Some $\varepsilon - \delta$ proofs
Thanks to Rob Bayer for this handout

Basic Strategy

1. Write down what you’re going to prove:
   - $\forall \varepsilon > 0 \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
   - $\forall \varepsilon > 0 \exists N : x > N \Rightarrow |f(x) - L| < \varepsilon$
   - $\forall M \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) > M$
   - $\forall M \exists N : x > N \Rightarrow f(x) > M$

2. Find $\delta$ or $N$
   (a) Locate $|x - a|$ or $x$
   (b) Turn everything else into constants by assuming $|x - a| < 1$ (or $\frac{1}{2}$ if that doesn’t work) or $x > 1$
   (c) Then $\delta = \min(1, \text{the } \delta \text{ you found})$ or $N = \max(1, \text{the } N \text{ you found})$ (use min for $-\infty$)

3. Prove your $\delta$ or $N$ works. This basically just means doing the same things over again but in the opposite order, and justifying each replacement based on your choice of $\delta$ or $N$

Examples

In addition to the examples in the book, here’s a few for reference:

1. $\lim_{x \to 2} x^2 + x = 6$
   We will show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - 2| < \delta \Rightarrow |x^2 + x - 6| < \varepsilon$
   Note $|x^2 + x - 6| = |x - 2| \cdot |x + 3|$, so we just need to turn $|x + 3|$ into a constant.
   If $|x - 2| < 1$, then $1 < x < 3$, so $4 < x + 3 < 6$ and thus $|x + 3| < 6$
   So $|x^2 + x - 6| < 6|x - 2|$ so we’ll guess $\delta = \min(1, \varepsilon/6)$
   **Claim:** $\forall \varepsilon > 0, 0 < |x - 2| < \delta = \min(1, \varepsilon/6) \Rightarrow |x^2 + x - 6| < \varepsilon$
   **Proof:**
   If $|x - 2| < \delta$, then $|x - 2| < 1$, so we know by previous work that $|x + 3| < 6$. Since $|x - 2| < \delta$ we also know $|x - 2| < \varepsilon/6$. Then we have:
   $|x^2 + x - 6| = |x - 2||x + 3| < 6|x - 2| < 6\frac{\varepsilon}{6} = \varepsilon$
   as was to be shown.

2. $\lim_{x \to \infty} \sqrt{x + 4} = \infty$
   We will show that for all $\forall M$ there exists $\exists N$ such that $\forall x > N \Rightarrow \sqrt{x + 4} > M$
   Let $M$ be given. Then we want $\sqrt{x + 4} > M$, which will happen if $x + 4 > M^2$, which will happen if $x > M^2 - 4$, so we guess $N = M^2 - 4$
Claim: \( \forall M, \ x > N^2 - 4 \Rightarrow \sqrt{x+4} > M \)

Proof: Suppose \( x > N = M^2 - 4 \). Then \( x + 4 > M^2 \), so \( \sqrt{x+4} > |M| \geq M \), as was to be shown.

3. \[ \lim_{x \to \infty} \frac{\sin x}{x} = 0 \]

We will show that \( \forall \varepsilon > 0 \exists N : x > N \Rightarrow |\frac{\sin x}{x} - 0| < \varepsilon \)

Let’s find \( N \): Since \( -1 < \sin x < 1 \), \( \left| \frac{\sin x}{x} \right| < \left| \frac{1}{x} \right| \). In order to drop the abs value bars, let’s assume \( x > 1 \) To get \( \frac{1}{x} < \varepsilon \), we need \( x > \frac{1}{\varepsilon} \), so we’ll guess \( N = \max(1, \frac{1}{\varepsilon}) \)

Claim: \( \forall \varepsilon > 0, \ x > \max(1, \frac{1}{\varepsilon}) \Rightarrow |\frac{\sin x}{x} - 0| < \varepsilon \)

Proof: Suppose \( x > \max(1, \frac{1}{\varepsilon}) \). Then we have \( x > 1 \), so \( \left| \frac{1}{x} \right| = \frac{1}{x} \) and

\[ \left| \frac{\sin x}{x} \right| < \frac{1}{x} < \frac{1}{1/\varepsilon} = \varepsilon \]

as was to be shown. (Note that since \( x > \frac{1}{\varepsilon} \), replacing the denominator \( x \) by \( \frac{1}{\varepsilon} \) makes the whole thing bigger.

4. \[ \lim_{x \to 0} \frac{1}{x^2} = \infty \]

We’ll show \( \forall M \exists \delta : 0 < |x - 0| < \delta \Rightarrow \frac{1}{x^2} > M \)

Since we’re trying to show this limit goes to \( +\infty \), we can safely assume \( M > 0 \). Then since we want \( \frac{1}{x^2} > M \), we can cross multiply (note all terms are positive) to get \( \frac{1}{M} > x^2 \). Taking square roots gives \( \sqrt{1/M} > |x| \), we’ll guess \( \delta = \sqrt{1/M} \)

Claim: \( \forall M, 0 < |x| < \sqrt{1/M} \Rightarrow \frac{1}{x^2} > M \)

If \( |x| < \sqrt{1/M} \), then squaring both sides gives \( x^2 < 1/M \), so cross multiplying (again: we may assume \( M > 0 \)) gives \( M < \frac{1}{x^2} \), as was to be shown.

5. \( f(x) = \frac{1}{x} \) is continuous at 1

We’ll show that \( \lim_{x \to 1} f(x) = f(1) = 1 \) by showing that \( \forall \varepsilon > 0 \exists \delta > 0 : 0 < |x - 1| < \delta \Rightarrow |\frac{1}{x} - 1| < \varepsilon \)

\[ \left| \frac{1}{x} - 1 \right| = \left| \frac{1-x}{x} \right| = \left| \frac{-(x-1)}{x} \right| = \left| \frac{x-1}{x} \right| \]

If \( |x-1| < 1 \), we get \( 0 < x < 2 \), so \( 0 < |x| < 2 \), and \( \frac{1}{2} < \frac{1}{|x|} < \infty \), which is not a helpful inequality. So let’s try \( |x-1| < \frac{1}{2} \) instead:

Then \( \frac{1}{2} < x < \frac{3}{2} \), so \( \frac{2}{3} < \frac{1}{|x|} < 2 \).

Thus \( \frac{|x-1|}{|x|} < 2|x-1| \) and we’ll try \( \delta = \min(\frac{1}{2}, \varepsilon/2) \)

Claim: \( \forall \varepsilon > 0, 0 < |x - 1| < \min(\frac{1}{2}, \varepsilon/2) \Rightarrow |\frac{1}{x} - 1| < \varepsilon \)

Proof: If \( |x-1| < \min(\frac{1}{2}, \varepsilon/2) \), then we know \( |x-1| < \frac{1}{2} \), so by previous work, \( \frac{1}{2} < \frac{1}{|x|} < 2 \). We also know that \( |x-1| < \varepsilon/2 \), so we can do the following:
\[
\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|} < 2|x - 1| < 2\epsilon/2 = \epsilon
\]

as was to be shown. Thus, \( \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1 \), so \( \frac{1}{x} \) is continuous at 1.

6. \( \lim_{x \to -1^+} \frac{5}{(x+1)^3} = \infty \)

We’ll show \( \forall M \exists \delta : 0 < x - (-1) < \delta \Rightarrow \frac{5}{(x+1)^3} > M \)

Note: the assumption that \( 0 < x - (-1) \) guarantees that \( x > -1 \).

Let’s find \( \delta \):

We want \( \frac{5}{(x+1)^3} > M \), which will happen when \( \frac{5}{M} > (x+1)^3 \). Note that since \( x > -1 \) and \( M > 0 \), multiplying and dividing by them did not change the direction of the inequality. Thus, we want \( \sqrt[3]{5/M} > x + 1 \), which means we want \( x - (-1) < \sqrt[3]{5/M} \), so we choose \( \delta = \sqrt[3]{5/M} \).

**Claim:** \( \forall M > 0, 0 < x - (-1) < \sqrt[3]{5/M} \Rightarrow \frac{5}{(x+1)^3} > M \)

**Proof:** Since \( 0 < x + 1 < \sqrt[3]{5/M} \), we can cube both sides to get \( (x + 1)^3 < \frac{5}{M} \). By our assumption that \( 0 < x + 1 \) and \( M > 0 \) we can multiply and divide to get \( M < \frac{5}{(x+1)^3} \) as was to be shown.