v)|^{1/2} \geq e^{-\sigma t}$, and have
$$a_t k_\theta v = \left(e^t\left(\tfrac{1-\cos\theta-\sqrt{2}\varepsilon\sin\theta}{2}\right), \tfrac{2\varepsilon\cos\theta-\sqrt{2}\sin\theta}{2}, e^{-t}\left(\tfrac{1+\cos\theta+\sqrt{2}\varepsilon\sin\theta}{2}\right)\right).$$

Since for $0<\theta<1/100$, we have

$$1-\cos\theta-\sqrt{2}\varepsilon\sin\theta=-\sqrt{2}\varepsilon\theta+\tfrac{\theta^2}{2}+O(\theta^3)\quad\text{and}$$
$$1+\cos\theta+\sqrt{2}\varepsilon\sin\theta=2+O(\varepsilon).$$

Let $\varepsilon'=\min\{\varepsilon,1/100\}$, the above implies

$$\begin{aligned}\int_K \|a_tkv\|^{-1-\delta}\,\mathrm{d}k &\ll \int_0^{e^{(-2+\sigma)t}} e^{(1+\delta)t}\,\mathrm{d}\theta+\int_{e^{(-2+\sigma)t}}^{\varepsilon'} e^{(-1-\delta)t}\varepsilon^{-1}\theta^{-1-\delta}\,\mathrm{d}\theta\\ &\ll e^{(-1+\sigma+\delta)t}+\delta^{-1}e^{(-1-\delta)t}e^{\sigma t}\theta^{-\delta}\Big]_{e^{(-2+\sigma)t}}^{\varepsilon'}\\ &\ll \delta^{-1}e^{(-1+\delta+\sigma)t}.\end{aligned}$$

As we claimed. □

In the bounds in Lemma 13.5, the multiplicative coefficient in front of $\|v\|^{-1-\delta}$ tends to zero as $t\to\infty$ (which is what we want), at the expense of giving a restriction on v. Without the restrictions imposed in Lemma 13.5, we have the following general (weaker) bound:

13.6. **Lemma.** *Let $v\in\mathbb{R}^3$, then*

$$\int_K \|a_tkv\|^{-1-\delta}\,\mathrm{d}k \ll e^{\delta t}\|v\|^{-1-\delta}$$

where the implied constant is absolute.

Proof. The proof is similar to Lemma 13.5. Indeed, we have

$$\begin{aligned}\int_K \|a_tkv\|^{-1-\delta}\,\mathrm{d}k &\ll \int_0^{e^{-t}} e^{(1+\delta)t}\,\mathrm{d}\theta+\int_{e^{-t}}^{\pi/4} e^{(-1-\delta)t}\theta^{-2-2\delta}\,\mathrm{d}\theta\\ &\ll e^{\delta t}\,\|v\|^{-1-\delta},\end{aligned}$$

as we claimed. □

Cusp functions of Margulis. Recall the functions

$$\alpha_1(g)=\{1/\|gv\|:0\neq v\in\mathbb{Z}^3\}$$
$$\alpha_2(g)=\left\{1/\|g^*(w_1\wedge w_2)\|:0\neq w_1\wedge w_2\in\wedge^2\mathbb{Z}^3\right\}$$

from [EMM98] and [EMM05].

For all $0\neq v\in\mathbb{R}^3$, all $0<\sigma<1$, and all $s>1$ let

$$I_v^s(\sigma)=\{k\in K:\|a_skg_Qv\|\le\sigma\}.$$

We will be interested in $I_v^s(\sigma)$ for vectors where $|Q_0(g_Qv)|\le\sigma^{10}$ and $1\ll\|g_Qv\|\le\sigma e^s$. In this range, we have

$$|I_v^s(\sigma)|\asymp(e^{-s}\sigma/\|g_Qv\|)^{1/2},\tag{13.4}$$

see e.g. [EMM05, Lemma A.6].

Let $\mathcal{P}$ denote the set of primitive vectors in $\mathbb{Z}^3$, and let

$$\mathcal{P}_\ell=\{v\in\mathcal{P}:e^{\ell-1}\le\|v\|<e^\ell\},$$

for every integer $\ell \geq 1$.

We make the following observation, which is a special case of [EMM98, Lemma 5.6] tailored to our applications here.

13.7. **Lemma.** *Let $0 < \sigma < 1$ and $s > 1$. Let $v_1, v_2 \in \mathcal{P}_\ell$ be so that $|Q_0(g_Q v_i)| \leq \sigma^{10}$, $\|g_Q v_i\| \leq \sigma e^s$, and $2I^s_{v_2}(\sigma) \cap 2I^s_{v_1}(\sigma) \neq \emptyset$. Then*

$$\alpha_1(a_s k g_Q \Gamma) \ll_c \sigma \alpha_2(a_s k g_Q) \quad \text{for all } k \in cI^s_{v_1}(\sigma),$$

for every constant c.

Proof. Let v_1 and v_2 be as in the statement. Then, in view of (13.4), we have

$$\|a_s k g_Q v_2\| \ll \sigma \quad \text{for all } k \in cI^s_{v_1}(\sigma).$$

For every $k \in cI^s_{v_1}(\sigma)$, fix a vector v_k so that

$$\alpha_1(a_s k g_Q \Gamma) = \|a_s k g_Q v_k\|^{-1}.$$

Let $P = \mathrm{Span}\{v_k, v\}$ where $v = v_1$ if $v_k = v_2$ and $v = v_2$ if $v_k \neq v_2$. Let u be the corresponding covector (which is unique up to sign). Then

$$\begin{aligned}\|(a_s k g_Q)^* u\| &\leq \|(a_s k g_Q)^*(v_k \wedge v)\| \\ &\leq \|a_s k g_Q v_k\| \, \|a_s k g_Q v\| \ll \sigma \alpha_1(a_s k g_Q)^{-1},\end{aligned}$$

for all $k \in cI^s_{v_1}(\sigma)$. Taking reciprocal, we have

$$\alpha_1(a_s k g_Q \Gamma) \ll \sigma \, \|(a_s k g_Q)^* u\|^{-1} \leq \sigma \alpha_2(a_s k g_Q \Gamma),$$

as it was claimed. □

We will also using the following theorem which is due to Eskin, Margulis and Mozes [EMM98].

13.8. **Theorem** ([EMM98], Thm. 3.2 and Thm. 3.3)**.** *Both of the following hold for $i = 1, 2$ and every $g \in G$.*

(1) For every $0 < p < 1$, we have

$$\sup_{s \geq 0} \int \alpha_i^p(a_s k g \Gamma)\, \mathrm{d}k \ll 1$$

(2) For every $s \geq 0$, we have

$$\int \alpha_i(a_s k g \Gamma)\, \mathrm{d}k \ll s.$$

The implied constant is $\ll (1-p)^{-1} \|g\|^\star$ in part (1) and $\ll \|g\|^\star$ in (2).

Proof. The first statement is [EMM98, Thm 3.2] and the second is [EMM98, Thm 3.3]. It follows from the proofs of these statements that the implied constant depends polynomially on $\alpha_i(g) \ll \|g\|^\star$ and *is* $\ll 1/(1-p)$ in the first case; see [EMM98, Lemma 5.1] for the dependence on p. □

We will also use the following elementary consequence of Theorem 13.8.

13.9. **Corollary.** *Let $\Theta \subset K$ be a set with $|\Theta| \le \sigma$. Then*

$$\int_\Theta \alpha_i^{1/2}(a_d kg\Gamma)\,\mathrm{d}k \ll \sigma^{1/4}$$

Proof. Let $\mathcal{B} = \{k \in K : \alpha_2(a_d k g_Q \Gamma) \le \sigma^{-1}\}$. Then

$$\begin{aligned}\int_\Theta \alpha_i^{1/2}(a_d kg\Gamma)\,\mathrm{d}k &\ll \int_{\Theta\cap\mathcal{B}} \alpha_i^{1/2}(a_d kg\Gamma)\,\mathrm{d}k + \sigma^{1/4}\int_{K\setminus\mathcal{B}} \alpha_i(a_t kg\Gamma)^{3/4}\,\mathrm{d}k\\ &\ll \sigma^{1/2} + \sigma^{1/4} \ll \sigma^{1/4}\end{aligned}$$

as we claimed. □

Approximation by rational forms. The following proposition is one of the main ingredients in the proof of Proposition 13.3.

13.10. **Proposition.** *There exists some D_1, A (absolute) so that if $\rho D_1 < \frac{1}{100}$, and $\ell > 0$ is large enough and the following is satisfied*

$$\#\{v \in \mathcal{P}_\ell : |Q_0(g_Q v)| \le e^{-A\rho\ell}\} \ge e^{(1-\rho)\ell}, \tag{13.5}$$

then at least one of the following holds:

(1) There exist planes $\{P_i : 1 \le i \le N\}$, for some $N \le e^{(1-2\rho)\ell}$, so that

$$\#\{v \in \mathcal{P}_\ell \setminus (\cup P_i) : |Q_0(g_Q v)| \le e^{-A\rho\ell}\} \le e^{(1-2\rho)\ell}.$$

Moreover, there are intervals $\{J_i : 1 \le i \le N\}$, with multiplicity at most two, so that $I_v^{(1+2\rho)\ell}(e^{-\rho\ell}) \cap J_i \ne \emptyset$ for every $v \in P_i$, and

$$\int_{cJ_i} \alpha_1(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k \ll_c e^{-\rho\ell}\int_{cJ_i} \alpha_2(a_{(1+2\rho)\ell} k g_Q\Gamma)\,\mathrm{d}k$$

for every constant c.

(2) There exists some $P \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|P\| \le e^{D_1\rho\ell}$ so that

$$\|Q - \lambda P\| \le e^{(-1+D_1\rho)\ell} \quad \textit{where } \lambda = (\det P)^{-1/3}.$$

Until the conclusion of the proof of Proposition 13.10, let $\eta = e^{-\rho\ell}$ and $s = (1+2\rho)\ell$, and write I_v for $I_v^s(\eta)$. We will also put

$$\mathcal{W}_\ell := \{v \in \mathcal{P}_\ell : |Q_0(g_Q v)| \le e^{-A\rho\ell}\}$$

13.11. **Lemma.** *Let the notation be as in Proposition 13.10. Then at least one of the following holds.*

(1) Part (1) in Proposition 13.10 holds.

(2) There exist $\mathcal{W} \subset \mathcal{W}_\ell$ with $\#\mathcal{W} \ge \eta^2 e^\ell$ so that

$$I_v \cap I_{v'} = \emptyset \quad \textit{for all } v \ne v' \in \mathcal{W}.$$

Proof. Suppose the claim in part (2) fails. Recall from (13.4) that

$$|I_v| \asymp e^{(-s-\ell)/2}\eta^{1/2} \asymp e^{-\ell}\eta^{3/2}, \tag{13.6}$$

where we used $s = (1+2\rho)\ell$, $\eta = e^{-\rho\ell}$, and $\|g_Q v\| \ll \|g_Q\|\|v\| \le \eta e^s$.

In consequence, we may choose $\mathcal{W}' \subset \mathcal{W}_\ell$ with $\#\mathcal{W}' < \eta^2 e^\ell$ so that $\{J_i\} = \{2I_v : v \in \mathcal{W}'\}$ covers $\cup_{\mathcal{W}_\ell} I_v$ with multiplicity at most two. Discard the intervals J_i which intersect only one I_v. For every remaining interval, J_i, we have

$$\|a_s k g_Q v\| \le \eta \quad \text{for all } v \text{ with } I_v \cap J_i \ne \emptyset \text{ and all } k \in I_v. \tag{13.7}$$

We draw both conclusions of part (1) in the lemma from (13.7). First note that (13.7) implies that if v, v' so that $I_v \cap J_i \ne \emptyset$ and $I_{v'} \cap J_i \ne \emptyset$, and we put $P_i = \mathrm{Span}\{v, v'\}$, then for every v'' so that $I_{v''} \cap J_i \ne \emptyset$, $v'' \in P_i$. Otherwise

$$1 \le |\det(a_s k g_Q v, a_s k g_Q v', a_s k g_Q v''| \ll \eta^3$$

which is a contradiction assuming s is large enough. Moreover,

$$\#\{v : v \notin \cup P_i\} \le \eta^2 e^\ell.$$

To see the second claim in part (1), note that if v, v' are so that $I_v \cap J_i \ne \emptyset$ and $I_{v'} \cap J_i \ne \emptyset$, the conditions in Lemma 13.7 are satisfied. Hence

$$\int_{cJ_i} \alpha_1(a_s k g_Q \Gamma)\, \mathrm{d}k \ll_c e^{-\rho s} \int_{cJ_i} \alpha_2(a_s k g_Q \Gamma)\, \mathrm{d}k,$$

as we claimed. □

If part (1) in Lemma 13.11 holds, the proof of Proposition 13.10 is complete. Thus We will assume part (2) in Lemma 13.11 holds, for some C which will be optimized later, for the rest of the proof.

Recall that $K \simeq \mathrm{SO}(2)$. Using this isomorphism, we identify K with $[0, 2\pi]$, and cover K with half open intervals $\{I_j\}$ where $|I_j| = 2\pi/N$ for $N = \lceil \eta^5 e^\ell \rceil$.

13.12. **Lemma.** *Let ℓ and $\mathcal{W}$ be as in part (2) in Lemma 13.11. For every j, let*

$$\mathcal{W}_{\ell,j} = \{v \in \mathcal{W}_\ell : |I_v \cap I_j| \ge |I_v|/2\},$$

and $\mathcal{J}_\ell = \{j : \#\mathcal{W}_{\ell,j} \ge \eta^{-2}\}$. Then

$$\#\mathcal{J}_\ell \ge \eta^9 e^\ell$$

Proof. Let $\mathcal{J}'_\ell = \{j : \#\mathcal{W}_{\ell,j} \le \eta^{-2}\}$. Then

$$\#\big(\textstyle\bigcup_{\mathcal{J}'} \mathcal{W}_{\ell,j}\big) \ll \eta^3 e^\ell.$$

Moreover, note that since $I_v \cap I_{v'} = \emptyset$ for all $v \ne v' \in \mathcal{W}$ and $|I_v| \asymp e^{-\ell}\eta^{3/2}$,

$$\#\mathcal{W}_{\ell,j} \ll |I_j|/\eta^{3/2} e^{-\ell} \ll \eta^{-7}.$$

These and $\#\mathcal{W} \ge \eta^2 e^\ell$ imply that

$$\#\mathcal{J}_\ell = \#\{j : \#\mathcal{W}_{\ell,j} \ge \eta^{-1}\} \ge \eta^9 e^\ell,$$

as we claimed in the lemma. □

13.13. **Lemma.** *With the notation be as in Lemma 13.12, let $j \in \mathcal{J}_\ell$ and let $\{v_1, \ldots, v_5\} \subset \mathcal{W}_{\ell,j}$ be so that $\left\|\frac{v_\alpha}{\|v_\alpha\|} - \frac{v_\beta}{\|v_\beta\|}\right\| \gg \eta^{-1/2} e^{-\ell}$. Then*

(1) No three of $\{v_1, \ldots, v_5\}$ *are co-planar.*
(2) Let $\gamma = (v_1, v_2, v_3)$*, then*

$$1 \le |\det \gamma| \le \eta^{-D} \quad \text{and} \quad \|\gamma^{-1} v_i\| \le \eta^{-D} \ \text{ for } i = 4, 5.$$

Proof. In the course of the proof, we will write g for g_Q to simplify the notation. Let us write $gv_i = v_i' + u_i$ where $\|u_i\| \ll e^{-\ell}\eta^A$ and $Q_0(v_i') = 0$. Now for all $v_\alpha, v_\beta \in \{v_i\}$, we have

$$Q_0(gv_\alpha, gv_\beta) = Q_0(v_\alpha', v_\beta') + O(\eta^A). \tag{13.8}$$

Moreover, since $\left\|\frac{v_\alpha}{\|v_\alpha\|} - \frac{v_\beta}{\|v_\beta\|}\right\| \gg \eta^{-1/2}e^{-\ell}$ and $v_\alpha, v_\beta \in \mathcal{W}_{\ell,j}$ we can write $v_\beta' = \lambda_{\alpha\beta} v_\alpha' + w_{\alpha\beta}'$ where $|\lambda_{\alpha\beta}| = O(1)$ and $\eta^{-1/2} \ll \|w_{\alpha\beta}\| \ll \eta^{-5}$ and $w_{\alpha\beta}' \perp v_\alpha'$ (with respect to the usual inner product). Hence,

$$gv_\beta = \lambda_{\alpha\beta} g v_\alpha + w_{\alpha\beta} \quad \text{where } w_{\alpha\beta} = w_{\alpha\beta}' + O(\eta^A).$$

More explicit, applying an element of K, we will assume $v_\alpha' = (\mathsf{x}, 0, 0)$. Then $w_{\alpha\beta}' = (0, \mathsf{y}, \mathsf{z})$ and $\eta^{-1/2} \ll \|w_{\alpha\beta}'\| \ll \eta^{-5}$. Moreover, $2\lambda_{\alpha\beta}\mathsf{z}\mathsf{x} - \mathsf{y}^2 = 0$ since $Q_0(v_\beta') = 0$, this implies $|\mathsf{y}| \gg \eta^{-1/4}$, thus $|\mathsf{z}| \gg \eta^{-1/2}/\|v_\alpha'\|$. This implies $|Q_0(v_\alpha', v_\beta')| \gg \eta^{-1/2}$ which in view of (13.8) gives $|Q_0(gv_\alpha, gv_\beta)| \gg \eta^{-1/2}$.

Now assume contrary to the claim in part (1), that $gv_3 = agv_1 + bgv_2$ (the argument in other cases is similar). Then

$$a\lambda_{31} + b\lambda_{32} = 1 \quad \text{and} \quad aw_{31} + bw_{32} = 0$$

Solving for a in the first equation, and replacing in the second we conlcude,

$$w_{31} = b(\lambda_{32} w_{31} - \lambda_{31} w_{32}).$$

Since $|\lambda_{\alpha\beta}| = O(1)$ and $\eta^{-1/2} \ll \|w_{\alpha\beta}\| \ll \eta^{-5}$, we conclude that $\eta^{10} \ll |b| \ll \eta^{-10}$. This and $a\lambda_{31} + b\lambda_{32} = 1$ imply $\eta^{10} \ll |a| \ll \eta^{-10}$. Recall that

$$|Q_0(gv_3)| = |a^2 Q_0(gv_1) + b^2 Q_0(gv_2) + 2ab Q_0(gv_1, gv_2)| \le \eta^A.$$

Since $|Q_0(gv_1, gv_2)| \gg \eta^{-1/2}$ and $\eta^{10} \ll |a|, |b| \ll \eta^{-10}$, we get a contradiction so long as $A \ge 40$ and η is small enough. The proof of (1) is complete.

We now turn to part (2). Let $I \subset \{w \in \mathbb{R}^3 : \|w\| = e^\ell, Q_0(w) = 0\}$ be an interval with $|I| \asymp \eta^{-5}$. Let $\{w_1, w_2, w_3\} \in I$ satisfy $\|w_i - w_j\| \gg \eta^{-1/2}$. Let $W = (w_1, w_2, w_3)$ which is non-singular by part (1). A direct computation shows that if $v' \in \{\lambda w : \lambda \in [e^{-2}, e^2], w \in I\}$, then $Wx' = v'$ has a solution $\|x'\| \ll \eta^{-\star}$.

As we did in part (1), let us write $gv_i = v_i' + u_i$ where $Q_0(v_i') = 0$ and $\|u_i\| \le e^{-\ell}\eta^A$. Since gv_1, gv_2, gv_3 lie in the $e^{-\ell}\eta^A$ neighborhood of an interval $I \subset \{w \in \mathbb{R}^3 : \|w\| = e^\ell, Q_0(w) = 0\}$ with $|I| \asymp \eta^{-5}$. Thus the volume of the tetrahedron spanned by gv_1, gv_2, gv_3 is $\ll \eta^{-\star}$. This and part (1) give

$$1 \le |\det \gamma| \ll \eta^{-\star} \tag{13.9}$$

Now applying the above discussion with $W' = (v_1', v_2', v_3')$ and $v' = v_4', v_5'$, there are $x_4', x_5' \in \mathbb{R}^3$ satisfying $\|x_i'\| \ll \eta^{-\star}$ so that $W'x_i' = v_i'$.

Since $g\gamma = W' + E$ where $\|E\| \ll e^{-\ell}\eta^{A}$. For $i = 4, 5$, let $x_i = x'_i + \bar{u}$ where $\bar{u} = (g\gamma)^{-1}(u_i - Ex'_i)$, then

$$\begin{aligned} g\gamma x_i &= g\gamma x'_i + g\gamma\bar{u} \\ &= W'x'_i + Ex'_i + u_i - Ex'_i = v'_i + u_i = gv_i \end{aligned}$$

moreover, $\|Ex'_i\| \ll e^{-\ell}\eta^{A-\star}$, $\|u_i\| \le e^{-\ell}\eta^{A}$, and $\|(g\gamma)^{-1}\| \ll e^{\ell}\eta^{-\star}$. Thus $\|\bar{u}\| \ll \eta^{A-\star}$. Altogether, we conclude that $\|x_i\| = \|\gamma^{-1}v_i\| \ll \eta^{-\star}$. This and (13.9), complete the proof of part (2). □

13.14. **Lemma.** *Let the notation be as in Lemma 13.12. Then for every $j \in \mathcal{J}_\ell$, there exists some $k_j \in K$ so that*

$$a_\ell k_j g_Q \Gamma = \epsilon_j g_j$$

where $\|\epsilon_j\| \ll \eta^{A-\star}$ and $\mathrm{vol}(Hg_j\Gamma) \ll \eta^{-\star}$.

Moreover, we can choose the collection $\{k_j\}$ as above so that the following holds. There exists $\mathcal{J}'_\ell \subset \mathcal{J}_\ell$ with $\#\mathcal{J}'_\ell \gg \eta^\star e^\ell$ so that if $j, j' \in \mathcal{J}'_\ell$ are distinct, then $\|k_j - k_{j'}\| \gg e^{-\ell}$.

Proof. Fix some $j \in \mathcal{J}_\ell$, and let $\{v_1, \ldots, v_5\} \subset \mathcal{W}_{\ell,j}$ be as in Lemma 13.13. Then conditions of Lemma 13.4 are satisfied with $\{v_1, \ldots, v_5\}$. Let

$$\gamma_1 = \gamma_1(j) = (v_1, v_2, v_3)$$

By the conclusion of Lemma 13.4 thus $\gamma_1^t Q \gamma_1 = P + E$ where

$$P \in \mathrm{Mat}_3(\mathbb{Z}), \quad \|P\| \ll \eta^{-\star}, \quad \text{and} \quad \|E\| \ll \eta^{A-\star}.$$

We conclude that there exists $\hat{g}_j \in G$, with $\|\hat{g}_j\| \ll \eta^{-\star}$ so that $(\hat{g}_j)^t Q_0 \hat{g}_j = P$. This implies: $H\hat{g}_j\Gamma$ is a periodic orbit with $\mathrm{vol}(H\hat{g}_j\Gamma) \ll \eta^{-\star}$.

Moreover, we conclude from the above that there exists some $h_j = k'_j \hat{a} k_j \in \mathrm{SO}(Q_0)$ and $\hat{\epsilon}_j \in G$ with $\|\hat{\epsilon}_j - I\| \ll \eta^{A-O(1)}$ so that

$$k'_j \hat{a} k_j g_Q \gamma_1 = \hat{\epsilon}_j \hat{g}_j. \tag{13.10}$$

Since $\|\hat{g}_j\| \ll \eta^{-\star}$, we conclude that

$$e^\ell \eta^\star \ll \|\gamma_1\|\eta^\star \ll \|\hat{a}\| \ll \|\gamma_1\|\eta^{-\star} \ll e^\ell \eta^{-\star}.$$

Let us write $\hat{a} = a_\ell a_\tau$ where $|\tau| = \star|\log\eta|$. Multiplying (13.10) by $(k'_j a_\tau)^{-1}$, we get

$$a_\ell k_j g_Q \gamma_1 = \epsilon_j g_j \quad \text{where } \|\epsilon_j - I\| \ll \eta^{A-\star} \text{ and } g_j = (k'_j a_\tau)^{-1}\hat{g}_j.$$

This establishes the first claim.

To see the second claim, let $j, j' \in \mathcal{J}_\ell$. Repeat the above argument with j and j', and let $k_j, k_{j'}$ and $\gamma_1(j)$ and $\gamma_1(j')$ be the elements as above. If $\|k_j - k_{j'}\| \ll e^{-\ell}$, then

$$a_\ell k_{j'} g_Q = g_{jj'} a_\ell k_j g_Q \quad \text{where} \quad \|g_{jj'}\| \ll 1.$$

This implies that

$$\|(\gamma_1(j))^{-1}\gamma_1(j')\| \ll \eta^{-\star}$$

Since $\#\mathcal{J}_\ell \gg \eta^9 e^s$, the collection of $\gamma_1(j)$ for $j\in\mathcal{J}_\ell$ is distinct, the final claim in the lemma follows. □

Proof of Proposition 13.10. Recall that $\eta = e^{-\rho\ell}$. Let $\mathcal{J}'_\ell$ and $\{k_j : j\in\mathcal{J}'_\ell\}$ be as in Lemma 13.14. Then $\#\mathcal{J}'_\ell \gg \eta^C e^\ell$, and for all $j\in\mathcal{J}'_\ell$,

$$a_\ell k_j g_Q\Gamma = \epsilon_j x_j$$

where $\mathrm{vol}(Hx_j)\le \eta^{-D'_1}$ and $\|\epsilon_j - I\|\le \eta^{A-D'_1}$, where D'_1 is absolute. Moreover, $|k_j-k_{j'}|\gg e^{-\ell}$ for all $j\ne j'$.

Let $\mathcal{N}=\{\hat g x : \mathrm{vol}(Hx)\le\eta^{-D'_1}, \|\hat g\|\le \eta^{A-D'_1}\}$. We conclude from the above that

$$|\{k\in K : a_\ell k g_Q\Gamma\in\mathcal{N}\}|\gg \eta^{D'_1}.$$

If A is large enough, we conclude that

$$g_Q=\epsilon g$$

where $\mathrm{vol}(Hg\Gamma)\ll\eta^{-D'_1}$, and $\|\epsilon - I\|\ll \eta^{-\star}e^{-\ell}$, see e.g. [LMW22, Prop. 4.6].

In particular, g^tQ_0g is rational form with height $\le\eta^{-D_1}=e^{-D_1\rho\ell}$, and

$$\|Q-g^tQ_0g\|\ll\|g\|^\star\|\epsilon-1\|\le e^{(-1+D_1\rho)\ell},$$

as we claimed. □

14. Proof of Proposition 13.3

We will complete the proof of Proposition 13.3 in this section; the proof relies on Proposition 13.10 and Theorem 13.8. If part (2) in Proposition 13.10 holds, with some $\ell\ge\rho t-1$, then part (2) in Proposition 13.3 holds and the proof is complete. Thus we assume part (2) in Proposition 13.10 does not hold for any $\ell\ge\rho t-1$.

Notation for the proof and Schmidt's Lemma. Recall that we identify $\wedge^2\mathbb{R}^3$ with the dual of $\mathbb{R}^3$, which will be identified with $\mathbb{R}^3$ using the standard inner product. Then for every $g\in G$

$$gv_1\wedge gv_2 = g^*(v_1\wedge v_2) \tag{14.1}$$

where $g^*=(g^t)^{-1}$; note that H is invariant under this involution. Also note that $Q^*(v)=Q_0(g^*v)$ for any $v\in\mathbb{R}^3$.

For every $u\in\wedge^2\mathbb{R}^3$, we will write

$$I_u^{*,t}(\sigma)=\{k\in K : \|(a_tkg_Q)^*u\|\le\sigma\},$$

and put $I_u^*:=I_u^{*,t}(e^{-\rho t})$. We will also write I_v for $I_v^t(e^{-\rho t})$.

We emphasize that I_v here is shorthand for $I_v^t(e^{-\rho t})$, and is does not represent a shorthand notation for $I_v^{(1+2\rho\ell)}(e^{-\rho\ell})$, which was used in the proof of Proposition 13.10.

By a variant of Schmidt's Lemma, see also [EMM98, Lemma 3.1], and the definition of $\tilde f$, we have

$$\tilde f(a_tk;g_Q)\ll\tilde\alpha(a_tk;g_Q) \tag{14.2}$$

where $\tilde{\alpha}(a_t k; g_Q) = \max\{\tilde{\alpha}_1(a_t k; g_Q), \tilde{\alpha}_2(a_t k, g_Q)\}$ and

$$\tilde{\alpha}_1(a_t k; g_Q) = \left\{\|a_t k g_Q v\|^{-1} : 0 \neq v \in \mathcal{N}_{Q,t}\right\}$$
$$\tilde{\alpha}_2(a_t k; g_Q) = \left\{\|(a_t k g_Q)^* u\|^{-1} : 0 \neq u \in \mathcal{N}_{Q^*,t}\right\}.$$

Recall the set $\tilde{\mathcal{C}}_t = \{k \in K : \tilde{f}(a_t k; g_Q) \geq A e^{A\rho t}\}$. Choosing A large enough to account for the the implied multiplicative constant in (14.2),

$$\tilde{\mathcal{C}}_t \subset \mathcal{C}_{t,1} \cup \mathcal{C}_{t,2}.$$

where $\mathcal{C}_{t,i} = \{k \in K : \tilde{\alpha}_i(a_t k; g_Q) \geq e^{A\rho t}\}$ for $i = 1, 2$. In view of 14.2 and the above, thus, it suffices to control

$$\int_{\mathcal{C}_{t,1}} \tilde{\alpha}_1(a_t k; g_Q)\, \mathrm{d}k \quad \text{and} \quad \int_{\mathcal{C}_{t,2}} \tilde{\alpha}_2(a_t k; g_Q)\, \mathrm{d}k$$

Let us set

$$\mathcal{W}_{t,1} := \{v \in \mathcal{N}_{Q,t} \cap \mathcal{P} : \|v\| \leq e^t, \exists k \in I_v, \|a_t k g_Q v\| \leq e^{-A\rho t}\}, \text{ and}$$
$$\mathcal{W}_{t,2} := \{u \in \mathcal{N}_{Q^*,t} \cap \mathcal{P} : \|v\| \leq e^t, \exists k \in I_u^*, \|(a_t k g_Q)^* u\| \leq e^{-A\rho t}\}.$$

Note that if $v \in \mathcal{W}_{t,1}$, then $I_v \cap \mathcal{C}_t \neq \emptyset$, thus $\|g_Q v\| \leq e^{(1-A\rho)t}$, similarly for covectors in $\mathcal{W}_{t,2}$. This implies $\|v\| \ll e^{(1-A\rho)t}$. For $i = 1, 2$ and every integer $\ell \leq t - (A-1)\rho t := t'$, let

$$\mathcal{W}_{t,i}(\ell) = \mathcal{W}_{t,i} \cap \mathcal{P}_\ell.$$

The sets $\mathcal{W}'_{t,i}(\ell)$. We will work with $\mathcal{W}_{t,1}$, the argument for $\mathcal{W}_{t,2}$ is similar.

Suppose now $v \in \mathcal{W}_{t,1}(\ell)$ for some $\ell \leq 1 + (1 - A\rho)t$. Recall from the definition of $\mathcal{W}_{t,1}$ that

$$\|a_t k g_Q v\| \leq e^{-A\rho t}, \quad \text{for some } k \in I_v;$$

thus, $|Q_0(a_t k g_Q v)| \ll e^{-2A\rho t} \leq e^{-A\rho t}$. Since $a_t k$ preserves the form Q_0,

$$|Q_0(g_Q v)| \leq e^{-A\rho t}. \tag{14.3}$$

Moreover, since v does not belong to any (ρ, A, t)-exceptional line we have $\|v\| \geq e^{\rho t}$. Since $v \in \mathcal{W}_{t,1}(\ell)$, we conclude that $\ell \geq \rho t - 1$.

For $i = 1, 2$, let

$$\mathcal{L}_i = \{\rho t - 1 \leq \ell \leq t' : \#\mathcal{W}_{t,i}(\ell) \geq e^{(1-\rho)\ell}\},$$

and let $\mathcal{L}'_i = \{\rho t - 1 \leq \ell \leq t' : \ell \notin \mathcal{L}_i\}$.

In view of (14.3), we have

$$\mathcal{W}_{t,i}(\ell) \subset \{v \in \mathcal{P}_\ell : |Q_0(g_Q v)| \leq e^{-A\rho\ell}\}.$$

This and the definition of $\mathcal{L}_1$, imply that for every $\ell \in \mathcal{L}_1$, we have

$$\#\{v \in \mathcal{P}_\ell : |Q_0(g_Q v)| \leq e^{-A\rho\ell}\} \geq e^{(1-\rho)\ell}.$$

Thus, the assumptions of Proposition 13.10 holds for any $\ell \in \mathcal{L}_1$. If part (2) in Proposition 13.10 held, the proof of Proposition 13.3 would be complete.

Therefore, we may assume that part (1) in Proposition 13.10 holds. That is: There exist planes $\{P_i : 1 \le i \le N\}$, for some $N \le e^{(1-2\rho)\ell}$, so that

$$\#\{v \in \mathcal{P}_\ell \setminus (\cup P_i) : |Q_0(g_Q v)| \le e^{-A\rho\ell}\} \le e^{(1-2\rho)\ell}.$$

Moreover, there are intervals $\{J_i : 1 \le i \le N\}$, with multiplicity at most two, so that $I_v^{(1+2\rho)\ell}(e^{-\rho\ell}) \cap J_i \ne \emptyset$ for such v; thus $I_v \subset cJ_i =: \hat{J}_i$ for an absolute $c \ge 1$. Furthermore, recall that

$$\int_{\hat{J}_i} \alpha_1(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k \ll e^{-\rho\ell} \int_{\hat{J}_i} \alpha_2(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k,$$

and note that $\{\hat{J}_i\}$ has bounded multiplicity depending on c.

For every $\ell \in \mathcal{L}_i'$, let $\mathcal{W}_{t,i}'(\ell) = \mathcal{W}_{t,i}(\ell)$, and for every $\ell \in \mathcal{L}_i$, let $\mathcal{W}_{t,i}'(\ell) = \mathcal{W}_{t,i} \setminus (\cup P_i)$. In either case

$$\#\mathcal{W}_{t,i}'(\ell) \le e^{(1-\rho)\ell}. \tag{14.4}$$

We put $\mathcal{W}_{t,1}' = \bigcup_\ell \mathcal{W}_{t,i}'(\ell)$.

The task is now two folds: we first show that for all $\rho t - 1 \le \ell \le t'$

$$\sum_{\mathcal{W}_{t,1}'} \int_{I_v} \|a_t k g_Q v\|^{-1-\delta}\,\mathrm{d}k \ll e^{-\star\rho^2 t};$$

this is an easy consequence of (14.4) and Lemma 13.6.

Then we show that for such ℓ,

$$\int_{\cup \hat{J}_i} \alpha_1(a_t k g_Q \Gamma) \ll e^{-\star\rho^2 t};$$

this step is more involved. It relies on Proposition 13.10, Theorem 13.8, and ideas similar to [EMM98].

Summing these estimates over all $\rho t - 1 \le \ell \le t'$, we get $\int_{\mathcal{C}_{t,1}} \alpha_1 \ll e^{-\star\rho^2 t}$ as we wanted to show. The details follow.

Contribution of $\mathcal{W}_{t,i}'$. Applying [EMM98, Lemma 5.5] we have

$$\sum_{\mathcal{W}_{t,1}'} \int_{I_v} \|a_t k g_Q v\|^{-1}\,\mathrm{d}k \ll \sum_{\lfloor \rho t - 1\rfloor}^{\lceil t'\rceil} \sum_{\mathcal{W}_{t,1}'(\ell)} \|v\|^{-1}$$

Using (14.4), one obtains

$$\sum_{\lfloor \rho t - 1\rfloor}^{\lceil t'\rceil} \sum_{\mathcal{W}_{t,1}'(\ell)} e^{\delta t} \|v\|^{-1} \ll \sum_{\ell \ge \rho t - 1} e^{-\ell} e^{(1-\rho)\ell} \ll e^{-\rho\ell/2}$$

Altogether, we conclude

$$\sum_{\mathcal{W}_{t,1}'} \int_{I_v} \|a_t k g_Q v\|^{-1-\delta}\,\mathrm{d}k \le e^{-\rho^2 t/2} \tag{14.5}$$

Similarly, $\sum_{\mathcal{W}_{t,2}'} \int_{I_u^*} \|(a_t k g_Q)^* u\|^{-1-\delta}\,\mathrm{d}k \le e^{-\rho^2 t/2}$.

Contribution of $\mathcal{W}_{t,i}(\ell) \setminus \mathcal{W}'_{t,i}(\ell)$. Again we will work with $i = 1$, the argument for $i = 2$ is similar. Let $\ell \in \mathcal{L}_1$ otherwise $\mathcal{W}_{t,1}(\ell) = \mathcal{W}'_{t,1}(\ell)$. As it was mentioned above, in view of part (1) in Proposition 13.10, there are intervals $\{J_i : 1 \le i \le N\}$, with multiplicity at most two, so that $I_v^{(1+2\rho)\ell}(e^{-\rho\ell}) \cap J_i \neq \emptyset$ for every $v \in \mathcal{W}_{t,1}(\ell)$. This implies $I_v \subset cJ_i =: \hat{J}_i$ for every such v, where $c \ge 1$ is absolute. Furthermore, recall that

$$\int_{\hat{J}_i} \alpha_1(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k \ll e^{-\rho\ell} \int_{\hat{J}_i} \alpha_2(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k, \tag{14.6}$$

and note that $\{\hat{J}_i\}$ has bounded multiplicity depending on c.

Sublemma. *We have*

$$\sum_{\hat{J}_i} \int_{\hat{J}_i} \alpha_1(a_t k g_Q \Gamma)\,\mathrm{d}k \ll e^{-\star\rho\ell}\Big(1 + \sum_{\hat{J}_i} \int_{\hat{J}_i} \alpha_2(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k\Big) \tag{14.7}$$

Let us first assume (14.7) and complete the proof of the proposition. Since $\sum 1_{\hat{J}_i} \ll 1_{\cup \hat{J}_i}$, (14.7) implies that

$$\begin{aligned}\sum_i \int_{\hat{J}_i} \alpha_1(a_t k g_Q \Gamma)\,\mathrm{d}k &\ll e^{-\star\rho\ell}\Big(1 + \int_{\cup\hat{J}_i} \alpha_2(a_{(1+2\rho)\ell} k g_Q \Gamma)\,\mathrm{d}k\Big) \\ &\ll e^{-\star\rho\ell}(1+2\rho)\ell \le e^{-\star\rho\ell},\end{aligned}$$

where we used part (2) in Theorem 13.8 in the second inequality.

Altogether, for every $\ell \in \mathcal{L}_1$,

$$\int_{\cup\hat{J}_i} \alpha_1(a_t k; g_Q)\,\mathrm{d}k \ll e^{-\star\rho\ell} \le e^{-\star\rho^2 t}. \tag{14.8}$$

where we used $\ell \ge \rho t - 1$ in the last inequality.

Now summing (14.5) and (14.8) over all $\rho t - 1 \le \ell \le t'$, we get

$$\int_{\mathcal{C}_{t,1}} \tilde{\alpha}_1(a_t k; g_Q \Gamma) \ll t e^{-\star\rho^2 t} \ll e^{-\star\rho^2 t}$$

Similarly, $\int_{\mathcal{C}_{t,2}} \tilde{\alpha}_2(a_t k; g_Q \Gamma) \ll e^{-\star\rho^2 t}$.

The proposition follows from these in view of (14.2). □

Proof of the Sublemma. It remains to prove (14.7). The argument is similar to the arguments pioneered in [EMM98], which are by now well known. To simplify the notation, let us write $s = (1+2\rho)\ell$ and put $x = g_Q\Gamma$.

Let $n_1 = \lceil 100\rho^{-3} \rceil$ and let $s_1 = (t-s)/n_1$. We claim that

$$\sum_i \int_{\hat{J}_i} \alpha_1(a_{ns_1+s} k x)\,\mathrm{d}k \le C^n \sum_i \int_{\hat{J}_i} \alpha_1(a_s k x)\,\mathrm{d}k + (C+1)^n e^{-\rho\ell/2}, \tag{14.9}$$

for all integers $0 \le n \le n_1$, where $C \ge 1$ is absolute.

Note that the sublemma follows from (14.9) and (14.6). Therefore, the task is to establish (14.9).

We first make the following observation

$$\sum_i \int_{\hat{J}_i} \alpha_2^{1/2}(a_d kx)\,\mathrm{d}k \ll e^{-3\rho\ell/4} \qquad \text{for all } d \geq 0. \tag{14.10}$$

To see (14.10), first recall from Proposition 13.10 that $\sum 1_{\hat{J}_i} \ll 1_{\cup \hat{J}_i}$, hence

$$\sum_i \int_{\hat{J}_i} \alpha_2^{1/2}(a_d kx)\,\mathrm{d}k \ll \int_{\bigcup \hat{J}_i} \alpha_2^{1/2}(a_d kx)\,\mathrm{d}k.$$

Moreover, the number of such intervals $\hat{J}_i$ is $\ll e^{(1-2\rho)\ell}$ and $|\hat{J}_i| \asymp e^{-\ell}e^{-3\rho\ell/2}$, see (13.6) and the paragraph following that equation. Thus

$$|\bigcup \hat{J}_i| \leq \sum |\hat{J}_i| \ll (e^{(1-2\rho)\ell} \cdot (e^{(-1-\frac{3}{2}\rho)\ell} \leq e^{-3\rho\ell}.$$

Hence (14.10) follows from Corollary 13.9, applied with $\Theta = \bigcup \hat{J}_i$.

Let us now return to the proof of (14.9), which will be completed using induction on n. The base case $n = 0$ is trivial.

Fix some $n \geq 0$ and some $\hat{J}_i$. Let $M = \lceil e^{ns_1+s}|\hat{J}_i|\rceil$ and $b = |\hat{J}_i|/M$. For every $k \in L$, let $h_k = a_{ns_1+s}ka_{-ns_1-s}$. Then

$$\int_{\hat{J}_i} \alpha_1(a_{(n+1)s_1+s}kx)\,\mathrm{d}k = \sum_j \int_0^b \alpha_1(a_{s_1}h_k a_{ns_1+s}k_j x)\,\mathrm{d}k, \tag{14.11}$$

where $k_j = jb$ for $0 \leq j \leq M-1$.

If $k = k_\theta$, see (13.1), then $h_k = h'_k u_{-e^{ns_1+s}\sin\theta/(1+\cos\theta)}$ where $\|h'_k - I\| \ll b$ is lower triangular. Using the change of variable $r = e^{ns_1+s}\frac{\sin\theta}{(1+\cos\theta)}$, we get

$$\int_0^b \alpha_1(a_{s_1}h_k a_{ns_1+s}k_j x)\,\mathrm{d}k \asymp e^{-ns_1-s}\int_0^{b'} \alpha_1(a_{s_1}u_{-r}a_{ns_1+s}k_j x)\,\mathrm{d}r. \tag{14.12}$$

where $b' = e^{ns_1+s}\frac{\sin b_0}{(1+\cos b_0)} \asymp 1$.

In view of [EMM98, Lemma 5.8], see also [EMM98, Lemma 5.5], we have

$$\begin{aligned}\int_0^{b'} \alpha_1(a_{s_1}u_{-r}a_{ns_1+s}k_j x)\,\mathrm{d}r \ll& \\ &\int_0^{b'} \alpha_1(u_{-r}a_{ns_1+s}k_j x)\,\mathrm{d}r + e^{2s_1}\int_0^{b'} \alpha_2^{1/2}(u_{-r}a_{ns_1+s}kx)\,\mathrm{d}r\end{aligned}$$

Multiplying the above by e^{-ns_1-s} and changing the variables from u_r back to h_k in the second line above, we conclude that

$$\begin{aligned}e^{-ns_1-s}\int_0^{b'} \alpha_1(a_{s_1}u_{-r}a_{ns_1+s}k_j x)\,\mathrm{d}r \ll& \\ \int_0^b \alpha_1(h_k a_{ns_1+s}k_j x)\,\mathrm{d}k + e^{2s_1}&\int_0^b \alpha_2^{1/2}(h_k a_{ns_1+s}k_j x)\,\mathrm{d}k = \\ &\int_0^b \alpha_1(a_{ns_1+s}kk_j x)\,\mathrm{d}k + e^{2s_1}\int_0^b \alpha_2^{1/2}(a_{ns_1+s}kk_j x)\,\mathrm{d}k.\end{aligned}$$

In view of (14.12) and (14.11), the above implies

$$\int_{\hat{J}_i} \alpha_1(a_{(n+1)s_1+s}kx)\,\mathrm{d}k \ll \sum_j \int_0^b \alpha_1(a_{ns_1+s}kk_jx)\,\mathrm{d}k \quad + \\ e^{2s_1}\sum_j \int_0^b \alpha_2^{1/2}(a_{ns_1+s}kk_jx)\,\mathrm{d}k.$$

Altogether, we have shown

$$\int_{\hat{J}_i}\alpha_1(a_{(n+1)s_1+s}kx)\,\mathrm{d}k \ll \int_{\hat{J}_i}\alpha_1(a_{ns_1+s}kx)\,\mathrm{d}k + e^{2s_1}\int_{\hat{J}_i}\alpha_2^{1/2}(a_{ns_1+s}kx)\,\mathrm{d}k.$$

Summing these over all $\hat{J}_i$, we get

$$\sum_{\hat{J}_i}\int_{\hat{J}_i} \alpha_1(a_{(n+1)s_1+s}kx)\,\mathrm{d}k \le C\sum_{\hat{J}_i}\int_{\hat{J}_i}\alpha_1(a_{ns_1+s}kx)\,\mathrm{d}k \quad + \\ Ce^{2s_1}\sum_{\hat{J}_i}\int_{\hat{J}_i}\alpha_2^{1/2}(a_{ns_1+s}kx)\,\mathrm{d}k \tag{14.13}$$

Recall now that $\ell \ge \rho t - 1$ and $s_1 \asymp \rho^3 t/100$. This and (14.10) imply that

$$e^{2s_1}\sum_{\hat{J}_i}\int_{\hat{J}_i}\alpha_2^{1/2}(a_{ns_1+s}kx)\,\mathrm{d}k \ll e^{\rho\ell/10}e^{-3\rho/4} \le e^{-\rho\ell/2}.$$

This, (14.13) and the inductive hypothesis finish the proof of (14.9). $\square$

15. Circular averages of Siegel tranforms

Let $f \in C_c(\mathbb{R}^3)$. For every $g \in G$, define

$$\hat{f}(g) = \sum_{v\in\mathbb{Z}^3} f(gv) \tag{15.1}$$

For any indefinite ternary quadratic form Q with $\det Q = 1$, we let $g_Q \in G$ be so that $Q(v) = Q_0(g_Qv)$, where $Q_0(\mathsf{x},\mathsf{y},\mathsf{z}) = 2\mathsf{x}\mathsf{z} - \mathsf{y}^2$.

In this section, we will use Proposition 13.3 and Theorem 12.1 to prove the following proposition. The proof of Theorem 2.3 will then be completed using this proposition.

15.1. **Proposition.** *There is an absolute constant M, and for every large enough E and $0 < \rho \le 10^{-4}$, there are ϱ_1, ϱ_2 (depending on ρ and E) so that if Q is as above and t is large enough, depending linearly on $\log(\|Q\|)$, the following holds.*

Assume that for every $Q' \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|Q'\| \le e^{\rho t}$ and all $\lambda \in \mathbb{R}$,

$$\left\|Q - \lambda Q'\right\| > \left\|Q'\right\|^{-E/M}. \tag{15.2}$$

There exists some C depending on E and polynomially on $\|Q\|$ so that the following holds. For any smooth function ξ on K, if

$$\Big|\int_K \hat{f}(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k - \int_K \xi\,\mathrm{d}k \int_X \hat{f}\,\mathrm{d}m_X\Big| > C\mathcal{S}(f)\mathcal{S}(\xi)e^{-\varrho_2 t}$$

then there is at least one $(\frac{\varrho_1}{4E}, E, t)$-exceptional line or plane, and at most four $(\frac{\varrho_1}{4E}, E, t)$-exceptional lines $\{L_i\}$ and at most four $(\frac{\varrho_1}{4E}, E, t)$-exceptional planes $\{P_i\}$. Moreover

$$\int_K \hat{f}(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k = \int_K \xi\,\mathrm{d}k \int_X \hat{f}\,\mathrm{d}m_X + \mathcal{M} + O(\mathcal{S}(f)\mathcal{S}(\xi)e^{-\varrho_2 t})$$

where

$$\mathcal{M} = \int_{\mathcal{C}} \hat{f}_{\mathrm{sp}}(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k$$

with

$$\hat{f}_{\mathrm{sp}}(a_t k g_Q \Gamma) = \sum_{v\in\mathbb{Z}^3\cap((\cup_i L_i)\cup(\cup_i P_i))} f(a_t k g_Q v)$$

$$\mathcal{C} = \Big\{k \in K : \hat{f}_{\mathrm{sp}}(a_t k g_Q \Gamma) \ge e^{\varrho_1 t}\Big\}$$

Before starting the proof, we recall the following well-known fact

15.2. **Lemma.** *Suppose there exists $g'\Gamma \in X$ with $\mathrm{vol}(Hg'\Gamma) \le R$, so that*

$$d(g_Q\Gamma, g'\Gamma) \le \beta$$

There exists an integral form Q' with $\|Q'\| \ll R^\star$ and some $\lambda \in \mathbb{R}$ so that

$$\|Q - \lambda Q'\| \ll R^\star \beta$$

Proof. Replacing g' by $hg'\gamma$ for some $h \in H$ and $\gamma \in \Gamma$ we may assume $\|g'\| \ll 1$. Similarly, we may assume $\|g_Q\| \ll 1$.

Since $\mathrm{vol}(Hg'\Gamma) \le R$, $Q_0 \circ g'$ is equivalent to an integral form Q'' with $\|Q''\| \ll R^\star$.

Since $d(g_Q\Gamma, g'\Gamma) \le \beta$, $g_Q = \epsilon g'\gamma'$ where $\|\epsilon - I\| \ll \beta$ and $\|\gamma'\| \ll 1$. Thus

$$\|Q - \lambda Q'\| \ll R^\star \beta$$

for some integral form Q' with $\|Q'\| \ll R^\star$ and some $\lambda \in \mathbb{R}$. □

Proof of Proposition 15.1. Let E and ρ be as in the statement, and let $t > 0$ be a large parameter. We will use Lemma 15.2 in the following form:

Let Q satisfy (15.2). There exists $E_1 \ge \max(4A_7, E)$, where A_7 is as in Theorem 12.1 so that the following holds. For all t so that $t > 4A_7 \log t$ and for every $x \in X$ with $\mathrm{vol}(Hx) \le e^{\rho t/E_1}$,

$$d(g\Gamma, x) > e^{-t}.$$

In view of this, part (1) in Theorem 12.1 (applied with $G = \mathrm{SL}_3(\mathbb{R})$, hence, $\mathsf{m} = 2$) holds with $R = e^{\rho t/E_1}$ and t. Indeed, since $\frac{A_7\rho}{E_1} \le \frac{1}{4}$ and $t^{A_7} \le e^{t/4}$,

$$R^{A_7} t^{A_7} e^{-2t} = e^{A_7\rho t/E_1} t^{A_7} e^{-2t} \le e^{-t};$$

hence, part (2) in Theorem 12.1 cannot hold.

For every S, let $1_{X_{1/S}} \le \varphi_S \le 1_{X_{1/(S+1)}}$ be smooth with $\mathcal{S}(\varphi_S) \ll S^\star$. Put $\hat{f}_S = \varphi_S \hat{f}$; we let L be so that $\mathcal{S}(\hat{f}_S) \ll S^L \mathcal{S}(f)$. Recall that $\alpha_1, \alpha_2 \gg S^\star$ in $X_{1/S}$, they belong to $L^2(X, m_X)$, and $\hat{f} \ll \max(\alpha_1, \alpha_2)$. Thus,

$$\int \hat{f}\,\mathrm{d}m_X = \int \hat{f}_S\,\mathrm{d}m_X + O(S^{-\star}). \tag{15.3}$$

Set $\varepsilon = \kappa_8 \rho/(2LE_1)$, where κ_8 is as in Theorem 12.1. We will show the claim in the theorem holds with

$$\varrho_1 = \varepsilon \quad \text{and} \quad \varrho_2 = \tfrac{\varepsilon^2}{32E^2}.$$

Apply Lemma 13.4 with $\eta^{-D} = e^{\frac{\varepsilon}{E}t}$, $\eta^A = e^{-\varepsilon t}$, and Q. If there are at least five $(\frac{\varepsilon}{4E}, E, t)$-exceptional lines, then Lemma 13.4 implies

$$\|Q - \lambda Q'\| \le \eta^{A - M'D} = e^{(-\varepsilon + \frac{M'\varepsilon}{E})t}$$

for some $Q' \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|Q'\| \le e^{\frac{M'\varepsilon}{E}t}$, where M' is absolute. Assuming E is large compared to this M', however, this contradicts (15.2) if $M \ge 2M'$. Thus, there are at most four $(\frac{\varepsilon}{4E}, E, t)$-special lines. Similarly, there are at most four $(\frac{\varepsilon}{4E}, E, t)$-exceptional planes.

Denote these lines and the planes (if they exist) by $\{L_i\}$ and $\{P_i\}$, respectively, and put $\mathrm{Exc} = (\cup_i L_i) \cup (\cup_i P_i)$. For every $k \in K$, write

$$\hat{f}(a_t k g_Q \Gamma) = \hat{f}_S(a_t k g_Q \Gamma) + \hat{f}_{\mathrm{cusp}}(a_t k g_Q \Gamma) + \hat{f}_{\mathrm{sp}}(a_t k g_Q \Gamma)$$

where $\hat{f}_S = \varphi_S \hat{f}$, $\hat{f}_{\mathrm{cusp}}$ is the contribution of $\mathbb{Z}^3 \setminus \mathrm{Exc}$ to $\hat{f} - \hat{f}_S$, and $\hat{f}_{\mathrm{sp}}$ is the contribution of $\mathbb{Z}^3 \cap \mathrm{Exc}$ to $\hat{f} - \hat{f}_S$.

By Theorem 12.1, applied with $R = e^{\rho t/E_1}$, for any $\xi \in C^\infty(K)$ we have

$$\begin{aligned} \Big| \int_K \hat{f}_S(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k - \int \xi\,\mathrm{d}k \int_X \hat{f}_S\,\mathrm{d}m_X \Big| &\ll \\ \mathcal{S}(\hat{f}_S)\mathcal{S}(\xi)e^{-\kappa_8 \rho t/E'} &\ll S^L \mathcal{S}(f)\mathcal{S}(\xi) e^{-\kappa_8 \rho t/E_1}. \end{aligned} \tag{15.4}$$

If we choose $S = e^{\varepsilon t} = e^{\kappa_8 \rho t/(2LE_1)}$, the above is $\ll \mathcal{S}(f)\mathcal{S}(\xi)e^{-\varepsilon t/2}$.

Moreover, by Theorem 13.8 applied with $p = 1/2$ and the Chebyshev's inequality, we have

$$\int_{\{k : a_t k g_Q \Gamma \notin X_{1/S}\}} S^{1/4}\,\mathrm{d}k \ll S^{-1/2} S^{1/4} = S^{-1/4}. \tag{15.5}$$

This and (15.4), reduce the problem to investigating the integral of $\hat{f} - \hat{f}_S = \hat{f}_{\mathrm{cusp}} + \hat{f}_{\mathrm{sp}}$ over $\hat{\mathcal{C}} := \{k \in K : \hat{f} - \hat{f}_S \ge ES^{1/4}\}$.

Let $\tilde{f}$ be as in (13.3) with $\varepsilon/4E$, E, and t. That is:

$$\tilde{f}(h; g_Q) = \sum_{v \in \mathcal{N}_{Q,t}} f(hv)$$

where $\mathcal{N}_{Q,t}$ denotes the set of vectors in $\mathbb{Z}^3$ which are not contained in any $(\varepsilon/4E, E, t)$-exceptional line or plane.

Let $\tilde{\mathcal{C}}_t = \{k \in K : \tilde{f}(a_t k; g_Q) \geq E e^{\varepsilon t/4} = ES^{1/4}\}$. By the definitions,

$$\int_{\hat{\mathcal{C}}} \hat{f}_{\mathrm{cusp}}(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k \leq \|\xi\|_\infty \int_{\tilde{\mathcal{C}}_t} \tilde{f}(a_t k; g_Q)\,\mathrm{d}k.$$

Thus Proposition 13.3, applied with E and $\varepsilon/4E$, implies

$$\int_{\tilde{\mathcal{C}}_t} \tilde{f}(a_t k; g_Q)\,\mathrm{d}k \ll e^{-\frac{\varepsilon^2 t}{32E^2}}.$$

From these two, we conclude that

$$\int_{\hat{\mathcal{C}}} \hat{f}_{\mathrm{cusp}}(a_t k g_Q \Gamma)\,\mathrm{d}k \ll \|\xi\|_\infty\, e^{-\frac{\varepsilon^2 t}{32E^2}}. \tag{15.6}$$

In view of (15.3), (15.4), (15.5) and (15.6), we have

$$\begin{aligned}\Big|\int_K \hat{f}(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k - \int_K \xi\,\mathrm{d}k \int_X \hat{f}\,\mathrm{d}m_X\Big|& \\ &= \int_{\mathcal{C}} \hat{f}_{\mathrm{sp}}(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k + O(\mathcal{S}(f)\mathcal{S}(\xi)e^{-\varepsilon^2 t/32E^2})\end{aligned}$$

where $\mathcal{C} = \{k \in K : \hat{f}_{\mathrm{sp}}(a_t k g_Q \Gamma) > E e^{\varepsilon t/4}\}$.

This completes the proof if we let $\varrho_1 = \varepsilon$ and $\varrho_2 = \varepsilon^2/32E^2$. □

The Diophantine condition on Q, as stated in (15.2), is primarily employed to derive the desired upper bound estimates for circular averages of the Siegel transform, namely the application of Proposition 13.3 in the proof. However, if one assumes the *weaker* condition on Q specified in Theorem 2.1, it is still possible to establish a lower bound estimate for these averages. This lower bound suffices to prove (a stronger version of) Theorem 2.1.

A simplified version of Proposition 15.1, which encapsulates this result, is presented below.

15.3. **Proposition.** *Let R be large, depending on $\|Q\|$, and let $T \geq R^A$. Assume that for every $Q' \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|Q'\| \leq R$ and all $\lambda \in \mathbb{R}$,*

$$\big\|Q - \lambda Q'\big\| > R^A (\log T)^A T^{-2}. \tag{15.7}$$

Then for all $f \in C_c^\infty(\mathbb{R}^3)$ and $\xi \in C_c^\infty(K)$ we have

$$\int_K \hat{f}(a_{\log T} k g_Q \Gamma)\xi(k)\,\mathrm{d}k \geq \int_K \xi\,\mathrm{d}k \int_X \hat{f}\,\mathrm{d}m_X + O(\mathcal{S}(f)\mathcal{S}(\xi)T^{-\kappa})$$

where A and κ are absolute.

Proof. The proof of this proposition is contained in the proof of Proposition 15.1, we explicate the proof for the convenience of the reader.

For every S, let $1_{X_{1/S}} \leq \varphi_S \leq 1_{X_{1/(S+1)}}$ be a smooth function with $\mathcal{S}(\varphi_S) \ll S^\star$. Put $\hat{f}_S = \varphi_S \hat{f}$; we let L be so that $\mathcal{S}(\hat{f}_S) \ll S^L \mathcal{S}(f)$.

In view of Lemma 15.2, if for some R' and some g' with $\mathrm{vol}(Hg'\Gamma) \leq R'$,

$$d(g_Q\Gamma, g'\Gamma) \leq R'^{A_7} (\log T)^{A_7} T^{-2},$$

then there exists $Q' \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|Q'\| \ll R'^{A'}$ and $\lambda \in \mathbb{R}$ so that

$$\|Q - \lambda Q'\| \le R'^{A_7 + A'}(\log T)^{A_7} T^{-2}.$$

Assuming $R' \asymp R^{1/A'}$ is chosen so that $\|Q'\| \ll R'^{A'}$ implies $\|Q'\| \le R$, the above contradicts (15.7) so long as A is large enough.

Therefore, by Theorem 12.1, applied with $G = \mathrm{SL}_2(\mathbb{R})$ (hence $\mathsf{m} = 2$) and $R' \asymp R^{1/A'}$, for any $\xi \in C^\infty(K)$,

$$\Big|\int_K \hat{f}_S(a_{\log T} k g_Q \Gamma)\xi(k)\,\mathrm{d}k - \int \xi\,\mathrm{d}k \int_X \hat{f}_S\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\hat{f}_S)\mathcal{S}(\xi) R'^{-\kappa_8} \ll S^L \mathcal{S}(f)\mathcal{S}(\xi) R^{-\kappa_8/A'}. \tag{15.8}$$

If we choose $S = R^\star$, the above is $\ll \mathcal{S}(f)\mathcal{S}(\xi)R^{-\star}$.

Moreover, recall from (15.3) that

$$\int \hat{f}\,\mathrm{d}m_X = \int \hat{f}_S\,\mathrm{d}m_X + O(S^{-\star})$$

Altogether, thus, we conclude

$$\begin{aligned}\int_K \hat{f}(a_{\log T} k g_Q \Gamma)\xi(k)\,\mathrm{d}k &\ge \int_K \hat{f}_S(a_t k g_Q \Gamma)\xi(k)\,\mathrm{d}k \\ &= \int \xi\,\mathrm{d}k \int_X \hat{f}\,\mathrm{d}m_X + \mathcal{S}(f)\mathcal{S}(\xi)R^{-\star},\end{aligned}$$

as it was claimed. □

16. Linear algebra and quadratic forms

In this section, we will use transitivity of the action of H on level sets $\{v : Q_0(v) = c\}$ to relate the counting problem in Theorem 2.3 to averages considered in Proposition 15.1. As mentioned before, the argument is similar to [DM91, EMM98].

Let us begin with the following

16.1. **Lemma** (cf. [EMM98], Lemma 3.4)**.** *Let $f \ge 0$ be a smooth function supported on $B_{\mathbb{R}^3}(0, R)$ (with $R \ge 1$) and let ξ be a smooth function on $\mathbb{S}^2$. Let $v \in \mathbb{R}^3$ and suppose $t \ge 2\log R$ and $e^t/2 \le \|v\| \le e^t$. Then*

$$\int f(a_t k v)\xi(k^{-1}\mathsf{e}_3)\,\mathrm{d}k = \frac{\sqrt{2}}{2\pi(1+O(e^{-t}))\|v\|} J_f(Q_0(v), e^{-t}\|v\|)\xi(\tfrac{v}{\|v\|}) + O\big(\mathcal{S}(f)\mathcal{S}(\xi)e^{-2t}\big)$$

where $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ is the standard basis for $\mathbb{R}^3$,

$$J_f(c,d) := \frac{1}{d}\int f\big(\tfrac{c+\mathsf{y}^2}{2d}, -\mathsf{y}, d\big)\,\mathrm{d}\mathsf{y} \qquad \textit{that is: } Q_0(\tfrac{c+\mathsf{y}^2}{2d}, \mathsf{y}, d) = c,$$

and the implied constants depend polynomially on R.

Moreover, there is some $C > 0$ depending on R so that if either $\|v\| > Ce^t$ or $|Q_0(v)| > C$, then both sides of the above equality equal zero without the error term $O\big(\mathcal{S}(f)e^{-4t}\big)$.

Proof. Put $\mathcal{K}_f(v) = \{k \in K : f(a_t kv) \neq 0\}$. Let $k \in \mathcal{K}_f(v)$, then

$$\begin{aligned} -R \leq \mathsf{x} :=& \langle a_t kv, \mathsf{e}_1\rangle = e^t\langle kv, \mathsf{e}_1\rangle \leq R \quad \text{and} \\ -R \leq \mathsf{y} :=& \langle a_t kv, \mathsf{e}_2\rangle = \langle kv, \mathsf{e}_2\rangle \leq R \end{aligned} \tag{16.1}$$

The estimates in (16.1) and $e^t/2 \leq \|v\| \leq e^t$ thus imply that there exists some C' which depends polynomially on R so that if we let

$$\mathcal{K}_R(v) = \{k \in K : \|k^{-1}\mathsf{e}_3 - \tfrac{v}{\|v\|}\| \leq C'e^{-t}\},$$

then $\mathcal{K}_f(v) \subset \mathcal{K}_R(v)$. Note that if $\mathcal{K}_R(v) = \emptyset$ or $\mathcal{K}_f(v) = \emptyset$, we still have $\mathcal{K}_f(v) \subset \mathcal{K}_R(v)$. Altogether, the range of integration is restricted to $\mathcal{K}_R(v)$.

We first note that for all $k \in \mathcal{K}_R(v)$, we have

$$|\xi(k^{-1}\mathsf{e}_3) - \xi(\tfrac{v}{\|v\|})| \ll \mathcal{S}(\xi)e^{-t} \tag{16.2}$$

Similarly, for all $k \in \mathcal{K}_R(v)$,

$$\mathsf{z} := \langle a_t kv, \mathsf{e}_3\rangle = e^{-t}\langle kv, \mathsf{e}_3\rangle = e^{-t}\|v\| + O(e^{-t}).$$

Let us now put

$$\mathsf{x}' = \tfrac{Q_0(v)+\mathsf{y}^2}{2e^{-t}\|v\|} = \tfrac{(2\mathsf{xz}-\mathsf{y}^2)+\mathsf{y}^2}{2e^{-t}\|v\|} = \tfrac{\mathsf{xz}}{e^{-t}\|v\|}.$$

Then $Q_0(\mathsf{x}', \mathsf{y}, e^{-t}\|v\|) = Q_0(v)$, $|\mathsf{x} - \mathsf{x}'| \ll e^{-t}$, and

$$|f(a_t kv) - f(\mathsf{x}', \mathsf{y}, e^{-t}\|v\|)| \ll \mathcal{S}(f_{c,\delta})e^{-t}.$$

In view of (16.2), thus, to complete the proof, we need to compute $\mathrm{d}\mathsf{y}$. First note that

$$\int f(a_t kv)\,\mathrm{d}k = \int f(a_t k'k_v v)\,\mathrm{d}k'.$$

Write $k_v v = (v_1, v_2, v_3)$, then $v_3 = (1 + O(e^{-t}))\|v\|$ and in the notation of (13.1),

$$\mathsf{y} = \langle k'k_v v, \mathsf{e}_2\rangle = v_1\tfrac{\sin\theta}{\sqrt{2}} + v_2\cos\theta - v_3\tfrac{\sin\theta}{\sqrt{2}}.$$

Thus $\mathrm{d}k = -\frac{\sqrt{2}}{2\pi((1+O(e^{-t}))\|v\|)}\,\mathrm{d}\mathsf{y}$ on $K_R(v)$, where we used $\cos\theta = 1 + O(e^{-2t})$ for $k'k_v \in K_R(v)$ and $v_3 = (1 + O(e^{-t}))\|v\|$. Altogether, we get

$$\begin{aligned} \int f(a_t kv)\,\mathrm{d}k &= \tfrac{\sqrt{2}}{2\pi((1+O(e^{-t}))\|v\|)}\int f(\mathsf{x}', -\mathsf{y}, e^{-t}\|v\|)\,\mathrm{d}\mathsf{y} + O\big(\mathcal{S}(f)e^{-2t}\big) \\ &= \tfrac{\sqrt{2}}{2\pi((1+O(e^{-t}))\|v\|)} J_f(Q_0(v), e^{-t}\|v\|) + O\big(\mathcal{S}(f)e^{-2t}\big). \end{aligned}$$

This and (16.2) complete the proof of the lemma; note that the last claim in the lemma follows from the above argument. $\square$

Fix some $0<\varepsilon<10^{-4}$ and cover $\mathbb{S}^2$ with disjoint half open cubes $\{\Omega'_{i,\varepsilon}\}$ of size between ε^2 and $\varepsilon^2/2$. For each i, let

$$\Omega_{i,\varepsilon}=(1-\varepsilon,1]\cdot(g_Q\Omega'_{i,\varepsilon}).$$

Also set $\sup_{\Omega'_{i,\varepsilon}}\|g_Qw\|=d_i$ and $d_i-C\varepsilon^2=\inf_{\Omega'_{i,\varepsilon}}\|g_Qw\|$, note that d_i and C depend on $\|g_Q\|$.

16.2. **Lemma.** *Let $c\in\mathbb{R}$ and $0<\varepsilon<10^{-4}$. There exist $f_i^\pm\in C_c^\infty(\mathbb{R}^3)$ and $\xi_i\in C_c^\infty(\mathbb{S}^2)$ satisfying $0\le f_i^-\le f_i^+\ll 1$, $\int f_i^+=(1+O(\varepsilon))\int f_i^-$, and $\mathcal{S}(\bullet)\ll\varepsilon^{-L}$ for $\bullet=f_i^\pm,\xi_i$ where L is absolute, so that*

$$J_{f_i^-}(Q_0(v),e^{-t}\|v\|)\xi_i(\tfrac{v}{\|v\|})\le 1_{\Omega_{i,\varepsilon}}(e^{-t}v)1_{[c-\varepsilon,c+\varepsilon]}(Q_0(v))\le J_{f_i^+}(Q_0(v),e^{-t}\|v\|)\xi_i(\tfrac{v}{\|v\|})$$

Proof. Let us set

$$\mathsf{B}_i^-=\{v=(\mathsf{x},\mathsf{y},\mathsf{z}):|\mathsf{y}|\le\tfrac12,(1-\varepsilon)d_i\le\mathsf{z}\le d_i-C\varepsilon^2,|Q_0(v)-c|\le\varepsilon\}$$
$$\mathsf{B}_i^+=\{v=(\mathsf{x},\mathsf{y},\mathsf{z}):|\mathsf{y}|\le\tfrac12,(1-\varepsilon)(d_i-C\varepsilon^2)\le\mathsf{z}\le d_i,|Q_0(v)-c|\le\varepsilon\}$$

Define B_i^{--} and B_i^{++} similarly, by replacing ε by $\varepsilon-C'\varepsilon^2$ and $\varepsilon+C'\varepsilon^2$ for a large C', depending on d_i and C, respectively.

Fix smooth functions $\tilde f_i^\pm$ satisfying $1_{\mathsf{B}_i^{--}}\le\tilde f_i^-\le 1_{\mathsf{B}_i^-}$, $1_{\mathsf{B}_i^+}\le\tilde f_i^+\le 1_{\mathsf{B}_i^{++}}$, and $\mathcal{S}(\tilde f_i^\pm)\ll\varepsilon^{-L}$. Put $f_i^\pm=\mathsf{z}\tilde f_i^\pm$.

Also fix smooth characteristic functions ξ_i so that

$$1_{\Xi_i}\le\xi_i\le 1_{\mathcal{N}_{C'\varepsilon^2}(\Xi_i)}\quad\text{and}\quad\mathcal{S}(\xi_i)\ll\varepsilon^{-L}$$

where $\mathcal{N}_\delta(\Xi_i)$ is the δ-neighborhood of $\Xi_i=\{\frac{w}{\|w\|}:w\in g_Q\Omega'_{i,\varepsilon}\}$.

Suppose now that v is so that $J_{f_i^-}(Q_0(v),e^{-t}\|v\|)\xi_i(\frac{v}{\|v\|})\neq 0$, then $v\in g_Q\Omega'_{i,\varepsilon}$. Moreover, using the definition

$$J_{f_i^-}(Q_0(v),e^{-t}\|v\|)=\tfrac{e^t}{\|v\|}\int f_i^-(\mathsf{x},-\mathsf{y},e^{-t}\|v\|)\,\mathrm{d}\mathsf{y}$$

where $Q_0(\mathsf{x},\mathsf{y},e^{-t}\|v\|)=Q_0(v)$, we have $f_i^-(\mathsf{x},-\mathsf{y},e^{-t}\|v\|)\neq 0$ for some x,y. Since $\frac{1}{\mathsf{z}}f_i^-\le 1_{B_i^-}$, we thus conclude

$$(1-\varepsilon)d_i\le e^{-t}\|v\|\le d_i-C\varepsilon^2\quad\text{and}\quad|Q_0(v)-c|\le\varepsilon$$

These imply that $e^{-t}v\in\Omega_{i,\varepsilon}$. Therefore,

$$1_{\Omega_{i,\varepsilon}}(e^{-t}v)1_{[c-\varepsilon,c+\varepsilon]}(Q_0(v))=1.$$

Since $J_{f_i^-}(Q_0(v),e^{-t}\|v\|)\xi(v/\|v\|)\le 1$ the lower bound follows.

We now establish the upper bound. Let $e^{-t}v\in\Omega_{i,\varepsilon}$ and $|Q_0(v)-c|\le\varepsilon$. Suppose $|\mathsf{y}|\le 1/2$, and let x be so that $Q_0(\mathsf{x},\mathsf{y},e^{-t}\|v\|)=Q_0(v)$. Then $(\mathsf{x},-\mathsf{y},e^{-t}v)\in\mathsf{B}_i^+$, thus $f_i^+(\mathsf{x},-\mathsf{y},e^{-t}\|v\|)=e^{-t}\|v\|$. Hence

$$J_{f_i^+}(Q_0(v),e^{-t}\|v\|)=\tfrac{e^t}{\|v\|}\int f_i^+(\mathsf{x},-\mathsf{y},e^{-t}\|v\|)\,\mathrm{d}\mathsf{y}=1,$$

as we claimed. □

We also record the following upper bound estimate whose proof relies on similar (but simpler) arguments. See [EMM98, Thm. 2.3], also [LMW23, Lemma 3.9] and a related estimate in [KKL23, Thm. 4].

16.3. **Lemma.** *Let $0 < \eta < 1$. Then for all $s \gg |\log \eta|$, we have*

$$\#\{v \in \mathbb{Z}^3 : \|v\| \le e^s, |Q(v)| \le M\} \ll e^{(1+\eta)s}$$

where the implied constants depend polynomially on M and $\|Q\|^{\pm 1}$.

Proof. As indicated above, this is proved in [EMM98, Thm. 2.3], the polynomial dependence stated above is implicit in loc. cit., and is made explicit in [LMW23, Lemma 3.9] and [KKL23, Thm. 4]. □

Recall that we cover $\mathbb{S}^2$ with disjoint half open cubes $\{\Omega'_{i,\varepsilon}\}$ of size between ε^2 and $\varepsilon^2/2$. For $t > 0$ and $a \le c \le b$, let

$$\mathcal{N}_i(c, e^t) = \#\{v \in \mathbb{Z}^3 : v \in ((1-\varepsilon)e^t, e^t] \cdot \Omega'_{i,\varepsilon}, |Q(v) - c| \le \varepsilon\}$$

The following lemma relates the lattice point counting to averages considered in Proposition 15.1; its proof relies on Lemma 16.1 and Lemma 16.2.

16.4. **Lemma.** *With the notation as in Lemma 16.2, let $f_i = f_i^+$ also let ξ_i be as in that lemma. Then for all $t \gg |\log \varepsilon|$, we have*

$$\tfrac{2\pi}{\sqrt{2}} e^t \int_K \hat{f}_i(a_t k g_Q \Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k = (1 + O(\varepsilon))N_i(c, e^t),$$

where the implied constants depend polynomially on $|a|, |b|$, and $\|g_Q\|^{\pm 1}$.

Proof. Let $t = \log T$. By Lemma 16.1, we have

$$\tfrac{2\pi}{\sqrt{2}} e^t \int f_i(a_t k v)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k = (1 + O(e^{-t}))J_{f_i}(Q_0(v), e^{-t}\|v\|)\tilde{\xi}_i(\tfrac{v}{\|v\|}) + O\big(\mathcal{S}(f_i)\mathcal{S}(\xi_i)e^{-t}\big) \tag{16.3}$$

Recall from Lemma 16.2 that $\Omega_{i,\varepsilon} = (1-\varepsilon, 1] \cdot (g_Q \Omega'_{i,\varepsilon})$. By that lemma

$$J_f(Q_0(v), e^{-t}\|v\|)\xi_i(\tfrac{v}{\|v\|}) = (1 + O(\varepsilon))1_{\Omega_{i,\varepsilon}}(e^{-t}v)1_{[c-\varepsilon, c+\varepsilon]}(Q_0(v)) \tag{16.4}$$

Using the last claim in Lemma 16.1, both sides of (16.3) are zero unless $\|v\| \ll e^t$ and $|Q_0(v)| \ll 1$. Therefore, using Lemma 16.3, with η small enough so that $\mathcal{S}(f_i)\mathcal{S}(\xi_i)e^{\eta t} < e^t\varepsilon^{10}$, and summing (16.3), we conclude

$$\tfrac{2\pi}{\sqrt{2}} e^t \int \hat{f}_i(a_t k g_Q \Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k = \tfrac{2\pi}{\sqrt{2}} e^t \int \sum_{v \in g_Q \mathbb{Z}^3} f_i(a_t k v)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k = (1 + O(e^{-t}))\sum J_{f_i}(Q_0(v), e^{-t}\|v\|)\tilde{\xi}_i(\tfrac{v}{\|v\|}) + O\big(\mathcal{S}(f_i)\mathcal{S}(\xi_i)e^{-t}\big)$$

This and (16.4) imply that

$$\tfrac{2\pi}{\sqrt{2}} e^t \int \hat{f}_i(a_t k g_Q \Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k = (1 + O(\varepsilon))N_i(c, e^t),$$

as we claimed. □

17. Proofs of Theorem 2.1 and Theorem 2.3

In this section, we will use Proposition 15.1 and Lemma 16.4 to complete the proof of Theorem 2.3. A similar, but simpler, argument based on Proposition 15.3 and and Lemma 16.4 will also be used to complete the proof of Theorem 2.1.

17.1. **Proof of Theorem 2.3.** Let A, δ be constants as in the statement of that theorem. Fix a large parameter T, and put $t = \log T$.

The following is our standing assumption in this section: for all $Q' \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|Q'\| \le T^\delta$ and all $\lambda \in \mathbb{R}$,

$$\|Q - \lambda Q'\| \ge \|Q'\|^{-A}. \tag{17.1}$$

17.2. **Lemma.** *There exists an absolute constant M', so that for all $\varrho < \delta/M'$ and all $\bar{E} > 5A + 5M'\varrho$. There are at most four lines $\mathcal{L} = \{L_i\}$ and at most four planes $\mathcal{P} = \{P_i\}$ so that if for some $t/5 \le \ell \le t$ any line L or plane P is $(\varrho, \bar{E}, \ell)$-exceptional then $L \in \mathcal{L}$ and $P \in \mathcal{P}$.*

Proof. We prove this for lines, the proof for planes is similar. Let $t/5 \le \ell \le t$. Then $e^{\varrho\ell} \le e^{\varrho t}$ and $e^{-\bar{E}\ell} \le e^{-\bar{E}t/5}$. Recall from Definition 13.1 that a line L is $(\varrho, \bar{E}, \ell)$-exceptional if $L \cap \mathbb{Z}^3$ is spanned by v where

$$\|v\| \le e^{\varrho\ell} \quad \text{and} \quad |Q(v)| \le e^{-\bar{E}\ell}$$

Hence any such line is $(\varrho, \bar{E}/5, t)$-exceptional.

Applying Lemma 13.4 with $\eta^{-D} = e^{\varrho t}$, $\eta^A = e^{-\varrho\bar{E}t/5}$, and Q, now implies that if there are at least five $(\varrho, \bar{E}/5, t)$-exceptional lines, then

$$\|Q - \lambda Q'\| \le \eta^{A - M'D} = e^{(-\varrho\bar{E} + 5M'\varrho)t/5}$$

for some $Q' \in \mathrm{Mat}_3(\mathbb{Z})$ with $\|Q'\| \le e^{M'\varrho t}$, where M' is absolute. Assuming $\bar{E}$ and ϱ are chosen as specified in the statement for this M', we derive a contradiction to (17.1). □

In what follows we will apply Proposition 15.1 with $E = \max(AM, 5A+5)$ and $\rho = \delta$. Then Lemma 17.2 implies that there are most four lines $\{L_i\}$ and at most four planes $\{P_i\}$ so that for any $t/5 \le \ell \le t$, any $(\varrho_1/4E, E, \ell)$-exceptional line or plane belongs to $\{L_i\}$ or $\{P_i\}$, respectively. Put $\mathcal{L} = \cup L_i$ and $\mathcal{P} = \cup P_i$.

The following basic lattice point count will be used in the argument

$$\#\{v \in \mathbb{Z}^3 : \|v\| \le e^{t/4}\} \le C_1' e^{3t/4}, \tag{17.2}$$

where C_1' is absolute.

Let $\varepsilon = e^{-\hat{\kappa}t}$ for some $\hat{\kappa}$ which will be optimized later. For all $t/5 \le \ell \le t$ and $a \le c \le b$, we put

$$\Upsilon_i(c, e^\ell) = \{v \in \mathbb{Z}^3 : v \in ((1-\varepsilon)e^\ell, e^\ell] \cdot \Omega'_{i,\varepsilon}, |Q(v) - c| \le \varepsilon\},$$

and $\mathcal{N}_i(c,e^\ell)=\#\Upsilon_i(c,e^\ell)$. Similarly, let
$$\Upsilon(c,e^\ell)=\{v\in\mathbb{Z}^3 : v\in((1-\varepsilon)e^\ell,e^\ell]\cdot\mathbb{S}^2, |Q(v)-c|\leq\varepsilon\},$$
and $\mathcal{N}(c,e^\ell)=\#\Upsilon(c,e^\ell)$.

By Lemma 16.4, there exist $f_{\ell,i,c}\in C_c^\infty(\mathbb{R}^3)$ and $\xi_i\in C_c^\infty(\mathbb{S}^2)$ so that

$$e^{-\ell}\mathcal{N}_i(c,e^\ell)=(1+O(\varepsilon))\tfrac{2\pi}{\sqrt{2}}\int\hat{f}_{\ell,i,c}(a_\ell k g_Q\Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k. \tag{17.3}$$

In view of (17.1), applying Proposition 15.1 with E and ρ as above,
$$\begin{aligned}\int\hat{f}_{\ell,i,c}(a_\ell k g_Q\Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k=\int_K\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k\int_{\mathbb{R}^3}f_{\ell,i,c}\,\mathrm{dLeb}&\\+\mathcal{R}'_{\ell,i,c}+O(\mathcal{S}(f_{\ell,i,c})\mathcal{S}(\xi_i)e^{-\varrho_2\ell})&\end{aligned}$$
where
$$\mathcal{R}'_{\ell,i,c}=\int\hat{f}^{\mathrm{sp}}_{\ell,i,c}(a_\ell k g_Q\Gamma)\xi(k)\,\mathrm{d}k$$
with
$$\hat{f}^{\mathrm{sp}}_{\ell,i,c}(a_\ell k g_Q\Gamma)=\sum_{w\in\mathbb{Z}^3\cap(\mathcal{L}\cup\mathcal{P})}f(a_\ell k g_Q w)$$
and we used $\int_X\hat{f}_{\ell,i,c}\,\mathrm{d}m_X=\int_{\mathbb{R}^3}f_{\ell,i,c}\,\mathrm{dLeb}$.

We note that Proposition 15.1 indeed gives a more precise information where the domain of integration in the definition of $\mathcal{R}'_{\ell,i,c}$ is restricted to
$$\mathcal{C}=\left\{k\in K:\hat{f}^{\mathrm{sp}}_{\ell,i,c}(a_\ell k g_Q\Gamma)\geq e^{\varrho_1\ell}\right\}.$$
However, the integral over the complement of $\mathcal{C}$ is $O(\mathcal{S}(f_{\ell,i,c})\mathcal{S}(\xi_i)e^{-\varrho_2\ell})$, see (15.5), and it is more convenient for us here to use the above formulation.

Using the definitions of $f_{\ell,i,c}$ and ξ_i, see Lemma 16.2, we have
$$\sum_i\int_K\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k\int_{\mathbb{R}^3}f_{\ell,i,c}\,\mathrm{dLeb}=C'\cdot\varepsilon^2+O(\varepsilon^3),$$
where the implied constant is $O((1+|c|^\star))$, see also [EMM98, Lemma 3.8].

Moreover, arguing as in the proof of Lemma 16.4, but only summing over $v=g_Q w$ for $w\in\mathbb{Z}^3\cap(\mathcal{L}\cup\mathcal{P})$, we conclude that
$$e^\ell\cdot\mathcal{R}'_{\ell,i,c}=(1+O(\varepsilon))\cdot\big(\#(\Upsilon_i(c,e^\ell)\cap(\mathcal{L}\cup\mathcal{P})\big).$$
Summing this over all i and using the fact that $\Omega'_{i,\varepsilon}$ are disjoint,
$$e^\ell\cdot\mathcal{R}'_{\ell,c}:=e^\ell\sum_i\mathcal{R}'_{\ell,i,c}=(1+O(\varepsilon))\big(\#(\Upsilon(c,e^\ell)\cap(\mathcal{L}\cup\mathcal{P})\big) \tag{17.4}$$

We will, as we may, choose $\hat{\kappa}$ in the definition $\varepsilon=e^{-\hat{\kappa}t}$ small enough compared to ϱ_2, then using (17.3) and the above discussion,
$$\mathcal{N}(c,e^\ell)=C\cdot\varepsilon^2\cdot e^\ell+\mathcal{R}'_{\ell,c}\cdot e^\ell+O(\varepsilon^3e^\ell);$$
we also used $t/5\leq\ell\leq t$.

Applying this with $c_j = a + j\varepsilon$ for all $1 \le j \le \lfloor \frac{b-a}{\varepsilon} \rfloor$ and all $t/5 \le \ell \le t$,

$$\#\{v \in \mathbb{Z}^3 : (1-\varepsilon)e^\ell \le \|v\| \le e^\ell, a \le Q(v) \le b\} = \frac{C}{2} \cdot \varepsilon \cdot (b-a) \cdot e^\ell + \mathcal{R}'_\ell \cdot e^\ell + O(\varepsilon^2 e^\ell), \tag{17.5}$$

where the implied constant is $O((1+|a|+|b|)^\star)$ and $\mathcal{R}'_\ell = \sum_j \mathcal{R}'_{\ell,c_j}$.

Apply (17.5) with $\ell = (1-j\varepsilon)t$ for $0 \le j \le \lceil \frac{4}{5\varepsilon} \rceil$. Summing the resulting estimate over $0 \le j \le \lceil \frac{4}{5\varepsilon} \rceil$ and using (17.2) for $\ell \le t/4$, we obtain a bound as stated in Theorem 2.3, but with $e^t \cdot \mathcal{R}'$ instead of $\mathcal{R}_T$ in the theorem, where $\mathcal{R}' = \sum_\ell \mathcal{R}'_\ell$. Note however that in view of (17.4), we have

$$e^t \cdot \mathcal{R}' = (1 + O(\varepsilon)) \cdot (\#\{v \in \mathbb{Z}^3 \cap (\mathcal{L} \cup \mathcal{P}) : \|v\| \le e^t, a \le Q(v) \le b\})$$

This establishes the theorem except for the definition of C_Q, and the estimates (2.1) and (2.2). See the remark following [EMM98, Lemma 3.8] for the definition of C_Q. The proof of (2.1) and (2.2) is the content of the following lemma, which completes the proof. □

17.3. **Lemma.** *Suppose L and P be rational lines and planes, respectively, and let* $\mathrm{Span}\{v\} = L \cap \mathbb{Z}^3$ *and* $\mathrm{Span}\{w, w'\} = P \cap \mathbb{Z}^3$. *Then*

(1) If $|Q(v)| > e^{(-2+4\theta)t}$, then

$$\#\{u \in \mathbb{Z}^3 \cap L : \|u\| \le e^t, a \le Q(u) \le b\} \ll e^{(1-O(\theta))t}$$

(2) If $|Q^*(w \wedge w')| > e^{(-2+4\theta)t}$

$$\#\{u \in \mathbb{Z}^3 \cap P : \|u\| \le e^t, a \le Q(u) \le b\} \ll e^{(1-O(\theta))t}$$

Proof. We will use Lemma 13.5 and an argument based on Lemma 16.4 to prove this. Though, a more hands-on proof is certainly possible. Particularly, for part (1), we have $Q(u) = n^2 Q(v)$ for all $u \in \mathbb{Z}^3 \cap L$, which immediately implies the claim in part (1).

We will prove part (2), proof of part (1) (using the following argument) is similar. First note that applying Lemma 16.3 with $\eta = \theta/5$,

$$\#\{u \in \mathbb{Z}^3 \cap P : \|u\| \le e^{(1-\frac{\theta}{4})t}, a \le Q(u) \le b\} \ll e^{(1+\frac{\theta}{5})(1-\frac{\theta}{4})t} \ll e^{(1-O(\theta))t} \tag{17.6}$$

In view of (17.6), we will consider $(1-\frac{\theta}{2})t \le \ell \le t$. Let us write $\bar{w} = w \wedge w'$ and recall that $|Q^*(\bar{w})| > e^{(-2+4\theta)t}$. Then

$$|Q^*(\bar{w})| > e^{(-2+2\theta)\ell} \quad \text{for all } (1-\tfrac{\theta}{2})t \le \ell \le t.$$

Therefore, by Lemma 13.5 applied with ℓ, $\sigma = 1-\theta$, and $\delta = \theta/10$, we have

$$\int \|(a_\ell k)^* \bar{w}\|^{-1} \, \mathrm{d}k \ll e^{(-2\theta/3)\ell} \ll e^{-\frac{\theta}{2}t}. \tag{17.7}$$

Let $\varepsilon = e^{-\hat{\kappa}t}$ for $\hat{\kappa} < \theta/10$. Arguing as in the above prove to relate the term $\mathcal{R}'$ to the number of points in $\mathcal{L} \cup \mathcal{P}$, we see that

$$\#\{u \in \mathbb{Z}^3 \cap P : (1-\varepsilon)e^\ell \le \|u\| \le e^\ell, a \le Q(u) \le b\}$$

is $\ll \sum_{c,i} \int \hat f^P_{\ell,i,c}(a_t k g_Q \Gamma)\,\mathrm{d}k$, where

$$\hat f^P_{\ell,i,c}(a_\ell k g_Q \Gamma) = \sum_{w\in\mathbb{Z}^3\cap P} f(a_\ell k g_Q u).$$

Moreover, by a variant of Schmidt's Lemma, we have

$$f^P_{\ell,i,c}(a_\ell k g_Q \Gamma) \ll \|(a_\ell k)^* \bar w\|^{-1}.$$

This together with (17.7), thus imply

$$\#\{u\in\mathbb{Z}^3\cap P : (1-\varepsilon)e^\ell \le \|u\| \le e^\ell, a\le Q(u)\le b\} \ll e^{-\frac{\theta}{2}t}.$$

Summing this over all $\ell = t(1-j\varepsilon)$ for $0\le j\le \lfloor \frac{\theta}{2\varepsilon}\rfloor$, and using $\hat\kappa < \theta/10$ and (17.6), the claim in part (2) follows. □

17.4. **Proof of Theorem 2.1.** We now use a simplified version of the above argument to prove the following

17.5. **Theorem.** *Let Q be an indefinite ternary quadratic form with* $\det Q = 1$*. For all R large enough, depending on $\|Q\|$, and all $T\ge R^{A_8}$ at least one of the following holds.*

(1) Let $a<b$, then we have

$$\#\big\{v\in\mathbb{Z}^3 : \|v\|\le T, a\le Q(v)\le b\big\} \ge \mathsf{C}_Q(b-a)T + (1+|a|+|b|)^N T R^{-\kappa_9}.$$

(2) There exists $Q'\in\mathrm{Mat}_3(\mathbb{R})$ with $\|Q'\|\le R$ so that

$$\big\|Q-\lambda Q'\big\| \le R^{A_8}(\log T)^{A_8}T^{-2} \qquad \textit{where } \lambda = (\det Q')^{-1/3}.$$

The constants N, A_8, and κ_9 are absolute, and

$$\mathsf{C}_Q = \int_L \frac{\mathrm{d}\sigma}{\|\nabla Q\|}$$

where $L=\{v\in\mathbb{R}^3 : \|v\|\le 1, Q(v)=0\}$ and $\mathrm{d}\sigma$ is the area element on L.

Proof. As noted above, the proof follows steps similar to those in the proof of Theorem 2.3 in §17.1, but replaces the application of Proposition 15.1 with Proposition 15.3. We will use the notation used in §17.1.

We will, as we may, assume throughout the argument that for all $Q'\in\mathrm{Mat}_3(\mathbb{R})$ with $\|Q'\|\le R$ and all $\lambda\in\mathbb{R}$

$$\big\|Q-\lambda Q'\big\| \le R^{\bar A}(\log T)^{\bar A}T^{-2} \tag{17.8}$$

for some $\bar A$, which will be determined later in the proof. Otherwise, part (2) in the theorem holds and the proof is complete.

Recall again the following basic lattice point count

$$\#\{v\in\mathbb{Z}^3 : \|v\|\le e^{t/4}\} \le C_1' e^{3t/4}, \tag{17.9}$$

where C_1' is absolute.

Let $\varepsilon = e^{-\hat\kappa t}$ for some $\hat\kappa$ which will be optimized later. For all $t/5 \le \ell \le t$ and $a \le c \le b$, we put

$$\Upsilon_i(c, e^\ell) = \{v \in \mathbb{Z}^3 : v \in ((1-\varepsilon)e^\ell, e^\ell] \cdot \Omega'_{i,\varepsilon}, |Q(v) - c| \le \varepsilon\},$$

and $\mathcal{N}_i(c, e^\ell) = \#\Upsilon_i(c, e^\ell)$. Similarly, let

$$\Upsilon(c, e^\ell) = \{v \in \mathbb{Z}^3 : v \in ((1-\varepsilon)e^\ell, e^\ell] \cdot \mathbb{S}^2, |Q(v) - c| \le \varepsilon\},$$

and $\mathcal{N}(c, e^\ell) = \#\Upsilon(c, e^\ell)$.

By Lemma 16.4, there exist $f_{\ell,i,c} \in C_c^\infty(\mathbb{R}^3)$ and $\xi_i \in C_c^\infty(\mathbb{S}^2)$ so that

$$e^{-\ell}\mathcal{N}_i(c, e^\ell) = (1 + O(\varepsilon))\tfrac{2\pi}{\sqrt{2}} \int \hat f_{\ell,i,c}(a_\ell k g_Q \Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k. \tag{17.10}$$

In view of (17.8), and assuming $\bar A$ is large enough, Proposition 15.3 implies

$$\int \hat f_{\ell,i,c}(a_\ell k g_Q \Gamma)\xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k \ge \int_K \xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k \int_{\mathbb{R}^3} f_{\ell,i,c}\,\mathrm{dLeb} \\ + O(\mathcal{S}(f_{\ell,i,c})\mathcal{S}(\xi_i)e^{-\kappa\ell})$$

where we used $\int_X \hat f_{\ell,i,c}\,\mathrm{d}m_X = \int_{\mathbb{R}^3} f_{\ell,i,c}\,\mathrm{dLeb}$.

Using the definitions of $f_{\ell,i,c}$ and ξ_i, see Lemma 16.2, we have

$$\sum_i \int_K \xi_i(k^{-1}\mathsf{e}_3)\,\mathrm{d}k \int_{\mathbb{R}^3} f_{\ell,i,c}\,\mathrm{dLeb} = C' \cdot \varepsilon^2 + O(\varepsilon^3),$$

where the implied constant is $O((1 + |c|^\star))$, see also [EMM98, Lemma 3.8].

We will, as we may, choose $\hat\kappa$ in the definition $\varepsilon = e^{-\hat\kappa t}$ small enough compared to κ, then using (17.3) and the above discussion,

$$\mathcal{N}(c, e^\ell) \ge C \cdot \varepsilon^2 \cdot e^\ell + O(\varepsilon^3 e^\ell);$$

we also used $t/5 \le \ell \le t$.

Applying this with $c_j = a + j\varepsilon$ for all $1 \le j \le \lfloor \frac{b-a}{\varepsilon} \rfloor$ and all $t/5 \le \ell \le t$,

$$\#\{v \in \mathbb{Z}^3 : (1-\varepsilon)e^\ell \le \|v\| \le e^\ell, a \le Q(v) \le b\} \ge \\ \tfrac{C}{2} \cdot \varepsilon \cdot (b-a) \cdot e^\ell + O(\varepsilon^2 e^\ell), \tag{17.11}$$

where the implied constant is $O((1 + |a| + |b|)^\star)$.

Apply (17.11) with $\ell = (1 - j\varepsilon)t$ for $0 \le j \le \lceil \frac{4}{5\varepsilon} \rceil$. Summing the resulting estimate over $0 \le j \le \lceil \frac{4}{5\varepsilon} \rceil$ and using (17.9) for $\ell \le t/4$, we obtain a bound as stated in part (1) of the theorem. For the definition of C_Q, see the remark following [EMM98, Lemma 3.8]. □

Proof of Theorem 2.1. This is a direct consequence of of Theorem 17.5. Indeed, let $\kappa_3 = \kappa_9/2N$ and let $A_4 = \kappa_9$. Now Theorem 17.5 applied with $[a, b] = [c - R^{-\kappa_3}, c + R^{-\kappa_3}]$ for any $|c| \le R^{\kappa_3}$ implies Theorem 2.1. □

Proof of Corollary 2.2. This is a direct consequence of Theorem 2.1. For more details, see [LM14, §12.3]. □

References

[And75] Peter P. Andre. k-regular elements in semisimple algebraic groups. *Trans. Amer. Math. Soc.*, 201:105–124, 1975. 8

[BFLM11] Jean Bourgain, Alex Furman, Elon Lindenstrauss, and Shahar Mozes. Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus. *J. Amer. Math. Soc.*, 24(1):231–280, 2011. 104

[DM90] S. G. Dani and G. A. Margulis. Values of quadratic forms at integral points: an elementary approach. *Enseign. Math. (2)*, 36(1-2):143–174, 1990. 6

[DM91] S. G. Dani and G. A. Margulis. Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 101(1):1–17, 1991. 4, 6, 71

[EM04] Alex Eskin and Gregory Margulis. Recurrence properties of random walks on finite volume homogeneous manifolds. In *Random walks and geometry*, pages 431–444. Walter de Gruyter, Berlin, 2004. 88

[EMM98] Alex Eskin, Gregory Margulis, and Shahar Mozes. Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture. *Ann. of Math. (2)*, 147(1):93–141, 1998. 3, 6, 7, 53, 56, 57, 62, 64, 65, 66, 71, 74, 76, 77, 79, 88

[EMM05] Alex Eskin, Gregory Margulis, and Shahar Mozes. Quadratic forms of signature $(2,2)$ and eigenvalue spacings on rectangular 2-tori. *Ann. of Math. (2)*, 161(2):679–725, 2005. 3, 6, 53, 56

[EMS96] Alex Eskin, Shahar Mozes, and Nimish Shah. Unipotent flows and counting lattice points on homogeneous varieties. *Ann. of Math. (2)*, 143(2):253–299, 1996. 49

[FKR24] Giovanni Forni, Adam Kanigowski, and Maksym Radziwiłł. Horocycle flow at product of two primes, 2024, 2409.16687. 5

[GGW22] Shengwen Gan, Shaoming Guo, and Hong Wang. A restricted projection problem for fractal sets in $\mathbb{R}^n$, 2022, 2211.09508. 4, 7, 16, 102, 103

[Kim24] Wooyeon Kim. Moments of Margulis functions and indefinite ternary quadratic forms, 2024, 2403.16563. 3, 6, 53

[KKL23] Dubi Kelmer, Alex Kontorovich, and Christopher Lutsko. Mean square of Eisenstein series, 2023, 2305.15162. 74

[KM96] D. Y. Kleinbock and G. A. Margulis. Bounded orbits of nonquasiunipotent flows on homogeneous spaces. In *Sinai's Moscow Seminar on Dynamical Systems*, volume 171 of *Amer. Math. Soc. Transl. Ser. 2*, pages 141–172. Amer. Math. Soc., Providence, RI, 1996. 12

[KM98] D. Y. Kleinbock and G. A. Margulis. Flows on homogeneous spaces and Diophantine approximation on manifolds. *Ann. of Math. (2)*, 148(1):339–360, 1998. 88

[LM14] Elon Lindenstrauss and Gregory Margulis. Effective estimates on indefinite ternary forms. *Israel J. Math.*, 203(1):445–499, 2014. 2, 6, 79

[LM23] E. Lindenstrauss and A. Mohammadi. Polynomial effective density in quotients of $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{H}^2$. *Invent. Math.*, 231(3):1141–1237, 2023. 4, 9, 10, 21, 26, 88, 95, 97

[LMM$^+$24] Elon Lindenstrauss, Gregory Margulis, Amir Mohammadi, Nimish Shah, and Andreas Wieser. An effective closing lemma for unipotent flows, 2024. 99

[LMMS19] Elon Lindenstrauss, Amir Mohammadi, Gregory Margulis, and Nimish Shah. Quantitative behavior of unipotent flows and an effective avoidance principle, 2019, arXiv:1904.00290. 4, 11, 45, 47, 48, 82, 83, 84, 85, 86, 95

[LMW22] Elon Lindenstrauss, Amir Mohammadi, and Zhiren Wang. Effective equidistribution for some one parameter unipotent flows. *arxiv.org/abs/2211.11099*,

2022. 3, 4, 7, 10, 21, 22, 24, 32, 33, 34, 45, 46, 47, 53, 62, 87, 88, 97, 104, 105, 107

[LMW23] Elon Lindenstrauss, Amir Mohammadi, and Zhiren Wang. Quantitative equidistribution and the local statistics of the spectrum of a flat torus. *J. Anal. Math.*, 151(1):181–234, 2023. 52, 53, 54, 74

[Mar89] G. A Margulis. Indefinite quadratic forms and unipotent flows on homogeneous spaces. *Dynamical systems and ergodic theory (Warsaw, 1986)*, 23:399–409, 1989. 6

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. 3, 7

[Rat90] Marina Ratner. On measure rigidity of unipotent subgroups of semisimple groups. *Acta Math.*, 165(3-4):229–309, 1990. 2

[Rat91a] Marina Ratner. On Raghunathan's measure conjecture. *Ann. of Math. (2)*, 134(3):545–607, 1991. 2

[Rat91b] Marina Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.*, 63(1):235–280, 1991. 2

[Sha91] Nimish A. Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.*, 289(2):315–334, 1991. 4

[SS22] Anthony Sanchez and Juno Seong. An avoidance principle and Margulis functions for expanding translates of unipotent orbits, 2022, 2206.12019. 88, 95

[Yan22] Lei Yang. Effective version of Ratner's equidistribution theorem for $\mathrm{SL}(3,\mathbb{R})$, 2022. arXiv:2208.02525. 4

Appendices

Appendix A. Unipotent trajectories and parabolic subgroups

In this section $\tilde{\mathbf{G}} \subset \mathrm{SL}_N$ denotes a semisimple $\mathbb{Q}$-subgroup. We let $\tilde{G} = \tilde{\mathbf{G}}(\mathbb{R})$ and $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{G})$. Also let $\tilde{\mathfrak{g}}_{\mathbb{Z}} := \tilde{\mathfrak{g}} \cap \mathfrak{sl}_N(\mathbb{Z})$, then $\tilde{\mathfrak{g}}$ has a natural $\mathbb{Q}$-structure, and $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ is a $\tilde{G} \cap \mathrm{SL}_N(\mathbb{Z})$-stable lattice in $\tilde{\mathfrak{g}}$. If there is no confusion, we will simply write gv for $\mathrm{Ad}(g)v$; similarly for the natural actions on $\wedge^\ell \tilde{\mathfrak{g}}$.

Fix a Euclidean norm $\|\cdot\|$ on $\mathrm{Mat}_N(\mathbb{R})$. This induces a norm on $\mathfrak{sl}_N(\mathbb{R})$ and on $\mathrm{SL}_N(\mathbb{R})$. We will also write $\|\cdot\|$ for the induced norms on exterior products of $\mathfrak{sl}_N(\mathbb{R})$. For $g \in \mathrm{SL}_N(\mathbb{R})$ we let $|g| = \max\{\|g\|, \|g^{-1}\|\}$.

Let $\tilde{\Gamma} \subset \tilde{G} \cap \mathrm{SL}_N(\mathbb{Z})$ be an arithmetic lattice in $\tilde{G}$; let $\tilde{X} = \tilde{G}/\tilde{\Gamma}$, and

$$\tilde{X}_\eta = \{g\tilde{\Gamma} \in \tilde{X} : \min_{0 \neq v \in \tilde{\mathfrak{g}}(\mathbb{Z})} \|gv\| \geq \eta\} \quad \text{for all } \eta > 0.$$

These are compact subsets of $\tilde{X}$, and any compact subset of $\tilde{X}$ is contained in $\tilde{X}_\eta$ for some $\eta > 0$.

If $\mathbf{L} \subset \tilde{\mathbf{G}}$ is a connected $\mathbb{Q}$-subgroup, we let $\mathbf{v}_L$ be a primitive integral vector on the line $\wedge^{\dim L}\mathrm{Lie}(L) \subset \wedge^{\dim L}\tilde{\mathfrak{g}}$ where $L = \mathbf{L}(\mathbb{R})$. Recall from [LMMS19] the definition of the height of $\mathbf{L}$

$$\mathrm{ht}(\mathbf{L}) = \|\mathbf{v}_L\|. \tag{A.1}$$

We also recall the following setting from [LMMS19]. *This notation will only be used in this section.* Let $U \subset \tilde{G}$ be a unipotent subgroup and let $\mathfrak{u} = \mathrm{Lie}(U)$. We fix a basis $\mathcal{B}_U$ of $\mathfrak{u}$ consisting of unit vectors and set $B_{\mathfrak{u}}(0,\delta) = \{\sum_{z \in \mathcal{B}_U} a_z z : |a_z| \leq \delta\}$ for $\delta > 0$ as well as $B_U(e) = \exp(B_{\mathfrak{u}}(0,1))$.

Let $\lambda : \mathfrak{u} \to \mathfrak{u}$ be an $\mathbb{R}$-diagonalizable expanding linear map (all eigenvalues have absolute value > 1). For any $k \in \mathbb{Z}$ and any $u = \exp(z) \in U$, we set $\lambda_k(u) = \exp(\lambda^k(z))$. We note that $\lambda_k \circ \lambda_\ell = \lambda_{k+\ell}$. We shall assume that there exists $k_0 \in \mathbb{N}$ such that for every integer $k > k_0$,

$$\exp\left(\lambda_{k-k_0}(B_{\mathfrak{u}}(0,1))\right)\exp\left(\lambda_{k-1}(B_{\mathfrak{u}}(0,1))\right) \subset \exp\left(\lambda_k(B_{\mathfrak{u}}(0,1))\right). \tag{A.2}$$

Since the exponential map $\exp : \mathfrak{u} \to U$ pushes the Lebesgue measure on $\mathfrak{u}$ to a Haar measure, denoted by $|\cdot|$, on U, for any measurable $B \subset U$

$$|\lambda_k(B)| = |\det(\lambda)|^k |B| \qquad \text{for all } k \in \mathbb{Z}. \tag{A.3}$$

To avoid cumbersome statements, we suppose throughout that any constant that is allowed to depend on $\mathrm{ht}(\tilde{\mathbf{G}})$ is also (implicitly) allowed to depend on $\|\lambda\|$, $\|\lambda^{-1}\|$, $\frac{|\lambda_1(B_U(e))|}{|B_U(e)|} = |\det(\lambda)|$, and k_0.

A.1. **Theorem.** *There exist A_9 depending on N and A_{10} depending on N and polynomially on $\mathrm{ht}(\tilde{\mathbf{G}})$ so that for any $g \in \tilde{G}$, $k \geq 1$, and any $0 < \varepsilon \leq 1/2$ at least one of the following holds.*

(1)

$$|\{u \in B_U(e) : \lambda_k(u)g\tilde{\Gamma} \notin \tilde{X}_\varepsilon\}| \leq A_{10}\varepsilon^{1/A_9}|B_U(e)|.$$

(2) There is a $\mathbb{Q}$-parabolic subgroup $\mathbf{P} \subset \tilde{\mathbf{G}}$ with $\mathrm{ht}(\mathbf{P}) \le A_{10}|g|^{A_9}\varepsilon^{1/A_9}$ so that all the following are satisfied

$$\|\lambda_k(u)g\mathbf{v}_P\| \le A_{10}\varepsilon^{1/A_9} \qquad \text{for all } u \in B_U(e), \tag{A.4a}$$

$$\|\lambda_k(u)g\mathbf{v}_{R_u(P)}\| \le A_{10}\varepsilon^{1/A_9} \qquad \text{for all } u \in B_U(e), \tag{A.4b}$$

$$\|\lambda_k(u)g\mathbf{v}_Q\| \le A_{10}\varepsilon^{1/A_9} \qquad \text{for all } u \in B_U(e), \tag{A.4c}$$

where $Q = \mathbf{Q}(\mathbb{R})$ for $\mathbf{Q} = [\mathbf{P}, \mathbf{P}]$.

This result strengthens [LMMS19, Thm. 6.3]. While the original theorem only established the existence of a unipotent subgroup, we now demonstrate that this subgroup may, in fact, be taken to be the unipotent radical of a $\mathbb{Q}$-parabolic subgroup. Indeed we will prove Theorem A.1, using [LMMS19, Thm. 5.3], which was also used in the proof of [LMMS19, Thm. 6.3], and the following theorem, which is of independent interest.

For every subspace $V \subset \tilde{\mathfrak{g}}$, let $V^\perp$ denote the orthogonal complement of V with respect to the Killing form $\mathsf{k}_{\tilde{\mathfrak{g}}}$ on $\tilde{\mathfrak{g}}$.

A.2. **Theorem.** *There are positive constants δ, C, depending on $\tilde{\mathbf{G}}$, such that if $g \in \tilde{G}$ is such that there exists a vector $v \in \tilde{\mathfrak{g}}_{\mathbb{Z}}$ with $\|gv\| \le \delta$, then*
(1) $\{w \in \tilde{\mathfrak{g}}_{\mathbb{Z}} : \|gw\| \le \delta^{1/C}\}$ generates a nilpotent subalgebra, $\mathfrak{v}$ say.
(2) $\mathfrak{v}^\perp \cap \mathrm{Ad}(g)\tilde{\mathfrak{g}}_Z$ is spanned by vectors of size $\le C\delta^{-1/C}$.
(3) $\mathfrak{v}^\perp$ generates a proper parabolic subalgebra of $\tilde{\mathfrak{g}}$.

Let us first assume Theorem A.2 and complete the proof of Theorem A.1.

Proof of Theorem A.1. Let E and D be constants as in [LMMS19, Thm. 5.3]. In particular, D depends only on N, and E on N and $\mathrm{ht}(\tilde{\mathbf{G}})$. We will show that Theorem A.1 holds with appropriately chosen $A_{10} \ge E$ and $A_9 \ge D$. Assume

$$|\{u \in B_U(e) : \lambda_k(u)g\tilde{\Gamma} \notin \tilde{X}_\varepsilon\}| > E\varepsilon^{1/D}|B_U(e)|.$$

Then [LMMS19, Thm. 5.2] implies that there is a rational subspace $\mathbf{W} \subset \tilde{\mathfrak{g}}$, say of dimension ℓ, so that

$$\sup_{u \in B_U(e)} \|\lambda_k(u)g\mathbf{v}_W\| = \hat{\varepsilon} \ll \varepsilon^\star, \tag{A.5}$$

where $\mathbf{v}_W$ denotes the primitive vector corresponding to $\mathbf{W}$.

For later use, let us record that Minkowski's theorem and the above imply that for every $u \in B_U(e)$, there exists a nilpotent vector $z_u \in \tilde{\mathfrak{g}}_{\mathbb{Z}}$ with

$$\|\lambda_k(u)gz_u\| \le \hat{\varepsilon}^{1/\dim\tilde{\mathfrak{g}}}. \tag{A.6}$$

Let the partial flag $\Delta_{\ell_1} \subset \cdots \subset \Delta_{\ell_d}$ and the function η be given as in [LMMS19, Thm. 5.3]. In particular, $\mathrm{rk}(\Delta_{\ell_i}) = \ell_i$ and $\eta(\ell_0), \ldots, \eta(\ell_{d+1}) \in (0, 1]$, with $\ell_0 = 0$ and $\ell_{d+1} = \dim\tilde{\mathfrak{g}}$, is defined by

$$\eta(\ell_0) = \eta(\ell_{d+1}) = 1$$
$$\eta(\ell_i) = \max_{u \in \mathsf{B}_U(e)} \|\lambda_k(u)\Delta_{\ell_i}\| \qquad \text{for } 1 \le i \le d;$$

as shown in [LMMS19], the function $\eta(\bullet)$ extends to a function $\eta : [0, \dim \tilde{\mathfrak{g}}] \to (0, 1]$ so that $-\log \eta : [0, \dim \tilde{\mathfrak{g}}] \to \mathbb{R}^+$ is concave and linear on each interval $[\ell_0, \ell_1], \ldots, [\ell_d, \ell_{d+1}]$.

For $0 < \tau < 1$, put

$$\mathrm{Exc}_\tau = \{u \in B_U(e) : \alpha_i(\lambda_k(u)g)^{-1} < \tau^i \eta(i)\},$$

where α_i's are also as in loc. cit. Then by [LMMS19, Thm. 5.3], we have $|\mathrm{Exc}_\tau| \le E\tau^{1/D}|B_U(e)|$, where D depends only on $\dim \tilde{\mathfrak{g}}$.

Moreover, by the last sentence in [LMMS19, Thm. 5.3], we can assume that $\Delta_{\ell_1} \subset \cdots \subset \Delta_{\ell_d}$ is so that

$$\eta(\mathrm{rk}(\Delta)) \le \max_{u \in \mathsf{B}_U(e)} \|(\lambda_k(u)\Delta)\|,$$

where $\Delta = \mathbf{W} \cap \tilde{\mathfrak{g}}_{\mathbb{Z}}$.

We now argue as in the proof of [LMMS19, Lemma 6.2]. Since $-\log \eta$ is concave, we have $\eta(1) \le \hat{\varepsilon}^{1/\ell}$. Let $\kappa = (4\ell \dim \tilde{\mathfrak{g}})^{-1}$, and let $\hat{\varepsilon}^{1/\ell} < \rho < 1$ be a parameter which will be optimized later. Let $1 \le s_0 < d$ be the largest integer so that $\frac{\eta(s_0)}{\eta(s_0-1)} \le \rho$. The choice of κ and $\eta(\dim \tilde{\mathfrak{g}}) = 1$ imply that there is some $m \ge 0$ so that $\frac{\eta(s_0+d)}{\eta(s_0+d-1)} \le \rho^{1-d\kappa}$ for all $0 \le d \le m$ and

$$\frac{\eta(s_0+m+1)}{\eta(s_0+m)} \ge \rho^{1-(m+1)\kappa}. \tag{A.7}$$

Put $s = s_0 + m$, and note that $s = \ell_j$ for some j. Let $\mathbf{V}$ be the subspace spanned by Δ_s, then

$$\|\lambda_k(u)g\mathbf{v}_V\| \le \eta(s) \le \rho^{1-m\kappa} \qquad \text{for all } u \in B_U(e)$$

Moreover, $\mathbf{V}$ is a nilpotent subalgebra of $\tilde{\mathfrak{g}}$, and we can choose a basis $\{v_1, \ldots, v_s\}$ for $\mathbf{V} \cap \tilde{\mathfrak{g}}_{\mathbb{Z}}$ so that

$$\|\lambda_k(u)gv_i\| \le \hat{A}\frac{\eta(i)}{\eta(i-1)} \le \hat{A}\rho^{1-m\kappa}, \tag{A.8}$$

where $\hat{A}$ depends only on $\dim \tilde{\mathfrak{g}}$, see the proof of [LMMS19, Lemma 6.2] for all these statements.

Put $\tau = \rho^{\kappa/(2\dim \tilde{\mathfrak{g}})}$. Then for all $u \in B_U(e) \setminus \mathrm{Exc}_\tau$ and any $w \in \tilde{\mathfrak{g}}_{\mathbb{Z}} \setminus \mathbf{V}$,

$$\begin{aligned}\tau^{s+1}\eta(s+1) &\le \|\lambda_k(u)g(\Delta_s + \mathbb{Z}w)\| \\ &\le \|\lambda_k(u)gw\|\|\lambda_k(u)g\Delta_s\| \le \|\lambda_k(u)gw\|\eta(s).\end{aligned}$$

This and (A.7) imply that

$$\begin{aligned}\|\lambda_k(u)gw\| &\ge \tau^{s+1}\rho^{1-(m+1)\kappa} \\ &> \rho^{1-(m+\frac{1}{2})\kappa} > \hat{A}\rho^{1-m\kappa}.\end{aligned} \tag{A.9}$$

In view of (A.8) and (A.9), for every $u \in B_U(e) \setminus \mathrm{Exc}_\tau$, the space

$$\mathrm{Ad}(\lambda_k(u)g)(\tilde{\mathfrak{g}}_{\mathbb{Z}} \cap \mathbf{V})$$

is spanned by $\{w \in \tilde{\mathfrak{g}}_{\mathbb{Z}} : \|\lambda_k(u)gw\| \le \hat{A}\rho^\theta\}$.

Apply Theorem A.2 with $\hat{A}\rho^{1-m\kappa}$ and $\lambda_k(u)g\mathbf{V}$. Then part (2) of that theorem implies that $\big(\mathrm{Ad}(\lambda_k(u)g)(\tilde{\mathfrak{g}}_{\mathbb{Z}}\cap\mathbf{V})\big)^{\perp}$ generates a proper parabolic subalgebra, $\mathrm{Ad}(\lambda_k(u)g)\mathrm{Lie}(P)$ say. Moreover, if we choose ρ to be a small enough power of ε, then (A.6) and part (2) in Theorem A.2 imply that

$$\|\lambda_k(u)g\mathbf{v}_P\| \le \rho^{-\star}\hat{\varepsilon}^{1/\dim\tilde{\mathfrak{g}}} \ll \varepsilon^{\star} \qquad \text{for all } u\in B_U(e)\setminus \mathrm{Exc}_\tau.$$

Since $u\mapsto \lambda_k(u)g\mathbf{v}_P$ is a polynomial map and $|\mathrm{Exc}_\tau| \ll \tau^{1/D}|B_U(e)|$, we conclude that

$$\|\lambda_k(u)g\mathbf{v}_P\| \ll \varepsilon^{\star} \qquad \text{for all } u\in B_U(e).$$

This finishes the proof of (A.4a).

To establish the claims in (A.4b) and (A.4c), we use (A.4a) and follow the argument in the proof of [LMMS19, Lemma 4.2]. Specifically, we show that for all $u\in B_U(e)\setminus\mathrm{Exc}_\tau$, $\mathrm{Ad}(\lambda_k(u)g)\mathrm{Lie}(Q)$ and $\mathrm{Ad}(\lambda_k(u)g)\mathrm{Lie}(R_u(P))$ are spanned by vectors of size $\ll \rho^{-\star}$. Combined with (A.6), this completes the proof, provided that ρ is chosen to be a sufficiently small power of ε. $\square$

A.3. **Proof of Theorem A.2.** We continue to use the above notation. The proof of Theorem A.2, relies on the following two lemmas.

A.4. **Lemma.** *Let $\mathfrak{v}\subset\tilde{\mathfrak{g}}$ be a nilpotent Lie subalgebra. Then there exists a parabolic subgroup P such that*

$$\mathfrak{v}\subset \mathrm{Lie}(R_u(P))\subset \mathrm{Lie}(P)\subset \mathfrak{v}^{\perp}.$$

Proof. Let $\mathfrak{v}_0=\mathfrak{v}$, and for $i\ge 1$, define $\mathfrak{v}_i$ to be the niladical of $N_{\tilde{\mathfrak{g}}}(\mathfrak{v}_{i-1})$. Then there exists some m so that $\mathfrak{v}_m=\mathrm{Lie}(R_u(P))$ for a parabolic subgroup P of $\tilde{G}$. Since $\mathfrak{v}_i\subset\mathfrak{v}_{i+1}$, we have

$$\mathfrak{v}\subset\mathrm{Lie}(R_u(P)).$$

This and the definition of the Killing form, $\mathsf{k}_{\tilde{\mathfrak{g}}}(v,w)=\mathrm{tr}(\mathrm{ad}v\circ\mathrm{ad}w)$, now imply that $\mathrm{Lie}(P)\subset\mathfrak{v}^{\perp}$. The proof is complete. $\square$

A.5. **Lemma.** *There is a positive constant C', depending on $\dim\tilde{\mathfrak{g}}$, so that for every $A\ge 2$ and $\delta\le A^{-C'}$ the following holds. If there is a vector $v\in\tilde{\mathfrak{g}}_{\mathbb{Z}}$ with $\|gv\|\le\delta$, then $\{w\in\tilde{\mathfrak{g}}_{\mathbb{Z}}:\|gw\|\le A\}$ generates a proper Lie subalgebra of $\tilde{\mathfrak{g}}$.*

Proof. Let $\mathfrak{l}$ denote the Lie algebra spanned by $\{w\in\tilde{\mathfrak{g}}_{\mathbb{Z}}:\|gw\|\le A\}$, and let $g\mathbf{L}g^{-1}$ be the corresponding subgroup. Then $\mathfrak{l}$ is spanned by vectors of size $\ll A^{\star}$, where the implied constants depend only on $\dim\tilde{\mathfrak{g}}$. Moreover $gv\in\mathfrak{l}$. Thus if δ is small enough, we have

$$\|g\mathbf{v}_L\| \ll A^{\star}\delta\le\delta^{1/2}, \tag{A.10}$$

which in particular implies that $\mathfrak{l}\ne\tilde{\mathfrak{g}}$. $\square$

Proof of Theorem A.2. Let $\delta_1=\delta^{\star}$ be a small power of δ which will be explicated later. Let us also put $\Lambda:=\mathrm{Ad}(g)\tilde{\mathfrak{g}}_{\mathbb{Z}}$. Then so long as δ is small enough, $\{w \in \Lambda : \|w\| \le \delta_1\}$ generate over $\mathbb{Z}$ a lattice in a nilpotent Lie subalgebra $\mathfrak{v} \subset \tilde{\mathfrak{g}}$. This establishes part (1) in the theorem.

We now prove part (2). That is: $\mathfrak{v}^\perp \cap \Lambda$ is spanned by elements of size $\ll \delta_1^{-1}$. Consider the vectors $v_1, v_2, \ldots, v_n$ that attain the ith successive minima of Λ, respectively. Let $m = \dim \mathfrak{v}$. Then $v_1, \ldots, v_m \in \mathfrak{v}$, and furthermore, $\|v_i\| > \delta_1$ holds for all $i > m$.

Let $v_1^*, \ldots, v_n^*$ be the dual basis with respect to the Killing form $\mathsf{k}_{\tilde{\mathfrak{g}}}$. Then $\langle v_1^*, \ldots, v_n^* \rangle \subset \Lambda/M$ where $M \ll 1$. Since $\mathsf{k}_{\tilde{\mathfrak{g}}}(v_i, v_j^*) = \delta_{ij}$, we deduce that $\|v_i^*\| \gg \|v_i\|^{-1}$. Moreover, $\prod \|v_i\| \asymp \prod \|v_i^*\|^{-1}$, therefore, $\|v_i^*\| \asymp \|v_i\|^{-1}$.

The orthogonal complement $\mathfrak{v}^\perp$ is spanned by the vectors $v_{m+1}^*, \ldots, v_n^*$, and therefore, increasing M if necessary, it is spanned by $\{w \in \Lambda : \|w\| \le M\|v_{m+1}\|^{-1}\}$. In other words, it is spanned by vectors in Λ of size $\ll \delta_1^{-1}$.

We now prove part (3) in the theorem. By applying Lemma A.5, see in particular (A.10), if δ_1 is a small enough power of δ, the Lie algebra generated by $\mathfrak{v}^\perp$ cannot be the entire algebra. Furthermore, by Lemma A.4, we know it must contain a parabolic subalgebra which contains $\mathfrak{v}$. Therefore, $\langle \mathfrak{v}^\perp \rangle$ is indeed a nontrivial parabolic subalgebra containing $\mathfrak{v}$. $\square$

A.6. **An S-arithmetic version of Theorem A.1.** We conclude this appendix with noting that Theorem A.1 can easily be adapted to the S-arithmetic setting as in [LMMS19], the only difference being that now A_{10} can depend also on $\#S$ and polynomially on the primes in S. For the record we state this explicitly; the straightforward modification of the proofs above to this more general context is left to the reader.

In addition to the semisimple $\mathbb{Q}$-subgroup. $\tilde{\mathbf{G}} \subset \mathrm{SL}_N$, we now also choose a finite set of places S of $\mathbb{Q}$ containing the infinite place. Let $\tilde{G} = \tilde{\mathbf{G}}(\mathbb{Q}_S)$ and $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{G})$. Also let $\tilde{\mathfrak{g}}_{\mathbb{Z}_S} := \tilde{\mathfrak{g}} \cap \mathfrak{sl}_N(\mathbb{Z}_S)$. Then $\tilde{\mathfrak{g}}$ has a natural $\mathbb{Q}$-structure, and $\tilde{\mathfrak{g}}_{\mathbb{Z}_S}$ is a $\tilde{G} \cap \mathrm{SL}_N(\mathbb{Z}_S)$-stable lattice in $\tilde{\mathfrak{g}}$. If there is no confusion, we will simply write gv for $\mathrm{Ad}(g)v$; similarly for the natural actions on $\wedge^\ell \tilde{\mathfrak{g}}$.

Let $\|\cdot\|_\infty$ denote the Euclidean norm on $\mathrm{Mat}_N(\mathbb{R})$, and for every finite place $v \in S$, let $\|\cdot\|_v$ be a maximum norm with respect to the standard basis for $\mathrm{Mat}_N(\mathbb{Q}_v)$ induced by the standard absolute value on $\mathbb{Q}_v$. Let $\|\cdot\|_S$ or simply $\|\cdot\|$ denote $\max_{v\in S}\|\cdot\|_v$ This induces a norm on $\mathfrak{sl}_N(\mathbb{Q}_S)$ and on $\mathrm{SL}_N(\mathbb{Q}_S)$. We will also write $\|\cdot\|$ for the induced norms on exterior products of $\mathfrak{sl}_N(\mathbb{R})$. For $g \in \mathrm{SL}_N(\mathbb{Q}_S)$ we let $|g| = \max\{\|g\|, \|g^{-1}\|\}$.

Let $\tilde{\Gamma} \subset \tilde{G} \cap \mathrm{SL}_N(\mathbb{Z}_S)$ be an arithmetic lattice in $\tilde{G}$; let $\tilde{X} = \tilde{G}/\tilde{\Gamma}$, and

$$\tilde{X}_\eta = \{g\tilde{\Gamma} \in \tilde{X} : \min_{0 \ne v \in \tilde{\mathfrak{g}}(\mathbb{Z})} \|gv\| \ge \eta\} \quad \text{for all } \eta > 0.$$

These are compact subsets of $\tilde{X}$, and any compact subset of $\tilde{X}$ is contained in $\tilde{X}_\eta$ for some $\eta > 0$.

As it was done in [LMMS19], let $U = \prod_{v\in S} U_v \subset \tilde{G}$ be a unipotent subgroup and let $\mathfrak{u} = \mathrm{Lie}(U)$. We fix a basis $\mathcal{B}_U$ of $\mathfrak{u}$ consisting of unit vectors and set $B_{\mathfrak{u}}(0, \delta) = \{\sum_{z\in\mathcal{B}_U} a_z z : |a_z|_S \le \delta\}$ for $\delta > 0$ as well as $B_U(e) = \exp(B_{\mathfrak{u}}(0,1))$.

Let $\lambda : \mathfrak{u} \to \mathfrak{u}$ be a $\mathbb{Q}_S$-diagonalizable expanding linear map satisfying (A.2) and (A.3).

The proof of Theorem A.1 generalizes *mutatis mutandis* to yield the following result.

A.7. **Theorem.** *There exist A_{11} depending on N, and A_{12} depending on N, $\#S$ and polynomialy on $\mathrm{ht}(\tilde{\mathbf{G}})$ and the primes in S so that for any $g \in \tilde{G}$, $k \geq 1$, and any $0 < \varepsilon \leq 1/2$ at least one of the following holds.*

(1)
$$|\{u \in B_U(e) : \lambda_k(u)g\tilde{\Gamma} \notin \tilde{X}_\varepsilon\}| \leq A_{12}\varepsilon^{1/A_{11}}.$$

(2) There is a $\mathbb{Q}$-parabolic subgroup $\mathbf{P} \subset \tilde{\mathbf{G}}$ with $\mathrm{ht}(\mathbf{P}) \leq A_{12}|g|^{A_{11}}\varepsilon^{1/A_{11}}$ so that all the following are satisfied

$$\|\lambda_k(u)g\mathbf{v}_P\| \leq A_{12}\varepsilon^{1/A_{11}} \qquad \text{for all } u \in B_U(e), \tag{A.11a}$$

$$\|\lambda_k(u)g\mathbf{v}_{R_u(P)}\| \leq A_{12}\varepsilon^{1/A_{11}} \qquad \text{for all } u \in B_U(e), \tag{A.11b}$$

$$\|\lambda_k(u)g\mathbf{v}_Q\| \leq A_{12}\varepsilon^{1/A_{11}} \qquad \text{for all } u \in B_U(e), \tag{A.11c}$$

where $Q = \mathbf{Q}(\mathbb{Q}_S)$ for $\mathbf{Q} = [\mathbf{P}, \mathbf{P}]$.

Appendix B. Avoidance principles: The proofs

This section contains proofs of results in §4. The proofs are by now standard, and we include them primarily for the convenience of the reader.

B.1. **Proof of Proposition 4.4.** In this section, we prove Proposition 4.4. The proof, which is essentially that of [LMW22, Prop. 4.6] mutatis mutandis, is based on the study of a certain Margulis function, see (B.5). We recall the details to explicate the necessary changes.

For every $d > 0$, define the probability measure σ_d on H by

$$\int \varphi(h)\,\mathrm{d}\sigma_d(h) = \frac{1}{3}\int_{-1}^{2} \varphi(a_d u_r)\,\mathrm{d}r.$$

Let us first remark on our choice of the interval $[-1, 2]$: We will define a function f_Y in (B.5) below. In Lemmas 13.5–B.5, certain estimates for

$$\int f_Y(h\,\bullet)\,\mathrm{d}(\sigma_{d_1} * \cdots * \sigma_{d_n})(h)$$

will be obtained, then in Lemma B.6, we will convert these estimates to similar estimates for

$$\int_0^1 f_Y(a_{d_1+\cdots+d_n}u_r\,\bullet)\,\mathrm{d}r.$$

The argument in Lemma B.6 is based on commutation relations between a_d and u_r; similar arguments have been used several times throughout the paper. Since the function f_Y can have a rather large Lipschitz constant, we will not appeal to continuity properties of f_Y in Lemma B.6. Instead, we will use the fact that $[0, 1] \subset [-1, 2] + r$ for any $|r| \leq 1/2$.

Recall that $\dim \mathfrak{r} = 2\mathsf{m}+1$, and that $\mathfrak{r}$ is $\mathrm{Ad}(H)$-irreducible. Moreover, recall that for every highest weight vector $w \in \mathfrak{r}$, we have

$$\mathrm{Ad}(a_t)w = e^{\mathsf{m}t}w. \tag{B.1}$$

We begin with the following linear algebra lemma.

B.2. **Lemma** (cf. Lemma A.1, [LMW22]). *Let $0 < \alpha \leq 1/(2\mathsf{m}+1)$. For all $0 \neq w \in \mathfrak{r}$, we have*

$$\int \|\mathrm{Ad}(h)w\|^{-\alpha}\,\mathrm{d}\sigma_d(h) \leq Ce^{-\alpha\mathsf{m}d}\|w\|^{-\alpha}$$

where C is an absolute constant.

Sketch of the proof. Standard representation theory of SL_2 implies that for every $\|w\| = 1$, we have

$$\mathrm{Ad}(a_t u_r)w = (e^{\mathsf{m}t}p_w(r), \ldots)$$

where $p_w(r) = \sum c_i r^i$ is a polynomial of degree $2\mathsf{m}$ with $\max\{|c_i|\} \gg 1$. Thus if we put

$$I(\varepsilon) = \{r \in [-1, 2] : \varepsilon/2 \leq p_w(r) \leq \varepsilon\},$$

then $|I(\varepsilon)| \ll \varepsilon^{1/(2\mathsf{m})}$ where the implied constant is absolute, see e.g. [KM98, Prop. 3.2].

Moreover, $\|\mathrm{Ad}(a_t u_r)w\| \geq e^{\mathsf{m}t}\varepsilon/2$, for every $r \in I(\varepsilon)$. The lemma follows if we dyadically decompose the domain and summing the resulting geometric series, which converges independently of α since $\alpha \leq \frac{1}{2\mathsf{m}+1} < \frac{1}{2\mathsf{m}}$. □

We will also use the following non-divergence result á la Eskin, Margulis, and Mozes [EMM98].

B.3. **Proposition** (cf. [EM04]). *There exist A_{13}, κ_{10} depending only on the dimension, and a function $\omega : X \to [2, \infty)$, so that both of the following hold for all $x \in X$ and all $d \geq B_\omega$*

$$\mathrm{inj}(x) \geq \omega(x)^{-A_{13}} \tag{B.2a}$$

$$\int \omega(hx)\,\mathrm{d}\sigma_d^{(\ell)}(h) \leq e^{-\kappa_{10}\ell d}\omega(x) + \bar{B}e^{A_{13}d} \tag{B.2b}$$

where $\sigma_d^{(\ell)}$ denotes the ℓ-fold convolution and $\bar{B} \geq 1$ depends only of X.

Proof. A function with these properties is constructed in [EM04]. More explicitly, see [SS22, Prop. 26] for (B.2a) and [SS22, Thm. 16] for (B.2b). □

Let $Y = Hy$ be a periodic orbit. For every $x \in X \setminus Y$, define

$$I_Y(x) = \{w \in \mathfrak{r} : 0 < \|w\| < {A_{13}}^{-1}\omega(x)^{-A_{13}}, \exp(w)x \in Y\}.$$

Increasing A_{13} if necessary, we have

$$\#I_Y(x) \leq E\mathrm{vol}(Y) \tag{B.3}$$

for a constant E depending only on X, see [SS22, Prop. 25] also [LM23, §9].

Again increasing A_{13} if necessary, for every $h = a_d u_r$ with $d \geq 0$, all $r \in [-1, 2]$, and for all $w \in \mathfrak{g}$ and all $x \in X$, we have

$$\| \operatorname{Ad}(h^{\pm 1}) w\| \leq A_{13} e^{A_{13} d/2} \|w\|, \tag{B.4a}$$

$$A_{13}{}^{-1} e^{-A_{13} d/2} \omega(x) \leq \omega(h^{\pm 1} x) \leq A_{13} e^{A_{13} d/2} \omega(x). \tag{B.4b}$$

Let $\alpha = \min\{1/(2\mathsf{m}+1), 1/A_{13}\}$, furthermore, we will replace κ_{10} with a smaller constant if necessary and assume that $\kappa_{10} \leq \alpha \mathsf{m}/2$.

Define

$$f_Y(x) = \begin{cases} \sum_{w \in I_Y(x)} \|w\|^{-\alpha} & I_Y(x) \neq \emptyset \\ \omega(x) & \text{otherwise} \end{cases}. \tag{B.5}$$

B.4. **Lemma.** *Let C be as in Proposition B.3, and let $d \geq B_\omega$. Then*

$$\int f(hx) \, \mathrm{d}\sigma_d(h) \leq C e^{-\alpha \mathsf{m} d} f_Y(x) + A_{13} e^d E \operatorname{vol}(Y) \cdot (e^{-\kappa_{10} d} \omega(x) + \bar{B} e^{A_{13} d})$$

where $\bar{B}$ is as in Proposition B.3.

Proof. Since Y is fixed throughout the argument, we drop it from the index in the notation, e.g., we will denote f_Y by f etc.

Let $d \geq 0$ and let $h = a_d u_r$ for some $r \in [-1, 2]$. Let $x \in X$. First, let us assume that there exists some $w \in I(hx)$ with

$$\|w\| < A_{13}{}^{-2} e^{-A_{13} d} \omega(hx) =: \Upsilon.$$

This in particular implies that both $I(hx)$ and $I(x)$ are non-empty. Hence,

$$\begin{aligned} f(hx) &= \sum_{w \in I(hx)} \|w\|^{-\alpha} = \sum_{\|w\| < \Upsilon} \|w\|^{-\alpha} + \sum_{\|w\| \geq \Upsilon} \|w\|^{-\alpha} \\ &\leq \sum_{w \in I(x)} \| \operatorname{Ad}(h) w\|^{-\alpha} + A_{13}{}^{2\alpha} e^{A_{13} \alpha d} \big(1 + \# I(hx)\big) \cdot \omega(hx)^\alpha \\ &\leq \sum_{w \in I(x)} \| \operatorname{Ad}(h) w\|^{-\alpha} + A_{13} e^d \big(1 + \# I(hx)\big) \cdot \omega(hx), \end{aligned} \tag{B.6}$$

in the last inequality, we used $0 < \alpha \leq \min\{1/2, 1/A_{13}\}$ and $\omega(\cdot) \geq 1$.

Note also that if $\|w\| \geq \Upsilon$ for all $w \in I(hx)$ (which in view of the choice of A_{13} includes the case $I(x) = \emptyset$) or if $I(hx) = \emptyset$, then

$$f(hx) \leq A_{13} e^d \big(1 + \# I(hx)\big) \cdot \omega(hx). \tag{B.7}$$

Averaging (B.6) and (B.7) over $[-1, 2]$ and using (B.3), we conclude that

$$\int f(hx) \, \mathrm{d}\sigma_d(h) \leq \sum_{w \in I(x)} \int \|hw\|^{-\alpha} \, \mathrm{d}\sigma_d(h) \quad + \\ A_{13} e^d E \operatorname{vol}(Y) \cdot \int \omega(hx) \, \mathrm{d}\sigma_d(h);$$

we replace the summation on the right by 0 if $I(x) = \emptyset$.

Thus by Lemma B.2 and Proposition B.3, we conclude that

$$\int f(hx)\,\mathrm{d}\sigma_d(h) \le Ce^{-\mathsf{m}\alpha d}\cdot \sum_{w\in I(x)} \|w\|^{-\alpha} \quad + \\ A_{13}e^d E\mathrm{vol}(Y)\cdot(e^{-\kappa_{10}d}\omega(x)+\bar{B}e^{A_{13}d})$$

again, the summation on the right by 0 if $I(x)=\emptyset$. Thus

$$\int f(hx)\,\mathrm{d}\sigma_d(h) \le Ce^{-\mathsf{m}\alpha d}f(x) + A_{13}e^d E\mathrm{vol}(Y)\cdot(e^{-\kappa_{10}d}\omega(x)+\bar{B}e^{A_{13}d}).$$

The proof is complete. □

B.5. **Lemma.** *Let C, A_{13}, and κ_{10} be as in Proposition B.3, and let*

$$\ell = \lceil 10(A_{13}+1)/\kappa_{10}\rceil.$$

There is an absolute constant T_0 so that the following holds. Let $T \ge T_0$ and define

$$d_i = \frac{\log T}{2^i \ell}$$

for all $i = 1,\ldots,k$ where k is the largest integer so that $d_k \ge \max\{B_\omega, \frac{\log(4C)}{\kappa_{10}}\}$ — note that $\frac{1}{2}\log\log T \le k \le 2\log\log T$. Then

$$\int f_Y(hx)\,\mathrm{d}\sigma^{(\ell)}_{d_1} * \cdots * \sigma^{(\ell)}_{d_k}(h) \le \\ (\log T)^{D_0'}T^{-\mathsf{m}\alpha}f(x) + B'\mathrm{vol}(Y)\big(T^{-\kappa_{11}}\omega(x)+1\big)$$

where D_0' and κ_{11} depend on dimension and B' depends on X.

Proof. Again since Y is fixed throughout the argument, we drop it from the index in the notation, e.g., we will denote f_Y by f etc.

Let us make some observations before starting the proof. Since $d_i \ge \frac{\log(4C)}{\kappa_{10}}$ for all i, and $\kappa_{10} \le \alpha\mathsf{m}/2$, the following holds

$$Ce^{-\alpha\mathsf{m}d_i} \le e^{-\kappa_{10}d_i} \le 1/4 \tag{B.8}$$

Moreover, we have the following two estimates:

$$5\sum_{j=i+1}^{k} d_j \ge 5\ell^{-1}\times 2^{-i-1}\log T \ge (2^i\ell)^{-1}\cdot\log T = d_i \tag{B.9}$$

By Lemma B.4, for all $d \ge B_\omega$, we have

$$\int f(hx)\,\mathrm{d}\sigma_d(h) \le \\ Ce^{-\alpha\mathsf{m}d}f(x) + A_{13}Ee^d\mathrm{vol}(Y)\cdot\big(e^{-\kappa_{10}d}\omega(x)+\bar{B}e^{A_{13}d}\big). \tag{B.10}$$

Let $\lambda = A_{13}E\bar{B}$. Iterating (B.10), ℓ-times, we conclude that

$$\int f(h_k \cdots h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdots \mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \leq$$
$$C^\ell e^{-\alpha \mathsf{m} \ell d_k} \int f(h_{k-1} \cdots h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdots \mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1}) \quad +$$
$$A_{13} E e^{d_k} \mathrm{vol}(Y)(\Xi_k + 2\bar{B} e^{A_{13} d_k})$$

we used $Ce^{-\alpha \mathsf{m} d_k} \leq e^{-\kappa_{10} d_k} \leq \frac{1}{4}$, see (B.8), to bound the ℓ-terms geometric sum by $2\bar{B}e^{A_{13}d_k}$, and

$$\Xi_k = \sum_{j=0}^{\ell-1} e^{-\kappa_{10} d_k(\ell-j)} \int \omega(h_k h_{k-1} \cdot \cdot h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdot \cdot \mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1})\, \mathrm{d}\sigma_{d_k}^{(j)}(h_k),$$

again we used $Ce^{-\alpha \mathsf{m} d_k} \leq e^{-\kappa_{10} d_k}$, in the definition of Ξ_k.

Note also that in view of the definition of λ, we have

$$A_{13} E e^{d_k} \mathrm{vol}(Y)(\Xi_k + 2\bar{B} e^{A_{13} d_k}) \leq \lambda \mathrm{vol}(Y) e^{(1+A_{13}) d_k}(\Xi_k + 2),$$

therefore, we conclude

$$\int f(h_k \cdots h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdots \mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \leq \tag{B.11}$$
$$C^\ell e^{-\alpha \mathsf{m} \ell d_k} \int f(h_{k-1} \cdots h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdots \mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1}) \quad +$$
$$\lambda \mathrm{vol}(Y) e^{(1+A_{13}) d_k}(\Xi_k + 2).$$

We will apply Proposition B.3, to bound Ξ_k from above. Let us begin by applying Proposition B.3, ℓ-times with d_k, then

$$\Xi_k \leq e^{-\kappa_{10} \ell d_k} \int \omega(h_{k-1} \cdot \cdot h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdot \cdot \mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1}) + \lambda e^{A_{13} d_k}$$

where we used $e^{-\kappa_{10} d_k} \leq 1/4$ and $\lambda = A_{13}E\bar{B} \geq 2\bar{B}$ to estimate the ℓ-terms geometric sum.

The goal now is to inductively apply Proposition B.3, ℓ times with d_i for all $1 \leq i \leq k-1$, in order to simplify the above estimate. Applying Proposition B.3, ℓ-times with d_{k-1}, we obtain from the above that

$$\Xi_k \leq e^{-\kappa_{10} \ell (d_k + d_{k-1})} \int \omega(h_{k-2} \cdot \cdot h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1) \cdot \cdot \mathrm{d}\sigma_{d_{k-2}}^{(\ell)}(h_{k-2}) \quad +$$
$$e^{-\kappa_{10} \ell d_k} \cdot (\lambda e^{A_{13} d_{k-1}}) + \lambda e^{A_{13} d_k}.$$

Put $\Theta_k = 0$, and for every $0 \le i < k$, let $\Theta_i = \sum_{j=i+1}^{k} d_j$. Continuing the above inequalities inductively, we conclude

$$\begin{aligned}\Xi_k &\le e^{-\kappa_{10}\ell\Theta_0}\omega(x) + \lambda(e^{A_{13}d_k} + \sum_{i=1}^{k-1} e^{-\kappa_{10}\ell\Theta_i}e^{A_{13}d_i})\\ &\le e^{-\kappa_{10}\ell\Theta_0}\omega(x) + \lambda(e^{A_{13}d_k} + \sum_{i=1}^{k-1} e^{-d_i}) \le e^{-\kappa_{10}\ell\Theta_0}\omega(x) + \lambda(e^{A_{13}d_k} + M)\end{aligned}$$

where we used $\kappa_{10}\ell\Theta_i \ge \frac{\kappa_{10}\ell}{5}\cdot(5\Theta_i) \ge (A_{13}+2)d_i$, see (B.9) and the definition of ℓ, in the second to last inequality and wrote $\sum e^{-d_i} \le M$ in the last inequality.

Iterating (B.11) and using the above analysis, we conclude

$$\int f(h_k\cdots h_1x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \le \\ C^{\ell k}e^{-\alpha\mathsf{m}\ell\Theta_0}f(x) + \lambda\mathrm{vol}(Y)\sum_{i=1}^{k} e^{-\kappa_{10}\ell\Theta_i}e^{(A_{13}+1)d_i}\big(\Xi_i + 2\big)$$

where for every $1 \le i \le k$, we have

$$\Xi_i = \sum_{j=0}^{\ell-1} e^{-\kappa_{10}d_i(\ell-j)}\int \omega(h_ih_{i-1}\cdot\cdot h_1x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdot\cdot\mathrm{d}\sigma_{d_{i-1}}^{(\ell)}(h_{i-1})\,\mathrm{d}\sigma_{d_i}^{(j)}(h_i).$$

Arguing as above, we have

$$\Xi_i \le e^{-\kappa_{10}\ell(\sum_{j=1}^{i} d_j)}\omega(x) + \lambda(e^{A_{13}d_i} + M).$$

Recall that $\Theta_i = \sum_{j=i+1}^{k} d_j$; therefore, we conclude that

$$\int f(h_k\cdots h_1x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \le \\ C^{\ell k}e^{-\alpha\mathsf{m}\ell\Theta_0}f(x) + e^{-\kappa_{10}\ell\Theta_0}\lambda\mathrm{vol}(Y)\omega(x)\textstyle\sum_{i=1}^{k} e^{(A_{13}+1)d_i} + \\ (M+2)\lambda^2\mathrm{vol}(Y)\displaystyle\sum_{i=1}^{k} e^{-\kappa_{10}\ell\Theta_i}e^{(2A_{13}+1)d_i}$$

Recall again from (B.9) and the definition of ℓ that $\kappa_{10}\ell\Theta_i \ge (2A_{13}+2)d_i$. Hence, the last term above is $\le B'\mathrm{vol}(Y)$ for an absolute constant $B' \ge \lambda$. Similarly, sine $\ell\sum d_i = \log T - O(1)$ second to last term is $\le B'\mathrm{vol}(Y)T^{-\kappa_{11}}$.

Moreover, $\ell\sum d_i = \log T - O(1)$ where the implied constant is absolute, and $k \le 2\log\log T$. Hence,

$$C^{\ell k}e^{-\mathsf{m}\alpha\ell(\sum_{i=1}^{k} d_i)} \le (\log T)^{1+\star\log C}T^{-\mathsf{m}\alpha}$$

so long as T is large enough. The proof of the lemma is complete. $\square$

B.6. **Lemma.** *Let the notation be as in Lemma B.5, in particular for every $T \geq T_0$ define $d_1, \ldots, d_k$ as in that lemma. Put $d(T) = \ell \sum d_i$, then*

$$\int_0^1 f_Y(a_{d(T)} u_r x)\,\mathrm{d}r \leq 3(\log T)^{D_0'} T^{-\alpha\mathsf{m}} f_Y(x) + B\mathrm{vol}(Y)\big(T^{-\kappa_{11}}\omega(x) + 1\big)$$

where $B \geq 1$ depends on X.

Proof. Again, since Y is fixed throughout the argument, we drop it from the index in the notation, e.g., we will denote f_Y by f etc.

By Lemma B.5, we have

$$\text{(B.12)}\quad \frac{1}{3^{\ell k}} \int_{-1}^{2} \cdots \int_{-1}^{2} f(a_{d_k} u_{r_{k,\ell}} \cdots a_{d_k} u_{r_{k,1}} \cdots a_{d_1} u_{r_{1,1}} x)\,\mathrm{d}r_{1,1} \cdots \mathrm{d}r_{k,\ell} \leq \\ (\log T)^{D_0'} T^{-\alpha\mathsf{m}} f(x) + B'\mathrm{vol}(Y)\big(T^{-\kappa_{11}}\omega(x) + 1\big)$$

Now, for every $(r_{k,\ell}, \ldots, r_{1,2}, r_{1,1}) \in [-1, 2]^{\ell k}$, we have

$$a_{d_k} u_{r_{k,\ell}} \cdots a_{d_k} u_{r_{k,1}} \cdots a_{d_1} u_{r_{1,1}} = a_{d(T)} u_{\varphi(\hat{r}) + r_{1,1}}$$

where $\hat{r} = (r_{k,\ell}, \ldots, r_{1,2})$ and $|\varphi(\hat{r})| \leq 0.2$.

In view of (B.12), there is $\hat{r} = (r_{k,\ell}, \ldots, r_{1,2}) \in [-1, 2]^{\ell k - 1}$ so that

$$\text{(B.13)}\quad \frac{1}{3} \int_{-1+\varphi(\hat{r})}^{2+\varphi(\hat{r})} f(a_{d(T)} u_r x)\,\mathrm{d}r \leq \\ (\log T)^{D_0'} T^{-\alpha\mathsf{m}} f(x) + B'\mathrm{vol}(Y)\big(T^{-\kappa_{11}}\omega(x) + 1\big).$$

Since $|\varphi(\hat{r})| \leq 0.2$, we have $[0, 1] \subset [-1, 2] + \varphi(\hat{r})$. Therefore, (B.13) and the fact that $f \geq 0$ imply that

$$\frac{1}{3} \int_0^1 f(a_{d(T)} u_r x)\,\mathrm{d}x \leq (\log T)^{D_0'} T^{-\alpha\mathsf{m}} f(x) + B'\mathrm{vol}(Y)\big(T^{-\kappa_{11}}\omega(x) + 1\big).$$

The lemma follows with $B = 3B'$. □

Proof of Proposition 4.4. Let $R \geq 1$ be a parameter and assume that $\mathrm{vol}(Y) \leq R$. Recall that for a periodic orbit Y, we put

$$f_Y(x) = \begin{cases} \sum_{w \in I_Y(x)} \|w\|^{-\alpha} & I_Y(x) \neq \emptyset \\ \omega(x) & \text{otherwise} \end{cases}.$$

Let $\psi(x_0) = \max\{d(x_0, Y)^{-\alpha}, \omega(x_0)\}$. Then

$$\text{(B.14)}\quad \psi(x_0) \ll f_{Y,d}(x_0) \ll \mathrm{vol}(Y)\psi(x_0),$$

where the implied constant depends only on X, see (B.3).

With the notation of Lemma B.5, let $T \geq T_0$ and $d_i = \frac{\log T}{2^i \ell}$ for $1 \leq i \leq k$. Then

$$\text{(B.15)}\quad \log T - \bar{b} \leq d(T) \leq \log T$$

where $\bar{b}$ is absolute.

Let $T_1 \geq T_0$ be so that $(\log T)^{D_0'} T^{-\alpha\mathsf{m}}$ is decreasing on $[T_1, \infty)$. Let

$$T_2 = \inf\{T \geq \max(T_1, \omega(x_0)^{1/\kappa_{11}}) : (\log T)^{D_0'} T^{-\alpha\mathsf{m}} \leq d(x_0, Y)^\alpha\}. \tag{B.16}$$

In other words, for all $T \geq T_2$, we have $T^{-\kappa_{11}}\omega(x_0) \leq 1$ and

$$(\log T)^{D_0'} T^{-\alpha\mathsf{m}} d(x_0, Y)^{-\alpha} \leq 1.$$

Furthermore, in view of (B.14) and since $\mathrm{vol}(Y) \leq R$, for all $T \geq T_2$,

$$(\log T)^{D_0'} T^{-\alpha\mathsf{m}} f_Y(x_0) \ll R(\log T)^{D_0'} T^{-\alpha\mathsf{m}} \psi(x_0)$$

In particular, using (B.14) again, we have $(\log T)^{D_0'} T^{-1/3} f_Y(x_0) \ll R$.

Altogether, we conclude that for all $T \geq T_2$, we have

$$3\log(T)^{D_0'} T^{-\alpha\mathsf{m}} f_Y(x_0) + B\mathrm{vol}(Y) T^{-\kappa_{11}}(\omega(x_0) + 1) \leq B_2 R \tag{B.17}$$

where B_2 is absolute.

Let $T \geq T_2$, and let $d(T) = \ell \sum d_i$ where d_i's are as above. Using (B.17) and Lemma B.6,

$$\int_0^1 f_Y(a_{d(T)} u_r x)\,\mathrm{d}r \leq B_2 R. \tag{B.18}$$

Let $D \geq 10$. Then by (B.18) we have

$$|\{r \in [0,1] : f_Y(a_{d(T)} u_r x_0) > B_2 R^D\}| \leq B_2 R / B_2 R^D \leq R^{-D+1}.$$

In view of (B.14), there some B_1 (depending on X) so that $d_X(a_s u_r x_0, Y) \leq B_1^{-1} R^{-D/\alpha}$ implies $f_Y(a_s u_r x_0) > B_2 R^D$ for all $s \geq 0$ and $r \in [0,1]$. Therefore, we conclude from the above that

$$\left|\left\{r \in [0,1] : d_X(a_{d(T)} u_r x_0, Y) \leq (B_1 R^{D/\alpha})^{-1}\right\}\right| \leq R^{-D+1}. \tag{B.19}$$

Let now $s \geq \log T_2$, then by (B.15) there exists some $T \geq T_2$ so that

$$d(T) - 2\bar{b} \leq s \leq d(T) + 2\bar{b}$$

For every $s \geq \log T_2$, let T_s be the minimum such T. Then (B.4a) implies that is $\hat{B} \geq 1$ (absolute) so that if $s \geq \log T_2$ and $r \in [0,1]$ are so that

$$d_X(a_s u_r x_0, Y) \leq (\hat{B} R^{D/\alpha})^{-1},$$

then $d_X(a_{d(T_s)} u_r x_0, Y) \leq (B_1 R^{D/\alpha})^{-1}$. This and (B.19), imply that

$$\left|\left\{r \in [0,1] : d_X(a_s u_r x_0, Y) \leq (\hat{B} R^{D/\alpha})^{-1}\right\}\right| \leq R^{-D+1} \tag{B.20}$$

Let C_3 be as in Proposition 4.2, increasing T_1 if necessary, we will assume $\log T_2 \geq \mathsf{m}_0 |\log(\mathrm{inj}(x_0))| + C_3$. Using Proposition 4.2, thus,

$$\left|\left\{r \in [0,1] : \mathrm{inj}(a_s u_r x) < \eta\right\}\right| < C_3 \eta^{1/\mathsf{m}_0} \tag{B.21}$$

for any $\eta > 0$ and all $s \geq \log T_2$.

Altogether, from (B.20) and (B.21) it follows that for any $s \geq \log T_2$,

$$\left|\left\{r \in [0,1] : \begin{array}{c} \mathrm{inj}(a_s u_r x) < \eta \quad \text{or} \\ d_X(a_s u_r x_0, Y) \leq (\hat{B} R^{D/\alpha})^{-1} \end{array}\right\}\right| \leq C_3 \eta^{1/\mathsf{m}_0} + R^{-D+1}. \tag{B.22}$$

In view of [SS22, Thm. 5], the number of periodic H-orbits with volume $\leq R$ in X is $\leq \hat{E}R^{\hat{D}}$ where $\hat{D}$ depends only on dimension and $\hat{E}$ depends on X. Let $D = \hat{D} + 10$ and $C_4 = \max\{\hat{E}, \hat{B}, C_3\}$. Then (B.22) implies

$$\left|\left\{r \in [0,1] : \begin{array}{c} \mathrm{inj}(a_s u_r x) < \eta \text{ or there exists } x \text{ with} \\ \mathrm{vol}(Hx) \leq R \text{ s.t. } d_X(a_s u_r x_0, x) \leq (C_4 R^{D/\alpha})^{-1} \end{array}\right\}\right| \leq C_4(\eta^{1/\mathsf{m}_0} + R^{-1}). \tag{B.23}$$

We now show that (B.23) implies the proposition. Suppose

$$d_X(x_0, x) \geq S^{-\mathsf{m}}(\log S)^{D_0'}$$

for every x with $\mathrm{vol}(Hx) \leq R$. Then by (B.16), we have

$$T_2 \leq \max\{S, \omega(x_0)^{1/\kappa_{11}}, T_1\}.$$

Therefore, if we let $D_0 = (\max\{D_0', \hat{D} + 10\})/\alpha$ and set $s_0 = C_3 + \log T_1$, then Proposition 4.4 follows from (B.23). □

B.7. **Proof of Proposition 4.6.** In what follows all the implied multiplicative constants depend only on X.

We begin by recalling some statements and lemmas which will be used in the proof. Let v_H be a unit vector on the line $\wedge^3\mathfrak{h} \subset \wedge^3\mathfrak{g}$.

B.8. **Lemma** (Cf. [LM23], Lemma 6.3)**.** *There exist A_{14}, A_{15}, and C_6 so that the following holds. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$, and let*

$$\delta \leq C_6{}^{-1}\big(\max\{\|\gamma_i^{\pm 1}\| : 1 \leq i \leq n\}\big)^{-A_{14}}.$$

Suppose there exists some $g \in G$ so that $\gamma_i g^{-1} v_H = \epsilon_i g^{-1} v_H$ for $i = 1, 2$ where $\|\epsilon_i - I\| \leq \delta$. Then, there is some $g' \in G$ such that

$$\|g' - g^{-1}\| \leq C_6\|g\|^{A_{15}}\delta\big(\max\{\|\gamma_i^{\pm 1}\| : 1 \leq i \leq n\}\big)^{A_{15}}$$

and $\gamma_i g' v_H = g' v_H$ for $i = 1, \ldots, n$.

B.9. **Lemma** (Cf. [LM23], Lemma 6.2)**.** *There exist C_7 and A_{16} depending on Γ, and A_{17} (depending on the dimension) so that the following holds. Let $\gamma_1, \gamma_2 \in \Gamma$ be two non-commuting elements. If $g \in G$ is so that $\gamma_i g^{-1} v_H = g^{-1} v_H$ for $i = 1, 2$, then $Hg\Gamma$ is a closed orbit with*

$$\mathrm{vol}(Hg\Gamma) \leq C_7\|g\|^{A_{17}}\big(\max\{\|\gamma_1^{\pm 1}\|, \|\gamma_2^{\pm 1}\|\}\big)^{A_{16}}.$$

The statements in [LM23, Lemma 6.2, and Lemma 6.3] assumed $|g_2| \ll 1$. However, the arguments work without any changes and yield Lemmas B.8 and B.9.

We also recall the following lemma which is a consequence of reduction theory. In this form, the lemma is a spacial case of [LMMS19, Lemma 2.8].

B.10. **Lemma.** *There exist D_2 (depending on m) and C_8 (depending on X) so that the following holds for all $0 < \eta < 1$. Let $g \in G$ be so that $g\Gamma \in X_\eta$. Then there is some $\gamma \in \Gamma$ so that*

$$\|g\gamma\| \leq C_8\eta^{-D_2}. \tag{B.24}$$

Finally, we recall Proposition 4.2: for all positive ε, every interval $J \subset [0,1]$, and every $x \in X$, we have

$$\left|\left\{r \in J : \operatorname{inj}(a_d u_r x) < \varepsilon^{\mathsf{m}_0}\right\}\right| < C_3 \varepsilon |J|, \tag{B.25}$$

so long as $d \geq \mathsf{m}_0 |\log(|J| \operatorname{inj}(x))| + C_3$.

For the rest of the argument, let

$$t \geq 100 D_2 \mathsf{m}_0 |\log(\eta \operatorname{inj}(x_1))| + C_3 \tag{B.26}$$

Let $r_1 \in [0,1]$ be so that $x_2 = a_t u_{r_1} x_1 \in X_\eta$. Write $x_2 = g_2 \Gamma$ where $|g_2| \ll \eta^{-D_2}$, see (B.24).

Also let k be a constant which will be explicated later and will depend only on m.

In the course of the proof, we will use $\mathsf{k}.$ to denote constants which depend on k. The notation $\hat{\mathsf{m}}.$ (and the previously used $\mathsf{m}.$) will be used for constants which depend on m but not on the choice of k above.

Let $\hat{\mathsf{m}}_0$ be a constant which depends on m so that

$$\|a_s\| \leq e^{\hat{\mathsf{m}}_0 s} \quad \text{for all } s \geq 1. \tag{B.27}$$

We will show that unless part (2) in the proposition holds, we have the following: for every such x_2, there exists $J(x_2) \subset [0,1]$ with $|[0,1] \setminus J(x_2)| \leq 200 C_3 \eta^{1/(2\mathsf{m}_0)}$ so that for all $r \in J(x_2)$, we have:

(a) $a_{\mathsf{k}t} u_r x_2 \in X_\eta$,
(b) the map $\mathsf{h} \mapsto \mathsf{h} a_{\mathsf{k}t} u_r x_2$ is injective over E_t, and
(c) for all $z \in \mathsf{E}_t.a_{\mathsf{k}t} u_r x_2$, we have $f_{t,\alpha}(z) \leq e^{Dt}$.

This will imply that part (1) in the proposition holds as

$$a_{\mathsf{k}t} u_r a_t u_{r'} x_1 = a_{\mathsf{k}+1t} u_{r'+e^{-t}r} x_1.$$

Assume contrary to the above claim that for some x_2 as above, there exists a subset $I'_{\mathrm{bad}} \subset [0,1]$ with $|I'_{\mathrm{bad}}| > 200 C_3 \eta^{1/(2\mathsf{m}_0)}$ so that one of (a), (b), or (c) above fails. Then in view of (B.25) applied with x_2 and $\mathsf{k}t$, there is a subset $I_{\mathrm{bad}} \subset [0,1]$ with $|I_{\mathrm{bad}}| \geq 100 C_3 \eta^{1/(2\mathsf{m}_0)}$ so that for all $r \in I_{\mathrm{bad}}$ we have $a_{\mathsf{k}t} u_r x_2 \in X_\eta$, but

- either the map $\mathsf{h} \mapsto \mathsf{h} a_{\mathsf{k}t} u_r x_2$ is not injective on E_t,
- or there exists $z \in \mathsf{E}_t.a_{\mathsf{k}t} u_r x_2$ so that $f_{t,\alpha}(z) > e^{Dt}$.

We will show that this implies part (2) in the proposition holds.

Let us first recall that $f_\tau : \mathsf{E}_\tau.y \to [1,\infty)$ is defined as follows

$$f_\tau(z) = \begin{cases} \sum_{0 \neq w \in I_\tau(z)} \|w\|^{-\alpha} & \text{if } I_\tau(z) \neq \{0\} \\ \operatorname{inj}(z)^{-\alpha} & \text{otherwise} \end{cases}.$$

where $0 < \alpha \leq 1$.

Finding lattice elements γ_r. We introduce the shorthand notation

$$h_r := a_{\mathsf{k}t} u_r, \quad \text{for any } r \in [0,1].$$

Let us first investigate the latter situation. That is: for $r \in I_{\mathrm{bad}}$ (recall that $h_r x_2 \in X_\eta$) there exists some $z = \mathsf{h}_1 h_r x_2 \in \mathsf{E}_t.h_r x_2$, so that $f_{t,\alpha}(z) > e^{Dt}$. Since $h_r x_2 \in X_\eta$, we have

$$\mathrm{inj}(\mathsf{h} h_r x_2) \gg \eta e^{-\mathsf{m}t}, \quad \text{for all } \mathsf{h} \in \mathsf{E}_t. \tag{B.28}$$

Using the definition of $f_{t,\alpha}$, thus, we conclude that if $I_t(z) = \{0\}$, then $f_{t,\alpha}(z) \ll \eta^{-1} e^{\mathsf{m}t}$. Since $t \geq 100 D_2 \mathsf{m}_0 |\log \eta|$, assuming $\mathsf{m}_0 \geq \mathsf{m}+1$, $D \geq \mathsf{m}+1$ and t is large enough, we conclude that $I_t(z) \neq \{0\}$. Moreover, using (B.28), we have $\# I_t(z) \ll \eta^{-\hat{\mathsf{m}}_1} e^{\hat{\mathsf{m}}_1 t}$, see [LM23, Lemma 6.4] or the similar estimate [LMW22, Lemma 8.1].

Altogether, if $D \geq \mathsf{m} + 1 + 2\hat{\mathsf{m}}_1$ and t is large enough, there exists some $w \in I_t(z)$ with

$$0 < \|w\| \leq e^{(-D+\hat{\mathsf{m}}_1+1)t} =: e^{-D't}.$$

The above implies that for some $w \in \mathfrak{r}$ with $\|w\| \leq e^{-D't}$ and $\mathsf{h}_1 \neq \mathsf{h}_2 \in \mathsf{E}_t$, we have $\exp(w)\mathsf{h}_1 h_r x_2 = \mathsf{h}_2 h_r x_2$. Thus

$$\exp(w_r) h_r^{-1} \mathsf{s}_r h_r x_2 = x_2 \tag{B.29}$$

where $\mathsf{s}_r = \mathsf{h}_2^{-1}\mathsf{h}_1$, $w_r = \mathrm{Ad}(h_r^{-1}\mathsf{h}_2^{-1})w$. In particular, $\|w_r\| \ll e^{(-D'+\hat{\mathsf{m}}_0\mathsf{k})t}$ where the implied constant depends only on m. Assuming t is large enough compared to the implied multiplicative constant,

$$0 < \|w_r\| \leq e^{(-D'+\hat{\mathsf{m}}_0\mathsf{k}+1)t}. \tag{B.30}$$

Recall that $x_2 = g_2\Gamma$ where $|g_2| \ll \eta^{-D_2}$, thus, (B.29) implies

$$\exp(w_r) h_r^{-1} \mathsf{s}_r h_r = g_2 \gamma_r g_2^{-1} \tag{B.31}$$

where $1 \neq \mathsf{s}_r \in H$ with $\|\mathsf{s}_r\| \ll e^{2\hat{\mathsf{m}}_0 t}$ and $e \neq \gamma_r \in \Gamma$.

Similarly, if for some $r \in I_{\mathrm{bad}}$, $\mathsf{h} \mapsto \mathsf{h} h_r x_2$ is not injective, then

$$h_r^{-1}\mathsf{s}_r h_r = g_2\gamma_r g_2^{-1} \neq e.$$

In this case we actually have $e \neq \gamma_r \in g_2^{-1} H g_2$ — we will not use this extra information in what follows.

Some properties of the elements γ_r. Recall that $\|g_2\| \ll \eta^{-D_2}$ and that $t \geq 100 D_2 \mathsf{m}_0 |\log\eta|$. Therefore, if we put $\mathsf{k}_1 = 2\hat{\mathsf{m}}_0(\mathsf{k}+1)+1$, then

$$\|\gamma_r^{\pm 1}\| \leq e^{\mathsf{k}_1 t} \tag{B.32}$$

where we assumed t is large compared to η and used (B.31).

We identify $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ with its image in H, however, when we write $\|\ \|$ the norm is in G but $|a_i|$ denotes the usual absolute value. With this notation, e.g., $h_r^{-1}\mathsf{s}_r h_r$ is represented by

$$u_{-r}\begin{pmatrix} a_1 & e^{-\mathsf{k}t}a_2 \\ e^{\mathsf{k}t}a_3 & a_4 \end{pmatrix} u_r$$

where $|a_i| \le 10e^t$.

Let $\xi > 0$ be so that $\|g\gamma g^{-1} - I\| \ge \xi\eta^{2D_2}$ for all $\gamma \in \Gamma \setminus \{1\}$ and $\|g\| \le C_8\eta^{-D_2}$, see (B.24). Then by (B.31), we have

$$\left\| u_{-r} \begin{pmatrix} a_1 & e^{-\mathsf{k}t}a_2 \\ e^{\mathsf{k}t}a_3 & a_4 \end{pmatrix} u_r - I \right\| \ge \eta^{D_2'}$$

for some D_2' depending only on m and D_2. This implies

$$\max\{e^{\mathsf{k}t}|a_3|, |a_1 - 1|, |a_4 - 1|\} \ge \eta^{D_2'}. \tag{B.33}$$

Note also that if $e^{\mathsf{k}t}|a_3| < \eta^{D_2'}$, then

$$|a_2a_3| \le 10\eta^{D_2'}e^{(-\mathsf{k}+1)t},$$

thus $|a_1a_4 - 1| \ll \eta^\star e^{(-\mathsf{k}+1)t}$, and (B.33) implies $|a_1 - a_4| \gg \eta^{D_2'}$. Altogether,

$$\max\{e^{\mathsf{k}t}|a_3|, |a_1 - a_4|\} \gg \eta^{D_2'}. \tag{B.34}$$

Since $|I_{\rm bad}| \ge 100C_3\eta^{1/(2\mathsf{m}_0)}$, there are two intervals $J, J' \subset [0, 1]$ with $d(J, J') \ge \eta^{1/(2\mathsf{m}_0)}$, $|J|, |J'| \ge \eta^{1/(2\mathsf{m}_0)}$, and

$$|J \cap I_{\rm bad}| \ge \eta^{1/\mathsf{m}_0} \quad \text{and} \quad |J' \cap I_{\rm bad}| \ge \eta^{1/\mathsf{m}_0}. \tag{B.35}$$

Put $J_\eta = J \cap I_{\rm bad}$.

Claim: There are $\gg e^{(\mathsf{k}-2)t/2}$ distinct elements in $\{\gamma_r : r \in J_\eta\}$.

Fix $r \in J_\eta$ as above, and consider the set of $r' \in J_\eta$ so that and $\gamma_r = \gamma_{r'}$. Then for each such r',

$$\begin{aligned} h_r^{-1}\mathsf{s}_r h_r &= \exp(-w_r)g_2\gamma_r g_2^{-1} = \exp(-w_r)\exp(w_{r'})h_{r'}^{-1}\mathsf{s}_{r'}h_{r'} \\ &= \exp(w_{rr'})h_{r'}^{-1}\mathsf{s}_{r'}h_{r'} \end{aligned}$$

where $w_{rr'} \in \mathfrak{g}$ and $\|w_{rr'}\| \ll e^{(-D'+\hat{\mathsf{m}}_0\mathsf{k})t}$.

Set $\tau = e^{\mathsf{k}t}(r' - r)$. Assuming $D' = \frac{D}{\mathsf{m}+1} - \hat{\mathsf{m}}_1 - 1$ is large enough, we conclude that

$$u_\tau \mathsf{s}_r u_{-\tau} = h_{r'}h_r^{-1}\,\mathsf{s}_r\,h_r h_{r'}^{-1} = \exp(\hat{w}_{rr'})\mathsf{s}_{r'}, \tag{B.36}$$

where $\|\hat{w}_{rr'}\| = \|\operatorname{Ad}(h_{r'})w_{rr'}\| \ll e^{(-D'+\hat{\mathsf{m}}_0\mathsf{k}+\mathsf{m}\mathsf{k})t}$. Finally, we compute

$$u_\tau\mathsf{s}_r u_{-\tau} = \begin{pmatrix} a_1 + a_3\tau & a_2 + (a_4 - a_1)\tau - a_3\tau^2 \\ a_3 & a_4 - a_3\tau \end{pmatrix}.$$

In view of (B.34), for every $r \in J_\eta$ the set of $r' \in J_\eta$ so that

$$|a_2e^{-\mathsf{k}t} + (a_4 - a_1)(r' - r) - a_3e^{\mathsf{k}t}(r' - r)^2| \le 10^4e^{(-\mathsf{k}+1)t} \tag{B.37}$$

has measure $\ll \eta^{-D_2'/2}e^{(-\mathsf{k}+1)t/2}$ since at least one of the coefficients of this quadratic polynomial is of size $\gg \eta^{D_2'}$. Let $J_{\eta,r}$ be the set of $r' \in J_\eta$ for which (B.37) holds.

If $r' \in J_\eta \setminus J_{\eta,r}$, then $|a_2 + (a_4 - a_1)\tau - a_3\tau^2| > 10^4e^t$ (recall that $\tau = e^{\mathsf{k}t}(r' - r)$), thus for all $r' \in J_\eta \setminus J_{\eta,r}$, we have

$$\|u_\tau\mathsf{s}_r u_{-\tau}\| > 10^4e^t > \|\exp(\hat{w}_{rr'})\mathsf{s}_{r'}\|,$$

in contradiction to (B.36).

In other words, for each $\gamma \in \Gamma$ the set of $r \in J_\eta$ for which $\gamma_r = \gamma$ has measure $\ll \eta^{-D'_2/2} e^{(-\mathsf{k}+1)t/2}$ and so the set $\{\gamma_r : r \in J_\eta\}$ has at least $\gg \eta^{(D'_2+1)/2} e^{(\mathsf{k}-1)t/2} \gg e^{(\mathsf{k}-2)t/2}$ distinct elements so long as $t \geq 100\mathsf{m}_0 \max(D'_2, D_2)|\log \eta|$, see (B.26). This establishes the claim.

Zariski closure of the group generated by $\{\gamma_r : r \in I_{\mathrm{bad}}\}$. Let $\mathbf{L}$ denote the Zariski closure of $\langle \gamma_r : r \in I_{\mathrm{bad}} \rangle$.

First note that by [LMM+24, Lemma 2.4], we have $[\mathbf{L} : \mathbf{L}^\circ] \ll 1$, where the implied constant depends only on the dimension. This and

$$\#\{\gamma_r : r \in I_{\mathrm{bad}}\} \gg e^{(\mathsf{k}-2)t/2}$$

imply that $\mathbf{L}^\circ \neq \{e\}$. Moreover, from $[\mathbf{L} : \mathbf{L}^\circ] \ll 1$ we also conclude that there exists $\gamma_1, \ldots, \gamma_n \subset \{\gamma_r : r \in I_{\mathrm{bad}}\}$, where n depends only the dimension, so that $\mathbf{L}$ equals the Zariski closure of $\langle \gamma_i : 1 \leq i \leq n \rangle$.

Recall now from (B.31) that $\exp(w_r) h_r^{-1} \mathsf{s}_r h_r = g_2 \gamma_r g_2^{-1}$, thus

$$\gamma_r . g_2^{-1} v_H = \exp(\mathrm{Ad}(g_2^{-1}) w_r) . g_2^{-1} v_H.$$

Moreover, since $\|w_r\| \leq e^{(-D' + \hat{\mathsf{m}}_0 \mathsf{k} + 1)t}$,

$$\|\mathrm{Ad}(g_2^{-1}) w_r\| \ll \eta^{-2\mathsf{m} D_1} e^{(-D' + \hat{\mathsf{m}}_0 \mathsf{k} + 1)t} \leq e^{(-D' + \hat{\mathsf{m}}_0 \mathsf{k} + 2)t}$$

for all $r \in I_{\mathrm{bad}}$. Recall that $\|\gamma_r^{\pm 1}\| \leq e^{\mathsf{k}_1 t}$. If $D' = \frac{D}{\mathsf{m}+1} - \hat{\mathsf{m}}_1 - 1$ is large enough, we may apply Lemma B.8, with $\{\gamma_1, \ldots, \gamma_n\}$, and conclude that there exists some $g_3 \in G$ with

$$\begin{aligned} \|g_2 - g_3\| &\leq C_6 \eta^{-2\mathsf{m} D_1 A_{15}} e^{(-D' + \hat{\mathsf{m}}_0 \mathsf{k} + 2 + A_{15} \mathsf{k}_1)t} \\ &\leq e^{(-D' + A_{15} \mathsf{k}_2)t}, \end{aligned} \tag{B.38}$$

so that $\gamma_i . g_3^{-1} v_H = g_3^{-1} v_H$ for all i, where k_3 depends only on m and we assumed t is large.

In view of the choice of $\{\gamma_i : 1 \leq i \leq n\}$, this implies $g . g_3^{-1} v_H = g_3^{-1} v_H$ for all $g \in \mathbf{L}(\mathbb{R})$. Hence,

$$\mathbf{L}(R) \subset g_3^{-1} N_G(H) g_3 \tag{B.39}$$

We also note that $[N_G(H) : H] \ll 1$ since $H \subset G$ is a maximal connected subgroup.

We now consider two possibilities for the elements $\{\gamma_r : r \in I_{\mathrm{bad}}\}$.

Case 1. $\mathbf{L}$ is commutative. Then $\mathbf{L}^\circ(\mathbb{R}) \subset g_3^{-1} H g_3$ is commutative and

$$\#\{\gamma \in \mathbf{L}^\circ(\mathbb{R}) : \|\gamma\| \leq e^{\mathsf{k}_1 t}\} \gg e^{(\mathsf{k}-2)t/2}.$$

Since for every torus $T \subset G$, we have $\#(B_T(e, R) \cap \Gamma) \ll (\log R)^2$, where the implied constant is absolute, $\mathbf{L}^\circ$ is unipotent and $\mathbf{L} \subset \mathbf{L}^\circ \cdot C_G$.

We also note that since $\mathbf{L}^\circ$ is a one dimensional unipotent subgroup,

$$\#\{\gamma \in \mathbf{L} : \|\gamma\| \leq 100 e^{(\mathsf{k}-2)t/3}\} \ll e^{(\mathsf{k}-2)t/3}.$$

Furthermore, there are $\gg e^{(\mathsf{k}-2)t/2}$ distinct elements γ_r with $r \in J_\eta$. Thus

$$\#\{\gamma_r : \|\gamma_r\| > 100e^{(\mathsf{k}-2)t/3} \text{ and } r \in J_\eta\} \gg e^{(\mathsf{k}-2)t/2}.$$

For every $r \in I_{\mathrm{bad}}$, let

$$\begin{pmatrix} a_{1,r} & a_{2,r} \\ a_{3,r} & a_{4,r} \end{pmatrix} \quad \text{where } |a_{j,r}| \le 10e^t$$

denote the element in $\mathrm{SL}_2(\mathbb{R})$ corresponding to $\mathsf{s}_r \in H$.

We will obtain an improvement of (B.33). Let $\xi\eta^{2D_2} \le \Upsilon \le e^{(\mathsf{k}-2)t/3}$ and assume that $\|g_2\gamma_r g_2^{-1} - I\| \ge \Upsilon$ — by the definition of ξ, this holds with $\Upsilon = \xi\eta^{2D_2}$ for all $r \in I_{\mathrm{bad}}$ and as we have just seen this also holds for with $\Upsilon = e^{(\mathsf{k}-2)t/3}$ for many choices of $r \in J_{\mathrm{bad}}$. Then by (B.31), we have

$$\left\| u_{-r} \begin{pmatrix} a_{1,r} & e^{-\mathsf{k}t}a_{2,r} \\ e^{\mathsf{k}t}a_{3,r} & a_{4,r} \end{pmatrix} u_r - I \right\| \ge 10\Upsilon' = O(\Upsilon^{1/\hat{\mathsf{m}}_0}) \tag{B.40}$$

where we increase $\hat{\mathsf{m}}_0$ in (B.27) if necessary so that above holds.

We claim

$$|a_{3,r}| \ge \Upsilon' e^{-\mathsf{k}t}. \tag{B.41}$$

To see this, first note that by (B.40) $\max\{e^{\mathsf{k}t}|a_{3,r}|, |a_{1,r}-1|, |a_{4,r}-1|\} \ge \Upsilon'$. Assume contrary to our claim that $|a_{3,r}| < \Upsilon' e^{-\mathsf{k}t}$. Then

$$\max\{|a_{1,r}-1|, |a_{4,r}-1|\} \ge \Upsilon'; \tag{B.42}$$

furthermore, we get $|a_{2,r}a_{3,r}| \ll \Upsilon' e^{(-\mathsf{k}+1)t}$. Thus,

$$|a_{1,r}a_{4,r} - 1| \ll \Upsilon' e^{(-\mathsf{k}+1)t} \ll e^{-\mathsf{k}t/2}. \tag{B.43}$$

Moreover, since $h_r^{-1}\mathsf{s}_r h_r$ is very nearly $g_2\gamma_r g_2^{-1}$, and the latter is either a unipotent element or its minus, we conclude that

$$\min(|a_{1,r} + a_{4,r} - 2|, |a_{1,r} + a_{4,r} + 2|) \ll e^{(-D'+\star)t}. \tag{B.44}$$

Equations (B.43) and (B.44) contradict (B.42) if t is large enough. Altogether, (B.41) holds.

We now show that Case 1 cannot occur. Since $\mathbf{L}^\circ$ is unipotent and $\mathbf{L} \subset \mathbf{L}^\circ \cdot C_{\mathbf{G}}$, we conclude from (B.39) combined with (B.31) and (B.38) that

$$u_{-r} \begin{pmatrix} a_{1,r} & e^{-\mathsf{k}t}a_{2,r} \\ e^{\mathsf{k}t}a_{3,r} & a_{4,r} \end{pmatrix} u_r \in \exp(-w_r)(hUh^{-1}) \cdot C_{\mathbf{G}} \tag{B.45}$$

for all $r \in I_{\mathrm{bad}}$, where $h \in H$ and $\|h\| \ll 1$. We show that this leads to a contradiction.

Recall the intervals J and J' from (B.35), and let $r_0 \in J' \cap I_{\mathrm{bad}}$. then $|r_0 - r| \ge \eta^{1/(2\mathsf{m}_0)}$ for all $r \in J_\eta$. Then, (B.45), yields that

$$u_{-r+r_0} \begin{pmatrix} a_{1,r} & e^{-\mathsf{k}t}a_{2,r} \\ e^{\mathsf{k}t}a_{3,r} & a_{4,r} \end{pmatrix} u_{r-r_0} \in \exp(-w_r')(u_{r_0}hUh^{-1}u_{-r_0}) \cdot C_{\mathbf{G}} \tag{B.46}$$

for all $r \in I_{\mathrm{bad}}$.

Let us write $u_{r_0}h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then for all $z \in \mathbb{R}$ we have

$$u_{r_0}h \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} h^{-1}u_{-r_0} = \begin{pmatrix} 1-acz & a^2z \\ -c^2z & 1+acz \end{pmatrix}.$$

Let $z_0 \in \mathbb{R}$ be so that

$$\begin{pmatrix} a_{1,r_0} & e^{-\mathsf{k}t}a_{2,r_0} \\ e^{\mathsf{k}t}a_{3,r_0} & a_{4,r_0} \end{pmatrix} = \pm\exp(-w_{r_0}) \begin{pmatrix} 1-acz_0 & a^2z_0 \\ -c^2z_0 & 1+acz_0 \end{pmatrix}.$$

By (B.41) applied with Υ' corresponding to $\Upsilon = \xi\eta^{2D_2}$, we have $|a_{3,r_0}| \geq \eta^\star e^{-\mathsf{k}t}$. Since

$$|a|, |b|, |c|, |d| \ll 1,$$

comparing the bottom left entries of the matrices, we get $|z_0| \gg \eta^{D_3}$. Now, since $|a_{2,r_0}| \leq 10e^t$, comparing the top right entries we conclude that

$$|a|^2 \ll \eta^{-D_3}e^{(-\mathsf{k}+1)t} \leq e^{-(\mathsf{k}+2)t}.$$

Since $\det(u_{r_0}h) = 1$, it follows that $|c|$ is also $\gg 1$.

Let now $r \in J_\eta$ be so that $\|\gamma_r\| \geq 100e^{(\mathsf{k}-2)t/3}$. We write $r_1 = r - r_0$, $a'_{2,r} = e^{-\mathsf{k}t}a_{2,r}$ and $a'_{3,r} = e^{\mathsf{k}t}a_{3,r}$. By (B.41), applied this time with Υ' corresponding to $\Upsilon = e^{(\mathsf{k}-2)t/3}$, we have $|a'_{3,r}| \geq \Upsilon' \geq e^{\frac{\mathsf{k}-2}{4\hat{\mathsf{m}}_0}t}$; note also that $|a'_{2,r}| \ll e^{(-\mathsf{k}+1)t}$. In view of (B.46), there exists $z_r \in \mathbb{R}$ so that

$$\begin{aligned} u_{-r_1}\begin{pmatrix} a_{1,r} & a'_{2,r} \\ a'_{3,r} & a_{4,r} \end{pmatrix} u_{r_1} &= \begin{pmatrix} a_{1,r} - r_1a'_{3,r} & a'_{2,r} + (a_{4,r} - a_{1,r})r_1 - a'_{3,r}r_1^2 \\ a'_{3,r} & a_{4,r} + r_1a'_{3,r} \end{pmatrix} \\ &= \pm\exp(-w'_r)\begin{pmatrix} 1-acz_r & a^2z_r \\ -c^2z_r & 1+acz_r \end{pmatrix}. \end{aligned}$$

Recall that $|a'_{3,r}| \geq e^{\frac{\mathsf{k}-2}{4\hat{\mathsf{m}}_0}t}$, $|a_{1,r}|, |a_{4,r}| \ll e^t$, and $|a'_{2,r}| \ll e^{(-\mathsf{k}+1)t}$; moreover $\eta^{1/(2\mathsf{m}_0)} \leq |r_1| \leq 1$ and by (B.26) $e^{t/10} \geq \eta^{-1}$. Thus, so long as $\mathsf{k} - 2 > 5\hat{\mathsf{m}}_0$,

$$0.1|a'_{3,r}|\eta^{1/\mathsf{m}_0} \leq |a'_{2,r} + (a_{4,r} - a_{1,r})r_1 - a'_{3,r}r_1^2| \leq 2|a'_{3,r}|.$$

Hence, since w'_r is small, $|c^2z_r|\eta \ll |a^2z_r| \ll |c^2z_r|$. On the other hand, using $r = r_0$, we already established $|a|^2 \leq e^{(-\mathsf{k}+2)t}$ and $|c| \gg 1$, thus $|a^2z_r| \ll e^{(-\mathsf{k}+2)t}|c^2z_r|$, which is a contradiction, see (B.26) again.

Altogether, we conclude that Case 1 cannot occur.

Case 2. $\mathbf{L}$ is not commutative. In other words, there are $r, r' \in I_{\mathrm{bad}}$ so that γ_r and $\gamma_{r'}$ do not commute.

Recall from (B.38) that

$$\gamma_r.g_3^{-1}v_H = g_3^{-1}v_H \quad \text{and} \quad \gamma_{r'}.g_3^{-1}v_H = g_3^{-1}v_H.$$

where $\|g_2 - g_3\| \leq e^{(-D'+A_{15}\mathsf{k}_2)t}$.

In view of Lemma B.9, thus, we have $Hg_3\Gamma$ is periodic and

$$\mathrm{vol}(Hg_3\Gamma) \leq C_7\eta^{-D_2A_{17}}\big(\max\{\|\gamma_r^{\pm1}\|, \|\gamma_{r'}^{\pm1}\|\}\big)^{A_{16}} \leq C_7e^{(1+A_{16}\mathsf{k}_1)t}.$$

Then for t large enough,

$$\mathrm{vol}(Hg_3\Gamma) \le e^{(2+A_{16})\mathsf{k}_1 t} \quad \text{and} \quad d_X(g_2\Gamma, g_3\Gamma) \le e^{(-D'+A_{15}\mathsf{k}_2)t},$$

where $D' = \frac{D}{\mathsf{m}+1} - \hat{\mathsf{m}}_1 - 1$.

Since $g_2\Gamma = x_2 = a_t u_{r_1} x_1$, part (2) in the proposition holds with $x' = (a_t u_{r_1})^{-1} g_3\Gamma$ and $D_1 = \max\{2 + A_{16}\mathsf{k}_1, 1 + \hat{\mathsf{m}}_1 + A_{15}\mathsf{k}_2\}$ so long as t is large enough. $\square$

Appendix C. Projection theorems

We begin by recalling a theorem of Gan, Guo, and Wang [GGW22]. Let

$$\xi(r) = \left(\tfrac{r}{1!}, \tfrac{r^2}{2!}, \ldots, \tfrac{r^n}{n!}\right) \subset \mathbb{R}^n.$$

For all $r \in [0,1]$ and all $1 \le d \le n$, let $\pi_{r,d} : \mathbb{R}^n \to \mathbb{R}^d$ denote the orthogonal projection into

$$\mathrm{Span}\{\xi'(r), \xi^{(2)}(r), \ldots, \xi^{(d)}(r)\}.$$

The following theorem follows from [GGW22, Thm.2.1], combined with a finitary adaptation of the argument presented in [GGW22, §2].

C.1. **Theorem.** *Let $1 \le d \le n$ and let $0 < \alpha \le d$. Let ρ be the uniform measure on a finite set $\Theta \subset B_{\mathbb{R}^n}(0,1)$ satisfying*

$$\rho(B_{\mathfrak{r}}(w,b) \cap \Theta) \le \Upsilon b^{\alpha} \qquad \text{for all } w \text{ and all } b \ge b_1$$

where $\Upsilon \ge 1$ and $0 < b_1 \le 1$.

Let $0 < \mathsf{c} < 10^{-4}\alpha$. For every $b \ge b_1$, there exists a subset $J_b \subset [0,1]$ with $|[0,1] \setminus J_b| \le C_{\mathsf{c}} b^{\star \mathsf{c}^2}$ so that the following holds. Let $r \in J_b$, then there exists a subset $\Theta_{b,r} \subset \Theta$ with

$$\rho(\Theta \setminus \Theta_{b,r}) \le C_{\mathsf{c}} b^{\star \mathsf{c}^2}$$

such that for all $w \in \Theta_{b,r}$, we have

$$\rho\big(\{w' \in \Theta : \|\pi_{r,d}(w) - \pi_{r,d}(w')\| \le b\}\big) \le C_{\mathsf{c}} \Upsilon b^{\alpha - \mathsf{c}}$$

where the implied constants are absolute.

Proof. Since $1 \le d \le n$ is fixed throughout the argument, we will simplify the notation by writing π_r instead of $\pi_{r,d}$.

For every $r \in [\frac{1}{2}, 1]$ and all $w \in \Theta$, define

$$m^b(\pi_r(w)) = \rho(\{w' : \|\pi_r(w) - \pi_r(w')\| \le b\}).$$

Let $E_1, E_2, \ldots$ be large constants to be specified later. Put $\varepsilon = \mathsf{c}/E_1$. For all $r \in [\frac{1}{2}, 1]$, set

$$\Theta_{\mathrm{Exc}}(r) = \{w : m^b(\pi_r(w)) \ge \Upsilon b^{\alpha - E_1\varepsilon}\}.$$

Suppose, contrary to the claim in Theorem C.1 that there exists a subset $I_{\mathrm{Exc}} \subset [\frac{1}{2}, 1]$ with $|I_{\mathrm{Exc}}| \ge E_2 b^{\varepsilon^2/2}$ so that for all $r \in I_{\mathrm{Exc}}$, we have

$$\rho(\Theta_{\mathrm{Exc}}(r)) \ge E_2 b^{\varepsilon^2/2}. \tag{C.1}$$

We will get a contradiction with [GGW22, Thm. 2.1], provided that E_1 and E_2 are large enough.

First note that, for every $r \in I_{\mathrm{Exc}}$, the number of b-boxes $\{\mathsf{B}_{i,r}\}$ required to cover $\pi_r(\Theta_{\mathrm{Exc}}(r))$ is $\le E_3\Upsilon^{-1}b^{-\alpha+E_1\varepsilon}$. Let $\mathbb{T}_{i,r} = \pi_r^{-1}(\mathsf{B}_{i,r}) \cap B_{\mathbb{R}^n}(0,1)$, and put $\mathcal{T}_r = \{\mathbb{T}_{i,r}\}$. Note that

$$\#\mathcal{T}_r \le E_3\Upsilon^{-1}b^{-\alpha+E_1\varepsilon}. \tag{C.2}$$

Let $\Lambda_b \subset I_{\mathrm{Exc}}$ be a maximal b-separated subset, and extend this to a maximal b-separated subset $\hat{\Lambda}_b$ of $[\frac{1}{2},1]$. Equip $\hat{\Lambda}_b \times \Theta$ with the product measure $\rho \times \sigma$, where σ denotes the uniform measure on $\hat{\Lambda}_b$. Let

$$\begin{aligned} F &= \{(r,w) \in \Lambda_b \times \Theta : m^b(\pi_r(w)) \ge \Upsilon b^{\alpha - E_1\varepsilon}\} \\ &= \{(r,w) \in \Lambda_b \times \Theta : w \in \Theta_{\mathrm{Exc}}(r)\}. \end{aligned}$$

In view of (C.1) and $|I_{\mathrm{Exc}}| \ge E_2 b^{\varepsilon^2/2}$, we have $\sigma \times \rho(F) \ge E_2^2 b^{\varepsilon^2}$.

For every $w \in \Theta$, let $F_w = \{r \in \Lambda_b : (r,w) \in F\}$, and set

$$\Theta' = \{w \in \Theta : \sigma(F_w) \ge E_2 b^{\varepsilon^2}\}.$$

Then, using Fubini's theorem, we conclude that $\rho(\Theta') \ge \frac{1}{2}E_2^2 b^{\varepsilon^2}$.

The above definitions thus imply

$$\sum_{r\in\Lambda_b} 1_{\mathbb{T}_r}(w) \ge E_3 b^{\varepsilon^2-1} \qquad \text{for all } w \in \Theta'$$

where $E_3 = O(E_2)$, for an absolute implied constant.

Let $\rho' = \rho|_{\Theta'}$. Applying [GGW22, Thm. 2.1] with ε^2, ρ' and $\{\mathcal{T}_r : r \in \Lambda_b\}$,

$$\begin{aligned} \sum_{r\in\Lambda_b} \#\mathcal{T}_r &\ge E_4(n,\mathsf{c},\alpha)\Upsilon^{-1}\rho'(\mathbb{R}^n)b^{-1-\alpha+E\varepsilon} \\ &\ge \tfrac{1}{2}E_2^2E_4(n,\mathsf{c},\alpha)\Upsilon^{-1}b^{\varepsilon^2}b^{-1-\alpha+E\varepsilon} \end{aligned}$$

where $E = 10^{10n}$ and in the second line we used $\rho'(\mathbb{R}^n) = \rho(\Theta') \ge \frac{1}{2}E_2^2 b^{\varepsilon^2}$.

Thus there exists some $r \in \Lambda_\delta$ so that

$$\#\mathcal{T}_r \ge \tfrac{1}{2}\Upsilon^{-1}E_4(n,\mathsf{c},\alpha)E_2^2 b^{-\alpha+(D+1)\varepsilon}. \tag{C.3}$$

Now comparing (C.3) and (C.2) we get a contradiction so long as E_1 is large enough and b is small enough. The proof is complete. $\square$

Recall that the n-dimensional ($n \ge 2$) irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ can be normalized so that for all $w \in \mathbb{R}^n$,

$$u_r w = \big(w\cdot\xi'(r), w\cdot\xi^{(2)}(r), \cdots, w\cdot\xi^{(n)}(r)\big)$$

Recall also that $\mathfrak{r} \simeq \mathbb{R}^{2\mathsf{m}+1}$ is an irreducible representation of $\mathrm{Ad}(H)$ where

- $\mathsf{m} = 1$ if $\mathbf{G}$ is isogeneous to $\mathrm{SO}(3,1)$ or $\mathrm{SL}_2 \times \mathrm{SL}_2$
- $\mathsf{m} = 2$ if $\mathbf{G}$ is isogeneous to SL_3 or $\mathrm{SU}(2,1)$
- $\mathsf{m} = 3$ if $\mathbf{G}$ is isogeneous to Sp_4
- $\mathsf{m} = 5$ if $\mathbf{G}$ is isogeneous to $\mathbf{G}_2$

Theorem C.1 thus applies to the adjoint action of u_r on these spaces. For every $1 \leq d \leq 2\mathsf{m}+1$, let $\mathfrak{r}_d$ denote the space spanned by vectors with weight $\mathsf{m}, \ldots, \mathsf{m}-d+1$. Let $\pi_d : \mathfrak{r} \to \mathfrak{r}_d$ denote the orthogonal projection.

C.2. **Theorem.** *Let $0 < \alpha, d \leq 2\mathsf{m}+1$, $d \in \mathbb{Z}$, and $0 < b_1 \leq 1$. Let ρ be the uniform measure on a finite set $\Theta \subset B_{\mathfrak{r}}(0,1)$ with*

$$\rho(B_{\mathfrak{r}}(w,b) \cap \Theta) \leq \Upsilon b^\alpha \qquad \textit{for all } w \textit{ and all } b \geq b_1$$

where $\Upsilon \geq 1$.

Let $0 < \mathsf{c} < 10^{-4}\alpha$. For every $b \geq b_1$, there exists a subset $J_b \subset [0,1]$ with $|[0,1] \setminus J_b| \leq C'_{\mathsf{c}} b^{\star \mathsf{c}^2}$ so that the following holds. Let $r \in J_b$, then there exists a subset $\Theta_{b,r} \subset \Theta$ with

$$\rho(\Theta \setminus \Theta_{b,r}) \leq C_{\mathsf{c}} b^{\star \mathsf{c}^2}$$

such that for all $w \in \Theta_{b,r}$ we have

$$\rho\big(\{w' \in \Theta : \|\pi_d(\mathrm{Ad}(u_r)w) - \pi_d(\mathrm{Ad}(u_r)w')\| \leq b\}\big) \leq C_{\mathsf{c}} \Upsilon b^{\min(\alpha,d)-\mathsf{c}}.$$

Appendix D. Measures and partitions of unity

In this appendix we collect some of the results from [LMW22, §6–8] which were used in this paper.

Regular tree decomposition. Let us recall the following discussion from [LMW22, §6] see also [BFLM11, Lemma 5.2]. Let $t, \mathsf{D}_0 \geq 1$ and $0 < \varepsilon < 1$ be three parameters: t is large and arbitrary, D_0 is moderate and fixed, and ε is small and fixed; in particular, our estimates are allowed to depend on D_0 and ε, but not on t. Let $\beta = e^{-\kappa t}$ for some κ satisfying $0 < \kappa(\mathsf{D}_0+1) \leq 10^{-6}\varepsilon$.

Let $F \subset B_{\mathfrak{r}}(0,1)$ satisfy that

$$e^{t/2} \leq \#F \leq e^{\mathsf{D}_0 t},$$

and assume that

$$\mathcal{G}^{(\alpha)}_{F,\delta}(w) \leq \Upsilon \qquad \text{for all } w \in F \tag{D.1}$$

where $\Upsilon > 0$ satisfies the following

$$\Upsilon \leq e^{(\mathsf{D}_0+1)t}. \tag{D.2}$$

Fix $\mathsf{L} \in \mathbb{N}$, large enough, so that both of the following hold

$$2^{-\mathsf{L}}(\mathsf{D}_0+1) < \kappa/100 \quad \text{ and } \quad (4\mathsf{m}+2)\mathsf{L} < 2^{\kappa \mathsf{L}/100}. \tag{D.3}$$

Define $k_0 := \lfloor (-\log_2 \beta)/\mathsf{L} \rfloor$ and $k_1 := \lceil (1 + \alpha^{-1}\log_2 \Upsilon)/\mathsf{L} \rceil + 1$; note that

$$2^{(\mathsf{L}k_1 - 1)\alpha} > \Upsilon. \tag{D.4}$$

For every $k_0 \leq k \leq k_1$, let $\mathcal{C}_{\mathsf{L}k}$ denote the collection of $2^{-\mathsf{L}k}$-cubes in $\mathfrak{r}$.

D.1. **Lemma.** *For all large enough t, we can write $F = F' \bigcup (\bigcup_{i=1}^N F_i)$ (a disjoint union) with*

$$\#F' < \beta^{1/4} \cdot (\#F) \quad \textit{and} \quad \#F_i \geq \beta^2 \cdot (\#F)$$

so that the following hold.

(1) For every i and every $k_0 - 10 \le k \le k_1$, there exists some τ_{ik} so that for every cube $\mathsf{C} \in \mathcal{C}_{\mathsf{L}k}$ we have

$$2^{\mathsf{L}(\tau_{ik}-2)} \le \#F_i \cap \mathsf{C} \le 2^{\mathsf{L}\tau_{ik}} \quad \textit{or} \quad F_i \cap \mathsf{C} = \emptyset. \tag{D.5}$$

(2) For every i, we have

$$\mathcal{G}^{(\alpha)}_{F_i,\delta}(w) \le \beta^{-2\mathsf{m}-2}\Upsilon \qquad \textit{for all } w \in F$$

Proof. Part (1) is proved in [LMW22, Lemma 6.4].

For part (2) see [LMW22, Lemma 6.5]. □

Covering lemmas. Fix $0 < \eta \le 0.01$ and let $\beta = \eta^2$. For $m \ge 0$, we introduce the shorthand notation Q^H_m for

$$\mathsf{Q}^H_{\eta,\beta^2,m} = \left\{u^-_s : |s| \le \beta^2 e^{-m}\right\} \cdot \left\{a_\tau : |\tau| \le \beta^2\right\} \cdot U_\eta, \tag{D.6}$$

where for every $\delta > 0$, let $U_\delta = \{u_r : |r| \le \delta\}$, see (3.6).

Define $\mathsf{Q}^G_m \subset G$ by thickening Q^H_m in the transversal direction as follows:

$$\mathsf{Q}^G_m := \mathsf{Q}^H_m \cdot \exp(B_{\mathfrak{r}}(0, 2\beta^2)). \tag{D.7}$$

D.2. **Lemma.** *For every $m \ge 0$, there exists a covering*

$$\left\{\mathsf{Q}^G_m.y_j : j \in \mathcal{J}_m, y_j \in X_{3\eta/2}\right\}$$

of $X_{2\eta}$ with multiplicity K, depending only on X. In particular, $\#\mathcal{J}_m \ll \eta^{-1}\beta^{-4\mathsf{m}-6}e^m$.

Proof. This is proved in [LMW22, Lemma 7.1], we recall the set up to explicate the bound claimed here. There exists a covering

$$\left\{\left(\mathsf{B}^{s,H}_{\beta^2} \cdot \mathsf{B}_\eta \cdot \exp(B_{\mathfrak{r}}(0, \beta^2))\right).\hat{y}_k : k \in \mathcal{K}, \hat{y}_k \in X_{2\eta}\right\}$$

of $X_{2\eta}$ with multiplicity $O(1)$ depending only on X.

Let us write $\bar{\mathsf{B}}^G_{\eta,\beta^2} = \mathsf{B}^{s,H}_{\beta^2} \cdot \mathsf{B}_\eta \cdot \exp(B_{\mathfrak{r}}(0, \beta^2))$. Then

$$\left(\bar{\mathsf{B}}^G_{0.1\eta,0.1\beta^2}\right)^{-1} \cdot \left(\bar{\mathsf{B}}^G_{0.1\eta,0.1\beta^2}\right) \subset \left(\bar{\mathsf{B}}^G_{c\eta,c\beta^2}\right), \tag{D.8}$$

where c depends only on m, see Lemma 3.2.

Let $\{\hat{y}_k \in X_{2\eta} : k \in \mathcal{K}\}$ be maximal with the following property

$$\bar{\mathsf{B}}^G_{0.01\eta,0.01\beta^2}.\hat{y}_i \cap \bar{\mathsf{B}}^G_{0.01\eta,0.01\beta^2}.\hat{y}_j = \emptyset \quad \text{ for all } i \ne j.$$

In view of (D.8) thus $\{\bar{\mathsf{B}}^G_{\eta,\beta^2}.\hat{y}_k : k \in \mathcal{K}\}$ covers $X_{2\eta}$ with multiplicity $O(1)$. Since $m_G(\bar{\mathsf{B}}^G_{\eta,\beta^2}) \asymp \eta\beta^{4\mathsf{m}+6}$, we that $\mathcal{K} \ll \eta^{-1}\beta^{-4\mathsf{m}-6}$.

The rest of the proof goes through as in [LMW22, Lemma 7.1]. □

Boxes and complexity. Let $\mathsf{prd} : \mathbb{R}^3 \to H$ be the map

$$\mathsf{prd}(s, \tau, r) = u_s^- a_\tau u_r.$$

A subset $\mathsf{D} \subset H$ will be called a *box* if there exist intervals $I^\bullet \subset \mathbb{R}$ (for $\bullet = \pm, 0$) so that

$$\mathsf{D} = \mathsf{prd}(I^- \times I^0 \times I^+).$$

We say $\Xi \subset H$ has complexity bounded by L (or at most L) if $\Xi = \bigcup_1^L \Xi_i$ where each Ξ_i is a box.

For every interval $I \subset \mathbb{R}$, let $\partial I = \partial_{100\eta|I|} I$ (recall that $\eta = \beta^{1/2}$), and put $\mathring{I} = I \setminus \partial I$. Given a box $\mathsf{D} = \mathsf{prd}(I^- \times I^0 \times I^+)$, we let

$$\mathring{\mathsf{D}} = \mathsf{mul}\big(\mathring{I}^- \times \mathring{I}^0 \times \mathring{I}^+\big) \quad \text{and} \tag{D.9a}$$

$$\partial\mathsf{D} = \mathsf{D} \setminus \mathring{\mathsf{D}}. \tag{D.9b}$$

More generally, if $\mathsf{D} = \mathsf{prd}(I^- \times I^0 \times I^+)$ is a box, and $\Xi \subset \mathsf{D}$ has complexity bounded by L, we define $\partial\Xi := \bigcup \partial\Xi_i$ and

$$\mathring{\Xi}_{\mathsf{D}} := \bigcup \mathring{\Xi}_i \tag{D.10}$$

where the union is taken over those i so that $\Xi_i = \mathsf{prd}(I_i^- \times I_i^0 \times I_i^+)$ with $|I_i^\bullet| \geq 100\eta |I^\bullet|$ for $\bullet = \pm, 0$.

D.3. **Admissible measures.** Let $\Lambda > 0$. Let $\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\}$. A probability measure $\mu_{\mathcal{E}}$ on $\mathcal{E}$ is said to be Λ-*admissible* if

$$\mu_{\mathcal{E}} = \frac{1}{\sum_{w\in F} \mu_w(X)} \sum_{w\in F} \mu_w$$

where for every $w \in F$, μ_w is a measure on $\mathsf{E}.\exp(w)y$ satisfying that if $\mathsf{h}\exp(w)y$ is in the support of μ_w

$$\mathrm{d}\mu_w(\mathsf{h}\exp(w)y) = \lambda\varrho_w(\mathsf{h})\,\mathrm{d}m_H(\mathsf{h}) \quad \text{where } 1/\Lambda \leq \varrho_w(\bullet) \leq \Lambda,$$

for some $\lambda > 0$ independent of $w \in F$. Moreover, there is a subset $\mathsf{E}_w = \bigcup_{i=1}^{\Lambda} \mathsf{E}_{w,i} \subset \mathsf{E}$ so that

(1) $\mu_w\big((\mathsf{E} \setminus \mathsf{E}_w).\exp(w)y\big) \leq \Lambda\beta\mu_w(\mathsf{E}.\exp(w)y)$,
(2) The complexity of $\mathsf{E}_{w,i}$ is bounded by Λ for all i, and
(3) $\mathrm{Lip}(\varrho_w|_{\mathsf{E}_{w,i}}) \leq \Lambda$ for all i.

Convex combination of measures. The following lemma was used several times in the paper, in particular, in the proof of Lemma 7.2.

D.4. **Lemma.** *Let $\ell, \mathsf{d} > 0$. Assume that $e^{-\ell/2} \leq \beta$ and that $h \mapsto hx$ is injective on $\mathsf{E} \cdot a_t \cdot \{u_r : r \in [0,1]\}$. Let*

$$\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\} \subset X_\eta$$

be equipped with an admissible measure $\mu_{\mathcal{E}}$. For every $r \in [0,1]$,

$$\int \varphi(a_{\mathsf{d}} u_s a_\ell u_r z)\,\mathrm{d}\mu_{\mathcal{E}}(z) = \sum_i c_i \int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z) + O(\beta^\star\,\mathrm{Lip}(\varphi)), \tag{D.11}$$

where $\mathcal{E}_i = \mathsf{E}.\{\exp(w)y_i : w \in F_i\} \subset X_\eta$ *and* $F_i \subset B_{\mathfrak{r}}(0, \beta)$ *satisfies*
(1) $\beta^{4\mathsf{m}+5} \cdot (\#F) \le F_i \le \beta^{4\mathsf{m}+4} \cdot (\#F)$, *and*
(2) $\mathsf{Q}_\ell^H.\exp(w)y_i \subset a_\ell u_r \mathcal{E}$ *for all* $w \in F_i$, *where*

$$\mathsf{Q}_\ell^H = \big\{u_s^- : |s| \le \beta^2 e^{-\ell}\big\} \cdot \{a_\tau : |\tau| \le \beta^2\} \cdot \{u_r : |r| \le \eta\}.$$

The implied constant depends on X.

Proof. This is proved in [LMW22, Lemma 8.9]. We provide road map to the argument to elucidate the claims made above regarding $F_{i,r}$.

To see part (1), and with the notation as in [LMW22, Lemma 8.7], we have $c_{i,r}^j \asymp N_{i,r}^j\big(e^{-\ell}\beta^4\eta\big) \cdot (\#F)^{-1}$. Therefore, if $c_{i,r}^j \ge \beta^{4\mathsf{m}+8}e^{-\ell}$, then

$$N_{i,r}^j \ge \beta^{4\mathsf{m}+4} \cdot (\#F).$$

Moreover, by Lemma D.2, $\#\mathcal{J}_\ell \ll \eta^{-1}\beta^{-4\mathsf{m}-6}e^\ell \le \beta^{-4\mathsf{m}-7}e^\ell$. Thus,

$$\sum_{c_{i,r}^j < \beta^{4\mathsf{m}+8}e^{-\ell}} c_{i,r}^j \le \beta.$$

One now defines $F_i := F_{i,r}^{j,m}$ where again we used the notation which is used after the conclusion of the proof of [LMW22, Lemma 8.7]. Then

$$\beta^{4\mathsf{m}+5} \cdot (\#F) \le \#F_i \le \beta^{4\mathsf{m}+4} \cdot (\#F),$$

see [LMW22, eq. (8.18)].

As it is argued in the proof of [LMW22, Lemma 8.9], the claim in (D.11) holds with $\mathcal{E}_i = \mathsf{E}.\{\exp(w)y_i : w \in F_i\}$, see [LMW22, eq. (8.21)].

Part (2) above is [LMW22, eq. (8.14)]. □

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