Wave Maps

A brief introduction to wave maps:

- **Definition**: Formally, wave maps are critical points of the Lagrangian

\[ \mathcal{L}(u, \partial u) = \int_{\mathbb{R}^{1+d}} \eta^{\alpha\beta} \langle \partial_\alpha u, \partial_\beta u \rangle_g \, dt \, dx \]

where \( u : (\mathbb{R}^{1+d}, \eta) \to (M, g) \). Here, \( \eta \) is the Minkowski metric on \( \mathbb{R}^{1+d} \) and \( (M, g) \) is a Riemannian manifold.

- **Intrinsic Formulation**: Critical points of \( \mathcal{L} \) satisfy the Euler-Lagrange equation

\[ \eta^{\alpha\beta} D_\alpha \partial_\beta u = 0 \]

- **Extrinsic Formulation**: If \( M \hookrightarrow \mathbb{R}^N \) is embedded, critical points are characterized by

\[ \Box u \perp T_u M \]
The Cauchy problem

Cauchy problem:

- **Intrinsic Formulation**: In local coordinates on \((M, g)\), the Cauchy problem for wave maps is

\[
\Box u^k = -\eta^{\alpha\beta} \Gamma^k_{ij}(u) \partial_\alpha u^i \partial_\beta u^j \\
(u, \partial_t u)|_{t=0} = (u_0, u_1)
\]

where \(\Gamma^k_{ij}\) are the Christoffel symbols on \(TM\).

- **Extrinsic Formulation**: In the embedded case, the Cauchy problem becomes

\[
\Box u = \eta^{\alpha\beta} S(u) (\partial_\alpha u, \partial_\beta u) \\
(u, \partial_t u)|_{t=0} = (u_0, u_1)
\]

where \(S\) is the second fundamental form of the embedding.
Energy conservation and scaling

- **Conservation of energy**: Wave maps exhibit a conserved energy

\[
E(u, \partial_t u)(t) = \int_{\mathbb{R}^d} \left( |\partial_t u|^2_g + |\nabla u|^2_g \right) dx = \text{const.}
\]

- **Scaling invariance**: Wave maps are invariant under the scaling

\[ u(t, x) \mapsto u(\lambda t, \lambda x). \]

- **Criticality**: The scaling invariance implies that the Cauchy problem is \( H^s \times H^{s-1} \) critical for \( s = \frac{d}{2} \), energy critical when \( d = 2 \) and energy supercritical for \( d > 2 \).
Equivariant wave maps: In the presence of symmetries, e.g., $M = S^d$, one can require

$$u \circ \rho = \rho^\ell \circ u$$

where $\rho \in SO(d)$ acts on $\mathbb{R}^d$ (resp. $S^d$) by rotation. The action on $S^d$ is rotation is about a fixed axis.

Foundational works:

- Shatah (1988): finite time blow-up (self-similar) for wave maps $u : \mathbb{R}^{1+d} \rightarrow S^d$ for $d \geq 3$.
- Shatah, Tahvildar-Zadeh (1994): Local theory, generalization of Shatah blow-up to rotationally symmetric, non-convex targets.
**Issue at hand:** Global well-posedness and scattering for 3d equivariant wave maps exterior to a ball.

**Exterior model:** We consider

\[ u : \mathbb{R}_t \times (\mathbb{R}^3 \setminus B) \rightarrow S^3 \]

with the Dirichlet boundary condition \( u(\partial B) = \text{north pole} \), and \( B = B(0, 1) \). Fixing equivariance class \( \ell = 1 \) we can write

\[ u : (t, r, \omega) \mapsto (\psi(t, r), \omega) \mapsto (\sin(\psi(t, r)) \cdot \omega, \cos(\psi(t, r))) \]

where \((r, \omega)\) are polar coordinates on \( \mathbb{R}^3 \) and \( \psi \) measures the azimuth angle from the north pole on \( S^3 \).
1-equivariant exterior Cauchy problem

Cauchy problem in the exterior setting:

\[
\begin{align*}
\psi_{tt} - \psi_{rr} - \frac{2}{r} \psi_r + \frac{\sin(2\psi)}{r^2} &= 0 \\
\psi(t, 1) &= 0 \quad \forall t \geq 0 \\
\vec{\psi}(0) := (\psi, \psi_t)|_{t=0} &= (\psi_0, \psi_1)
\end{align*}
\] (1)

Conserved energy:

\[
\mathcal{E}(\vec{\psi}) = \int_1^\infty \left[ \frac{1}{2} (\psi_t^2 + \psi_r^2) + \frac{\sin^2\psi}{r^2} \right] r^2 \, dr
\]
1-equivariant exterior Cauchy problem

Cauchy problem in the exterior setting:

\[ \psi_{tt} - \psi_{rr} - \frac{2}{r} \psi_r + \frac{\sin(2\psi)}{r^2} = 0 \]  

(1)

\[ \psi(t, 1) = 0 \quad \forall t \geq 0 \]

\[ \vec{\psi}(0) := (\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1) \]

Conserved energy:

\[ E(\vec{\psi}) = \int_1^{\infty} \left[ \frac{1}{2} (\psi_t^2 + \psi_r^2) + \frac{\sin^2 \psi}{r^2} \right] r^2 \, dr \]

- Finite energy + continuous dependence on a time interval \( I \)
  \[ \Rightarrow \psi(t, \infty) = n\pi \] for some \( n \in \mathbb{N} \), for every \( t \in I \).
  \[ \Rightarrow \] every wave map has a fixed topological degree.

- The natural space for the solution in the energy class defined by \( n = 0 \) is \( \mathcal{H} := \dot{H}_0^1 \times L^2(1, \infty) \) with the norm
  \[ \|\vec{\psi}\|_{\mathcal{H}}^2 = \int_1^{\infty} (\psi_t^2 + \psi_r^2) r^2 \, dr \]


Scattering for wave maps exterior to a ball
Harmonic Maps

Why is the exterior 3d problem interesting? Removing a ball gives rise to a family of nontrivial harmonic maps $Q_n$ indexed by the topological degree $n$.

Harmonic maps: A “degree $n$” harmonic map in this context is a solution to the following problem:

$$Q_{rr} + \frac{2}{r} Q_r = \frac{\sin(2Q)}{r^2}$$

$$Q(1) = 0, \quad Q(\infty) = n\pi$$

- $n = 0$: In the zero topological class we have $Q \equiv 0$.
- $n \geq 1$: After the change of variables $t = \log(r)$, $x(t) := Q(r)$, set $y = \dot{x}$ and (2) becomes the autonomous system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2x) \end{pmatrix}$$

$$x(0) = 0, \quad x(\infty) = n\pi$$
Figure: The red flow line is a depiction of the harmonic map $Q_1$ which connects the north pole to the south pole, i.e., $Q(1) = 0$ and $Q(\infty) = \pi$

- This is the equation of a damped pendulum.
- 3d non-exterior problem there are no harmonic maps...
2d Harmonic Maps?

2d Harmonic maps equation: In 2d, the exterior harmonic map equation reduces to the equation of a simple pendulum

\[ \ddot{x} = \frac{1}{2} \sin(2x), \quad x(0) = 0, \quad x(\infty) = n\pi \]

Figure: The red flow line is a depiction of the harmonic map $Q$ for the non-exterior problem which connects the north pole to the south pole.
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Back to the 3d exterior model:

**Soliton Resolution Conjecture:** Informally, this conjecture asserts that given “generic” initial data for a dispersive equation with global solutions, the long term behavior of the global evolution should eventually resolve into a superposition of solitons and a radiation component that decays.

- The 3d exterior wave map problem was proposed by Bizon, Chmaj, Maliborksi (2011), as a simple model to study relaxation to the ground states (given by the harmonic maps).

- Removing a ball breaks the scaling symmetry!

- B-C-M make the simple observation that removing the origin effectively renders the 3d Cauchy problem subcritical. (3d equiv. wave maps to the sphere are supercritical in non-exterior case). Global existence becomes a triviality.

- Numerical simulations suggest that in each energy class defined by the topological class, \( \psi(\infty) = n\pi \), every solution scatters to the unique harmonic map \( Q_n \) in that class.
### Main Results

**Theorem 1 (L, Schlag, 2011)**

\[ n = 0, \; Q_0 = 0. \]  
For any smooth energy data \( (\psi_0, \psi_1) \in \mathcal{H} \), there exists a global smooth evolution \( \vec{\psi} \) to (1). Furthermore, \( \vec{\psi} \) scatters to 0 in the sense that the energy of \( \vec{\psi} \) on any arbitrary, but compact region vanishes as \( t \to \infty \).

**Theorem 2 (L, Schlag, 2011)**

\[ n \geq 1, \; Q_n = Q. \]  
There exists \( \varepsilon > 0 \) such that for all smooth data \( (\psi_0, \psi_1) \in \mathcal{H}_n \) such that
\[
\| (\psi_0, \psi_1) - (Q, 0) \|_{\mathcal{H}} < \varepsilon
\]

the unique solution \( \psi \) to (1) with data \( (\psi_0, \psi_1) \) exists globally in time and scatters to \( Q \) as \( t \to \infty \).
**Scattering:** Here scattering can be phrased as follows: There exists $(\varphi, \varphi_t)$ such that

$$(\psi, \psi_t) = (Q_n, 0) + (\varphi, \varphi_t) + o_H(1) \quad \text{as} \quad t \to \infty$$

where $\bar{\varphi}$ solves the linearized equation

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r} \varphi_r + \frac{2}{r^2} \varphi = 0$$

$$\varphi(t, 1) = 0$$

Cote, Kenig, Merle (2008) prove scattering for 2d wave maps (non-exterior) for data with energy slightly above $\mathcal{E}(Q, 0)$ via the celebrated Kenig-Merle concentration-compactness/rigidity method, Kenig, Merle (2006 Invent.), (2008 Acta.). We also employ the Kenig-Merle method here.
Kenig-Merle method: We outline the proof of Theorem 1. Let

\[ S_+ = \{(\psi_0, \psi_1) \in \mathcal{H} | \vec{\psi}(t) \text{ exists globally and scatters as } t \to +\infty\} \]

We claim that \( S_+ = \mathcal{H} \). This is proved via the following outline:

- **(Small data result)**: Small data global existence and scattering, proving \( S_+ \) is not empty.

- **(Concentration Compactness)**: If Theorem 1 fails, i.e., if \( S_+ \neq \mathcal{H} \), then there exists a nonzero energy solution \( \vec{\psi} \) to (1) (called the critical element) such that the trajectory

\[ K_+ = \{\vec{\psi}(t) | t \geq 0\} \]

is precompact in \( \mathcal{H} \).

- **(Rigidity Argument)**: If a global evolution \( \vec{\psi} \) has the property that the trajectory, \( K_+ \), is pre-compact in \( \mathcal{H} \), then \( \psi \equiv 0 \).
Small data scattering: The small data global existence and scattering result follows from the Smith-Sogge Strichartz estimates for 5d linear exterior wave equations (2000 CPDE) after the following reduction: Set $u := \frac{\psi}{r}$. Then $u$ satisfies the following equation:

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + \frac{\sin(2ru) - 2ru}{r^3} = 0$$

(4)

$$u(1, t) = 0$$

- By Hardy’s inequality the map $\psi \mapsto \frac{\psi}{r}$ defines an isomorphism between $\mathcal{H}$ and $\dot{H}^1_0 \times L^2(\mathbb{R}^5 \setminus B)$, hence a small data global existence and scattering result for (4) implies the same result for (1) in $\mathcal{H}$.

- As usual, one can show that a solution $u$ scatters to a free wave $\iff ||u||_S < \infty$ where $S$ is a suitably chosen Strichartz norm. In this case, $S = L^3_t L^6_x$. 

Concentration Compactness:

- small data theory $\implies S_+$ contains a small ball around zero. Hence, if Theorem 1 fails, there is a bounded sequence of data $\vec{u}_n := (u^0_n, u^1_n) \in \mathcal{H}$ such that

$$\|\vec{u}_n\|_{\mathcal{H}} \to E_* > 0, \quad \text{and} \quad \|u_n\|_{S} \to \infty$$

One assumes that $E_*$ is minimal with this property.

- Naively, we would like to “pass to the limit” in the $u_n$ and obtain an element $u_*$ with $\|\vec{u}_*\|_{\mathcal{H}} = E_*$ and $\|u_*\|_{S} = \infty$.

- However, the symmetries of the equation present an obstacle to compactness. Namely,
  1. the $u_n$ can be arbitrarily translated in time.
  2. the $u_n$ might split into individual waves which become arbitrarily separated in space-time as $n \to \infty$. 
Bahouri-Gerard Decomposition

\{u_n\} a seq. of free radial waves bounded in \(\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^5)\).

Passing to a subsequence, \(\exists\) a seq. of free solutions \(v^j\) bounded in \(\mathcal{H}\), and seq.'s of times \(t^j_n \in \mathbb{R}\) such that for \(\gamma^k_n\) defined by

\[ u_n(t) = \sum_{1 \leq j < k} v^j(t + t^j_n) + \gamma^k_n(t) \quad (5) \]

we have for any \(j < k\), \(\tilde{\gamma}^k_n(-t^j_n) \rightarrow 0\) weakly in \(\mathcal{H}\) as \(n \rightarrow \infty\),

\[ \lim_{n \to \infty} |t^j_n - t^k_n| = \infty \quad \text{and the errors } \gamma^k_n \text{ vanish asymptotically} \]

\[ \lim_{k \to \infty} \limsup_{n \to \infty} \|\gamma^k_n\|_{(L^\infty_t L^p_x \cap L^3_t L^6_x)(\mathbb{R} \times \mathbb{R}^5)} = 0 \quad \forall \frac{10}{3} < p < \infty \quad (6) \]

Moreover, we have orthogonality of the free energy

\[ \|\tilde{u}_n\|^2_{\mathcal{H}} = \sum_{1 \leq j < k} \|\tilde{v}^j\|^2_{\mathcal{H}} + \|\tilde{\gamma}^k_n\|^2_{\mathcal{H}} + o(1) \quad \text{as } n \to \infty \quad (7) \]
Figure: a schematic description of the concentration-compactness decomposition
The minimality of $E_*$ allows one to conclude that for our sequence $\{\vec{u}_n\}$ there can be only one non-vanishing profile $\nu^j$, say, $\nu^1$.

Indeed, the general idea is that if there were two nonzero profiles $\nu^1$ and $\nu^2$, one can conclude via the orthogonality of the energies that the corresponding non-linear profiles $U^1$ and $U^2$ each have energy less than $E_*$ which means that $U^1$ and $U^2$ both scatter as $t \to \infty$ with uniformly controlled $S$ norms.

A perturbation lemma now allows one to conclude the same for the $u_n$ which is a contradiction.

This allows us to obtain the limiting "critical element", $u_*$, with $\|\vec{u}_*\|_H = E_*$ and $\|u_*\|_S = \infty$.

The pre-compactness of the trajectory $K_+ = \{u_*(t) \mid t \geq 0\}$ is then obtained via another application of Bahouri-Gerard.
Rigidity: The concentration compactness procedure produces a nonzero solution $\psi$ with a compact trajectory, $K_+ = \{\vec{\psi}(t) \mid t \geq 0\}$, in the event that Theorem 1 fails. The goal now is to show that any such solution is identically zero, which is a contradiction.

- One should note that in contrast to the $2d$ scattering result of Cote, Kenig, Merle we do not need an upper bound on the energy to carry out a rigidity argument.

- Indeed, we show that the nonlinear functional $\mathcal{L}$ associated to the virial identity is globally coercive in $\mathcal{H}$.

- This will involve a detailed analysis of the phase-portrait for the Euler-Lagrange equations associated to the virial functional.
Virial Inequality

The key ingredient in the “rigidity argument” is the following virial inequality. In what follows $\chi_R(r) = \chi(\frac{r}{R})$ is a smooth cut-off function that is 1 on $[1, R]$ and zero for $r \geq 2R$. If $\vec{\psi} \in H$ is a solution to (1), then for all $T \in \mathbb{R}$

$$\left\langle \chi_R \dot{\psi} \mid r \psi_r + \frac{29}{20} \psi \right\rangle \bigg|_0^T \leq \int_0^T \mathcal{L}(\psi) + O \left( \mathcal{E}_R^\infty (\vec{\psi}) \right) \, dt \quad (8)$$

Where $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ is defined by

$$\mathcal{L}(\psi) := -\int_1^\infty \left( \frac{1}{20} \psi^2 + \frac{19}{20} \psi_r^2 \right) r^2 \, dr$$

$$+ \int_1^\infty \left( \sin^2(\psi) - \frac{29}{20} \psi \sin(2\psi) \right) \, dr$$
When combined with the virial inequality, the following lemma is enough to prove that the only compact trajectory is $\psi \equiv 0$.

**Lemma 1**

Let $L : H \to \mathbb{R}$ be defined as in the previous slide. Then for every $\vec{\psi} = (\psi(t), \dot{\psi}(t)) \in H$ we have

$$L(\vec{\psi}) \leq -\frac{1}{20} \int_{1}^{\infty} \left( \dot{\psi}^2 + \psi^2_r \right) r^2 \, dr \leq -\frac{1}{180} \mathcal{E}(\vec{\psi})$$

- Lemma 1 means that the nonlinear virial functional $L$ is **globally coercive** on the energy space.
Proof of Theorem 1

Indeed, applying Lemma 1 to the critical element $\vec{\psi}$ and plugging this into (8) gives

$$\left\langle \chi_R \dot{\psi} \mid r\psi_r + \frac{29}{20}\psi \right\rangle \bigg|_0^T \leq -\int_0^T \frac{\mathcal{E}(\vec{\psi})}{180} + O\left(\mathcal{E}_R^\infty(\vec{\psi})\right) \, dt \quad (9)$$

- By the pre-compactness $K_+$ we can choose $R$ large enough so that $\mathcal{E}_R^\infty(\vec{\psi})$ is small uniformly in $t \geq 0$. Hence the right-hand-side of (9) is $\leq -cT\mathcal{E}(\vec{\psi})$.
- The left-hand-side of (9) is $O(R\mathcal{E}(\vec{\psi}))$. Hence, for every $T$ we have

$$T\mathcal{E}(\vec{\psi}) \leq C R \mathcal{E}(\vec{\psi})$$

which is a contradiction since $\psi$ is global. This proves Theorem 1. It remains to establish Lemma 1.
Proof of Lemma 1

Proof of Lemma 1: Observe that

\[ \mathcal{L}(\bar{\psi}) = -\frac{1}{20} \int_1^\infty \left( \dot{\psi}^2 + \psi_r^2 \right) r^2 \, dr + \Lambda(\psi) \]

where

\[ \Lambda(\psi) := -\frac{9}{10} \int_1^\infty \psi_r^2 \, r^2 \, dr + \int_1^\infty \left( \sin^2(\psi) - \frac{29}{20} \psi \sin(2\psi) \right) \, dr \]

\[ = -\frac{5}{9} E(\psi) + N(\psi) \]

It suffices to show that

\[ \Lambda(\psi) \leq 0 \quad \text{for every} \quad \psi \in \dot{H}^1_0(1, \infty) \]  \quad (10)

We prove (10) first on the subspace \( A_R := \dot{H}^1_0(1, R) \) for every \( R \) and then extend to all of \( \dot{H}^1_0(1, \infty) \) by an approximation argument.
Euler-Lagrange equation

Again, we want to prove that

\[ \Lambda(\psi) \leq 0 \quad \text{for every} \quad \psi \in \mathcal{A}_R \quad (11) \]

We claim that \( \psi \equiv 0 \) is the unique maximizer for \( \Lambda|_{\mathcal{A}_R} \) for every \( R \).

- After establishing the existence, we obtain the Euler-Lagrange equation for a maximizer.

\[ \psi_{rr} + \frac{2}{r}\psi_r = \frac{1}{r^2}f(\psi) \quad (12) \]

\[ \psi(1) = 0, \, \psi(R) = 0 \]

where \( f(x) := \frac{1}{4}\sin(2x) + \frac{29}{18}x\cos(2x) \).

- Setting \( t = \log(r) \) and defining \( x(t) := \psi(r) \) we obtain the following autonomous differential equation for \( x \):

\[ \ddot{x} + \dot{x} = f(x) \quad (13) \]

\[ x(0) = 0, \, x(\log(R)) = 0 \]
The proof now follows from the following lemma:

**Lemma 2**

Let $f(x) := \frac{1}{4} \sin(2x) + \frac{20}{18} x \cos(2x)$. Suppose that $x(t)$ is a solution to

$$\ddot{x} + \dot{x} = f(x)$$

and suppose that $x(0) = 0$ and that there exists a $T > 0$ such that $x(T) = 0$. Then $x \equiv 0$.

- We remark that the conclusion of Lemma 2 is extremely sensitive to the exact form of $f$. Lemma 2 is false if $f$ is replaced by $\frac{3}{2} f$. 
Lemma 2 will be established via a detailed analysis of the phase portrait of (14). To begin, we set $y = \dot{x}$ and rewrite (14) as the following autonomous system:

$$
\dot{v} := \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + f(x) \end{pmatrix} =: N(v) \tag{15}
$$

Let $x_j$ be the zeros of $f$. Note $f, -x_j = x_{-j}$. $v_j := (x_j, 0)$ are then fixed points of (15).

Each $v_j$ is a **hyperbolic** fixed point and one can show that

1. $v_j$ is a **sink** if $j$ is odd
2. $v_j$ is a **saddle** if $j$ is even
Figure: A depiction of the phase portrait associated to (15). The red and green flow lines correspond to the saddles at the fixed points $v_j$ for $j$ even and the blue flow lines represent the sinks at the $v_j$ for $j$ odd.
Figure: If Lemma 2 is false, then there would be a trajectory as depicted by the purple line in the above schematic with \( v(0) = (0, v_0) \) and \( v(T) = (0, v_1) \). Our goal is to rule out such a trajectory.
Figure: The figure above represents a slice of the phase portrait associated to (15). The red flow lines represent the unstable manifolds, $W^u_j$, associated to the $v_j$, and the green flow lines represent the stable manifolds, $W^s_j$, associated to the $v_j$. 
Proving that the form of the red trajectories corresponding to the unstable manifolds as depicted in the previous slide is a delicate matter. For this we will need to construct suitable Lyapunov functionals. We will also need the following:

**Key identity:**

\[
\frac{1}{2}(y^2(t_1) - y^2(t_0)) + \int_{t_0}^{t_1} y^2(s) \, ds = F(x(t_1)) - F(x(t_0)) \quad (*)
\]

where \( F(x) := \frac{5}{18} \cos(2x) + \frac{29}{36} x \sin(2x) \) is a primitive for \( f \). This is obtained by multiplying the equation (14) by \( \dot{x} \) and integrating from \( t_0 \) to \( t_1 \).

The form of the green trajectories corresponding to the stable manifolds is clear once we have established that the red trajectories have the desired form.
**Figure:** The red trajectory corresponds to the unstable manifold at $\nu_2$. The green region $\Sigma$ is Lyapunov in the sense that $\partial \Sigma$ is repulsive with respect to the forward flow of our vector field (15).
Lyapunov Functional

For the sake of finding a contradiction assume that the red trajectory does not fall into the sink at \( v_1 \). Then there exists a time \( T \) such that \( v_2^- (T) = (0, y_2(T)) \). Using the identity (*) with \( t_0 = -\infty, t_1 = T \) we have

\[
\frac{1}{2} y_2^2(T) + \int_{-\infty}^{T} y_2^2(s) \, ds = F(0) - F(x_2) < 2.18 \quad (16)
\]

Now we use the fact that since \( \partial \Sigma \) is Lyapunov the trajectory \( v_2^- \) cannot enter \( \Sigma \) and hence the integral on the left-hand-side of (16) is greater than the area of \( \Sigma \), i.e.,

\[
2.21 < \text{Area}(\Sigma) < \int_{-\infty}^{T} y_2^2(s) \, ds \quad (17)
\]

\( 2.18 < 2.21 \) but only by \(.03!\)
The region $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ pictured above has the property that $\partial \Sigma$ is repulsive with respect to the unstable manifold $W^u_{-2}$. 

**Figure:** The region $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ pictured above has the property that $\partial \Sigma$ is repulsive with respect to the unstable manifold $W^u_{-2}$. 


Scattering for wave maps exterior to a ball
Construction of $\partial \Sigma$

To construct $\Sigma$, we define three polynomials, $p_1, p_2, p_3$. As an example, the first polynomial $p_1$ as a function of $x$ is defined by:

$$p_1(x) := -\frac{3}{1000} + \frac{110}{47} \left(x + \frac{43}{18}\right) - \frac{89}{222} \left(x + \frac{43}{18}\right)^2 - \frac{23}{42} \left(x + \frac{43}{18}\right)^3$$

$$+ \frac{7}{85} \left(x + \frac{43}{18}\right)^4 + \frac{8}{303} \left(x + \frac{43}{18}\right)^5 - \frac{1}{446} \left(x + \frac{43}{18}\right)^6$$

$$- \frac{1}{760} \left(x + \frac{43}{18}\right)^7 + \frac{1}{4035} \left(x + \frac{43}{18}\right)^8 - \frac{1}{13999} \left(x + \frac{43}{18}\right)^9$$

We set $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. $\Sigma_1$ is defined by

$$\Sigma_1 := \left\{ (x, y) \in \Omega_{-1} \mid -\frac{43}{18} + \frac{3}{1000} < x < -\frac{3}{5}, 0 < y < p_1 \left(-\frac{3}{5}\right) \right\}$$
Figure: A schematic depiction of the flow in the first strip using Maple.
Figure: A schematic depiction of the flow in the second strip using Maple.
To deal with the trajectories emanating from the $v_j$ for even $j > 4$ we shift and rescale our equation (15) via the following renormalization. For each $j \in \mathbb{N}$, $\varepsilon \in \mathbb{R}$ we define $\zeta$ and $\eta$ via

$$
\begin{align*}
x(t) &=: 2^{j-1} \frac{1}{4} \pi + \zeta(\varepsilon^{-1} t) \\
y(t) &=: \varepsilon^{-1} \eta(\varepsilon^{-1} t)
\end{align*}
$$

Define $z_j := 2^{j-1} \frac{1}{4} \pi$. Then (15) implies the following system of equations for $\zeta, \eta$

$$
\begin{pmatrix}
\dot{\zeta} \\
\dot{\eta}
\end{pmatrix} = 
\begin{pmatrix}
\eta \\
-\varepsilon \eta + \varepsilon^2 f(z_j + \zeta)
\end{pmatrix} \tag{19}
$$

where $\dot{} = \frac{d}{ds}$ where $s = \varepsilon^{-1} t$. 

Set \( g(\zeta) := \frac{1}{4} \cos(2\zeta) - \frac{29}{18} \zeta \sin(2\zeta) \). And for even \( j > 4 \) set

\[
\varepsilon := \sqrt{\frac{72}{29\pi(2j - 1)}}
\]

Observe \( \varepsilon < \frac{7}{20} \) for \( j \geq 6 \). Then (19) becomes

\[
\begin{pmatrix}
\dot{\zeta} \\
\dot{\eta}
\end{pmatrix} =
\begin{pmatrix}
\sin(2\zeta) - \varepsilon \eta - \varepsilon^2 g(\zeta)
\end{pmatrix}
\] (20)

- Note that (20) is the equation governing the motion of a damped pendulum with a small perturbative term \( \varepsilon^2 g(\zeta) \), and in the limit as \( \varepsilon \to 0 \), (20) is exactly the equation of a simple pendulum.

- After this renormalization, the proof follows the same general outline—phase plane analysis, construction of a Lyapunov functional—as for the first two strips.
Figure: A schematic depiction of the flow for the renormalized equation.
Theorem 1 (Lawrie, S., 2011)

\( n = 0, \ Q_0 = 0 \). For any smooth energy data \((\psi_0, \psi_1) \in \mathcal{H}\), there exists a global smooth evolution \(\vec{\psi}\) to \((1)\). Furthermore, \(\vec{\psi}\) scatters to 0 in the sense that the energy of \(\vec{\psi}\) on any arbitrary, but compact region vanishes as \(t \to \infty\).

Theorem 2 (Lawrie, S., 2011)

\( n \geq 1, \ Q_n = Q \). There exists \(\varepsilon > 0\) such that for all smooth data \((\psi_0, \psi_1) \in \mathcal{H}_n\) such that

\[ \|(\psi_0, \psi_1) - (Q, 0)\|_{\mathcal{H}} < \varepsilon \]

the unique solution \(\psi\) to \((1)\) with data \((\psi_0, \psi_1)\) exists globally in time and scatters to \(Q\) as \(t \to \infty\).
Remarks regarding Theorem 2

Theorem 2 is proved by establishing Strichartz estimates for the wave equation exterior to a ball perturbed by a radial potential $V$ which arises from the linearization of the problem about the harmonic map $Q_n$. To be precise we prove Strichartz estimates for

$$(\partial_{tt} - \Delta_5 + V)u = F$$

$u(t, 1) = 0, \quad (u(0), u_t(0)) = (u_0, u_1) \in \dot{H}_0^1 \times L^2(\mathbb{R}_*^5, \text{radial})$

$$V(r) = \frac{2}{r^2} (\cos(2Q(r)) - 1)$$

One can show that $Q_n(r) = n\pi - O(r^{-2})$ as $r \to \infty$ so $V$ decays like $r^{-6}$ as $r \to \infty$.

The idea is to extend the exterior Strichartz estimates of Metcalfe, Smith, and Sogge to this setting via local energy estimates. It is crucial that the operator $-\Delta + V$ has no negative spectrum, and no eigenvalue or resonance at 0.
Conjecture of Bizon, Chmaj, Maliborksi: Numerical simulations suggest that Theorem 2 holds with $\varepsilon = \infty$.

This currently appears out of reach. The main difficulty with the implementation of the Kenig-Merle method lies with the coercivity of the virial functional centered at the harmonic maps $Q_n$.

Euler-Lagrange equations involve $Q_n$, hence cannot be transformed into an autonomous system.

$Q_n$ is not explicit.
Thank you!
p.s. the slides from this talk can be found on my webpage:
math.uchicago.edu/~alawrie