Wave maps from the hyperbolic plane

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Wave maps

The *wave map equation* is a generalization of the wave equation to (Riemannian) manifold-valued maps $\Phi : (M, \mathbf{g}) \to (N, \mathbf{h})$. The action is given by

$$\mathcal{S}[\Phi] = \int \mathbf{g}^{\mu\nu} \langle (\mathrm{d}\Phi)_\mu, (\mathrm{d}\Phi)_\nu \rangle_{\mathbf{h}} \, \mathrm{dVol}_{\mathbf{g}}$$

and wave maps are formal critical points of this action.

If M is of the product form $M = \mathbb{R} \times \widetilde{M}$ (or more generally, if there exists a time-like Killing vector field), then there exists a *coercive conserved energy* (or the *Hamiltonian*) for the system, which takes the form

$$\mathcal{E}[(\Phi, \dot{\Phi})](t) = \frac{1}{2} \int_{\{t\} \times \widetilde{M}} \langle \dot{\Phi}, \dot{\Phi} \rangle_{\mathbf{h}}(t) + \widetilde{\mathbf{g}}^{ij} \langle (\mathrm{d}\Phi)_i, (\mathrm{d}\Phi)_j \rangle_{\mathbf{h}}(t) \, \mathrm{dVol}_{\widetilde{\mathbf{g}}} \, .$$

When $M = \mathbb{R}^{1+2}$, the energy is *invariant* under the scaling symmetry of the equation. This fact underscores the special nature of the $(1+2)$-dimensional wave maps.

Wave maps from $\mathbb{R}^{1+2}$

There has been fantastic progress over the past two decades on this equation, and a good deal is known in the case $M = \mathbb{R}^{1+2}$.

Theorem (Wave maps from $\mathbb{R}^{1+2}$)

Consider the initial value problem (IVP) for the wave map equation in the case $M = \mathbb{R}^{1+2}$ with initial data of energy $E < \infty$.

1. If $(N, \mathbf{h})$ is negatively curved (i.e., all sectional curvatures < 0), then the IVP is globally well-posed and the solution scatters to a constant map.
2. In general, global well-posedness of the IVP and scattering to a constant map holds if $E < \mathcal{E}[(Q, 0)]$, where Q is the lowest energy non-trivial harmonic map $\mathbb{R}^2 \to N$.
3. There exists a solution which blows up in finite time in the case $(N, \mathbf{h}) = (\mathbb{S}^2, \mathbf{g}_{\mathbb{S}^2})$ and $E > \mathcal{E}[(Q, 0)]$.

Wave maps from $\mathbb{R}^{1+2}$, continued

Remarks

- Part 1 is a consequence of Part 2, as there are no non-trivial finite energy harmonic maps from $\mathbb{R}^2$ to negatively curved targets (Eells-Sampson).
- Note the key role played by *finite energy harmonic maps* from $\mathbb{R}^2$ to N. Harmonic maps are time-independent solutions to the wave map equation.

The theorem in the form we stated is due to

- Parts 1, 2: Sterbenz–Tataru, Krieger–Schlag, Tao
- Part 3: Krieger–Schlag–Tataru, Rodnianski–Sterbenz, Raphaël–Rodnianski

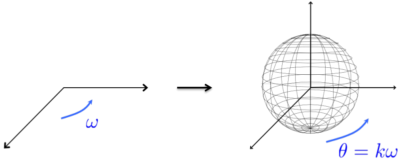
Other work on wave maps on $\mathbb{R}^{1+2}$ include

- **Low regularity theory.** Klainerman–Machedon, Klainerman–Selberg, Tataru, Tao, Klainerman–Rodnianski, Nahmod–Stefanov–Uhlenbeck, Shatah–Struwe, Krieger, etc.
- **Wave maps under symmetry.** Christodoulou–Tahvildar-Zadeh, Shatah–Tahvildar-Zadeh, Müller–Struwe, Struwe, Côte, Côte–Kenig–L.–Schlag, etc.

Equivariant wave maps from $\mathbb{R} \times \mathbb{H}^2$

Consider wave maps Φ from $M = \mathbb{R} \times \mathbb{H}^2$ to $(N, \mathbf{h})$. For simplicity, we make further assumptions:

- $N = \mathbb{H}^2$ or $\mathbb{S}^2$, with geod. polar coord. (ψ, ω), $ds^2 = d\psi^2 + g^2(\psi) d\omega^2$ where $g(\psi) = \sinh \psi$ when $N = \mathbb{H}^2$ and $g(\psi) = \sin \psi$ when $N = \mathbb{S}^2$.
- Φ is k-equivariant: $\Phi(t, r, \omega) = (\psi(t, r), k\omega)$



Then the wave map equation becomes

$$\partial_t^2 \psi - \frac{1}{\sinh r} \partial_r (\sinh r \partial_r \psi) + k^2 \frac{g(\psi) g'(\psi)}{\sinh^2 r} = 0 \tag{1}$$

The conserved energy takes the form

$$\mathcal{E}[\psi, \partial_t \psi](t) = \frac{1}{2} \int \left((\partial_t \psi)^2(t, r) + (\partial_r \psi)^2(t, r) + \frac{k^2 g(\psi(t, r))^2}{\sinh^2 r} \right) \sinh r \, \mathrm{d}r \tag{2}$$

Equivariant wave maps from $\mathbb{R} \times \mathbb{H}^2$: The Cauchy problem

Restrictions on finite energy data: For example, consider the Cauchy problem for $k = 1$ and $N = \mathbb{S}^2$, $g(\psi) = \sin\psi$:

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \coth r \psi_r + \frac{\sin 2\psi}{2\sinh^2 r} = 0, \quad \vec{\psi}(0) = (\psi_0, \psi_1) \\ &\mathcal{E}(\psi_0, \psi_1) = \frac{1}{2}\int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2\psi}{\sinh^2 r}\right) \sinh r \, \mathrm{d}r \end{aligned} \tag{3}$$

Finite energy requires:

- $\psi_0(0) = 0$ (or more generally $m\pi$, $m \in \mathbb{Z}$).
- There exists $\alpha \in \mathbb{R}$ so that $\lim_{r\to\infty} \psi_0(r) = \alpha$. Endpoint α is *fixed* by the evolution. It is thus natural to consider the Cauchy problem for data within disjoint energy classes

$$\mathcal{E}_\alpha := \{(\psi_0, \psi_1) \mid \mathcal{E}(\psi_0, \psi_1) < \infty, \ \psi_0(0) = 0, \ \psi_0(\infty) = \alpha\}$$

- Contrast with equivariant wave maps $\mathbb{R}^{1+2} \to \mathbb{S}^2$ – endpoint must be $n\pi$, which means every finite energy Euclidean wave map into $\mathbb{S}^2$ has a fixed integer valued topological degree.

Features of the model

Because the volume of the sphere of radius r grows exponentially in r on $\mathbb{H}^d$, linear waves on $\mathbb{H}^d$ exhibit improved dispersion in the long term. This aspect and its effect on nonlinear dispersive PDEs have been well-studied by focusing on model equations such as NLW and NLS:

- NLW on $\mathbb{R} \times \mathbb{H}^d$:

  $$\Box_{\mathbb{R}\times\mathbb{H}^d} u = \pm |u|^{p-1} u$$

  Anker–Pierfelice, Metcalfe–Taylor, L.-O.-S., etc.
- Shifted NLW on $\mathbb{R} \times \mathbb{H}^d$: Tataru, Anker–Pierfelice–Vallarino, Shen–Staffilani, Shen etc.
- NLS on $\mathbb{R} \times \mathbb{H}^d$: Banica, Banica–Carles–Staffilani, Banica–Carles–Duyckaerts, Banica–Duyckaerts, Ionescu–Staffilani, Ionescu–Pausader–Staffilani, Borthwick-Marzuola, etc.

Features of the model, continued.

There are new features in the wave map model that arise from the interplay of the hyperbolic geometry and the nonlinear structure of the wave map equation. Observe:

- $\mathbb{H}^2$ is *conformally equivalent* to the flat disk $\mathbb{D}^2$
- Two dimensional harmonic maps, as well as their respective energy, are *conformally invariant*.

Combining these two facts, we obtain an abundance of finite energy harmonic maps, which are of different nature than those on $\mathbb{R}^2$. We are motivated to pursue the following goal.

Main goal

Understand the long term dynamics of finite energy wave maps from $\mathbb{R} \times \mathbb{H}^2$, for which these harmonic maps are expected to play a key role.

Asymptotic stability of harmonic maps to $\mathbb{H}^2$

To illustrate the difference between the case $M = \mathbb{R} \times \mathbb{H}^2$ and the flat case, we present our first result when the target is $N = \mathbb{H}^2$.

Theorem 1 (L.-O.-S., '14)

Consider the 1*-equivariant WM from* $\mathbb{R} \times \mathbb{H}^2$ *to* $\mathbb{H}^2$*.*

1. *There exists a one-parameter family* $\{P_\lambda\}_{\lambda \in [0,1)}$ *of finite energy harmonic maps, given by the formula*

$$P_\lambda = 2\mathrm{arctanh}\left(\lambda \tanh\left(\frac{r}{2}\right)\right) \quad \text{for } \lambda \in [0,1).$$

   We have $\mathcal{E}[(P_\lambda, 0)] \nearrow \infty$ *as* $\lambda \to 1$*. There are no other finite energy harmonic maps.*

2. *Each* P_λ *is asymptotically stable under perturbations in* $\mathcal{H}_0$*, i.e., for any data of the form* $(P_\lambda + \psi_0, \psi_1)$*, where* $\psi_0, \psi_1 \in C_0^\infty(\mathbb{H}^2)$ *with*

$$\|(\psi_0, \psi_1)\|_{\mathcal{H}_0} := \int \left(|\psi_1|^2 + |\partial_r \psi_0|^2 + \frac{|\psi_0|^2}{\sinh^2 r} \right) \sinh r\, dr \ll 1,$$

   the solution ψ *to the IVP scatters to* P_λ *as* $t \to \pm\infty$*.*

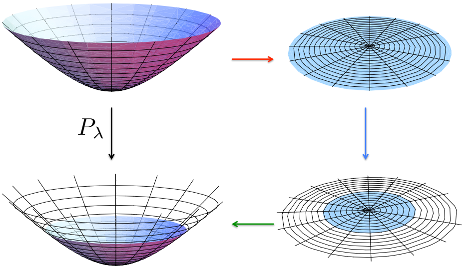
Remark. It is simple to extend this theorem to $k \geq 2$.

Proof of Theorem 1

Proof of Part 1. Recall the expression

$$P_\lambda(r) = 2\mathrm{arctanh}\,(\lambda\,\tanh(\frac{r}{2})).$$

That $P_\lambda(r)$ for $\lambda \in [0, 1)$ is a finite energy harmonic map is evident from:



That these are all of the finite energy harmonic maps in the first equivariance class follows from an ODE argument.

Proof of Theorem 1, continued

Proof of Part 2. The idea is to study spectral properties of the linearized operator about P_λ. For convenience, we conjugate the problem to $L^2[0,\infty)$ by

$$\phi := \sinh^{\frac{1}{2}} r(\psi - P_\lambda).$$

Then the linearized operator takes the form

$$\mathcal{L}_\lambda \phi = -\phi'' + \frac{3}{4}\frac{1}{\sinh^2 r}\phi + \frac{1}{4}\phi + U_\lambda \phi \tag{4}$$

where $U_\lambda \geq 0$. Furthermore,

$$\mathcal{L}_0 := -\partial_r^2 + \frac{3}{4}\frac{1}{\sinh^2 r} + \frac{1}{4} \simeq -\triangle_{\mathbb{H}^4} - 2.$$

Spectrum of $\mathcal{L}_0$ is purely abs. cont. and given by $[1/4, \infty)$. By Sturm comparison, $\mathcal{L}_\lambda$ has no eigenvalues or resonances. From this spectral information *local energy decay* and *Strichartz estimates* follow. Now Part 2 follows by the usual Picard iteration argument.

Remark. We prove $(1+4)$-dimensional Strichartz estimates, thanks to the extra repulsive potential from the term $\dfrac{g(\psi)g'(\psi)}{\sinh^2 r} = \dfrac{\sinh 2\psi}{2\sinh^2 r}$.

Further questions in the case $N = \mathbb{H}^2$

In order for blow up to happen in our problem, there must be a non-trivial harmonic map $\mathbb{R}^2 \to N$. Since this is false for $N = \mathbb{H}^2$, we have GWP in this case. The following conjecture about asymptotic behavior is reasonable:

Conjecture (Soliton resolution for equivariant wave maps $\mathbb{H}^2 \to \mathbb{H}^2$)

Consider the IVP for the 1-equivariant WM from $\mathbb{R} \times \mathbb{H}^2$ to $\mathbb{H}^2$, with finite energy initial data (ψ_0, ψ_1). Let $\lambda = \tanh \frac{\psi_0(\infty)}{2} \in [0, 1)$. Then the IVP is globally well-posed, and the solution scatters to P_λ as $t \to \pm\infty$.

Using the celebrated concentration compactness/rigidity approach of Kenig-Merle, ('06, '08), we were able to make partial progress on the problem. One of the key technical ingredients is a Bahouri-Gérard type profile decomposition established in a recent preprint, (L.O.S. '14), following recent work of Ionescu, Pausader, Staffilani on the NLS.

Theorem 2 (L.-O.-S., forthcoming)

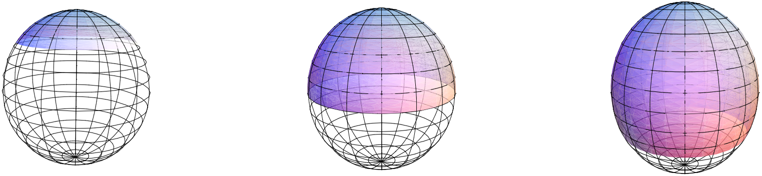
The above conjecture holds for initial data with $0 \leq \lambda < \Lambda$, *where* $\Lambda = 0.56\ldots$

Harmonic maps to $\mathbb{S}^2$

When the target is $N = \mathbb{S}^2$ and the equivariance index is $k = 1$, we have a one parameter family of finite energy harmonic maps $\{Q_\lambda\}_{\lambda \in [0,\infty)}$, given by the formula

$$Q_\lambda(r) = 2\arctan(\lambda \tanh(\frac{r}{2})) \tag{5}$$

Note that $Q_0(\infty) = 0$, $Q_1(\infty) = \frac{\pi}{2}$ (equator) and $Q_\lambda(\infty) \to \pi$ (south pole) as $\lambda \to \infty$. The range of Q_λ for different values of λ is plotted below:



$\lambda = \frac{1}{2}$ $\qquad$ $\lambda = 1$ $\qquad$ $\lambda = \sqrt{3}$

Existence of a gap eigenvalue in the case $N = \mathbb{S}^2$

In the case $N = \mathbb{S}^2$, there is an additional feature, which is in stark contrast to the flat case.

Theorem 3 (L.-O.-S., '14)

Consider the 1*-equivariant WM from* $\mathbb{R} \times \mathbb{H}^2$ *to* $\mathbb{S}^2$*.*

1. *The maps* Q_λ $(\lambda \in [0, \infty))$ *are harmonic maps of energy*

$$\mathcal{E}[Q_\lambda, 0] = 2\frac{\lambda^2}{\lambda^2 + 1} \nearrow 2 = \mathcal{E}_{\mathbb{R}\times\mathbb{R}^2}[Q, 0].$$

   There are no other finite energy harmonic maps.

2. *For* $\lambda < 1 + \delta$ *(say* $\delta \leq 0.095$*),* Q_λ *is asymptotically stable under perturbations in* $\mathcal{H}_0$*.*
3. *For* $\lambda \gg 1$*, the linearized operator* $\mathcal{L}_\lambda$ *about* Q_λ *has a gap eigenvalue* $\mu_\lambda^2 \in (0, 1/4)$*. This eigenvalue is simple and unique. Moreover,*

$$\mu_\lambda^2 \to 0 \quad \text{as } \lambda \to \infty.$$

Existence of a gap eigenvalue in the case $N = \mathbb{S}^2$

Remarks.

- Recall that when $\lambda = 1$, the range of Q_λ is exactly the upper hemisphere, which is *geodesically convex*. Although we have not used this fact directly in the proof, it is likely to be the geometric reason for non-existence of a gap eigenvalue in this case. See also Theorem 4 later.
- An eigenfunction gives rise to a non-decaying solution to the linearized wave equation $\partial_t^2 \phi + \mathcal{L}_\lambda \phi = 0$. Therefore, asymptotic stability *fails* at the level of the linearized equation for $\lambda \gg 1$.
- In the flat case, the linearized operator about the ground state Q has a *resonance* at 0, which is generated by the scaling symmetry of the problem. Part 3 is, roughly speaking, a consequence of the interplay between this resonance near $r = 0$ and the global geometry of $\mathbb{H}^2$, more specifically, the spectral gap of $-\triangle_{\mathbb{H}^2}$ on $\mathbb{H}^2$.

Wave maps to $\mathbb{S}^2$ in higher equivariance classes

A natural question is whether we can prove existence of gap eigenvalues in other models. Recently, we have been able to give an affirmative answer for k-equivariant wave maps for arbitrary $k \geq 1$, as well as for the equivariant energy critical Yang-Mills equation on $4d$ hyperbolic space. In the case of k-equivariant wave maps, we have

Theorem 4 (L.-O.-S., forthcoming)

Consider the k-equivariant WM from $\mathbb{R} \times \mathbb{H}^2$ to $\mathbb{S}^2$ with any $k \geq 1$.

1. *The maps*
$$Q_\lambda^{(k)} = 2\arctan(\lambda \tanh^k(\frac{r}{2})).$$
are the only finite energy harmonic maps in this setting.
2. *For $\lambda < 1 + \delta^{(k)}$, the linearized operator $\mathcal{L}_\lambda^{(k)}$ about $Q_\lambda^{(k)}$ does not have any eigenvalues or resonances.*
3. *For $\lambda \gg 1$, the operator $\mathcal{L}_\lambda^{(k)}$ has a gap eigenvalue $(\mu_\lambda^{(k)})^2 \in (0, 1/4)$. This eigenvalue is simple and unique, and we also have*
$$(\mu_\lambda^{(k)})^2 \to 0 \quad \text{as } \lambda \to \infty.$$

Implications of the gap eigenvalue for nonlinear dynamics

We return to the case $k = 1$, $\lambda \gg 1$. Although the gap eigenvalue precludes a proof of asymptotically stability via a pertubative argument based on linear theory, the "usual mechanisms" for instability of Q_λ are not present at the nonlinear level.

Wave maps from $\mathbb{R}^{1+2} \to \mathbb{S}^2$: $\exists$ unique harmonic map $Q_{\rm euc}(r) = 2\arctan r$. Top. deg. $= 1$. Unstable. Small perturbations within degree class can lead to finite time blow-up (KST, RR, RS).

- Struwe's Bubbling Thm: If $\psi(t)$ blows up at $t = 1$, then $\exists$ seq. $t_n \to 1$, $\lambda_n = o(1 - t_n)$ so that the rescaled seq.

$$\psi_n(t, r) = \psi(t_n + \lambda_n t, \lambda_n r) \to Q_{\rm euc} \quad \text{locally in } H^1_{\rm loc}((-1, 1) \times \mathbb{R}^2)$$

Euclidean bubbles for $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{S}^2$ Identical result holds for hyp. wave maps $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{S}^2$. Euc. HM $Q_{\rm euc}$ is bubbled. Why? Blow-up occurs via energy concentration at $r = 0$ (at tip of light cone) $\Rightarrow$ behavior is Euclidean.

- $\Rightarrow$ finite time blow up requires at least $\mathcal{E}_{\rm euc}[Q_{\rm euc}, 0] = 2$ worth of energy. But $\mathcal{E}[Q_\lambda, 0] = \frac{2\lambda^2}{1+\lambda^2} < 2$. Hence small perturbations of Q_λ lead to global solutions.

Moreover, no global scaling symmetry.

Implications of the gap eigenvalue for nonlinear dynamics, continued

Recalling that $\mathcal{E}[Q_\lambda, 0] < 2$, where 2 is the energy of the Euclidean ground state $Q_{\rm euc}$ (we have normalized away 2π). Therefore, at least for small energy perturbations of Q_λ, finite time blow up is impossible, and hence the solution exists globally. The remaining question is thus:

Question (Stability of Q_λ for $\lambda \gg 1$)

What happens to the solution ψ to the IVP for data (ψ_0, ψ_1) such that $\|(\psi_0, \psi_1) - (Q_\lambda, 0)\|_{\mathcal{H}_0} \ll 1$ as $t \to \pm\infty$?

- One possibility is *Lyapunov stability without asymptotic stability*, i.e., ψ does not converge to Q_λ, even locally in energy.
- Another possibility is *asymptotic stability*, i.e., ψ *does* converge to Q_λ in some norm (e.g., local convergence in energy). The mechanism would be *radiative damping*, which refers to leaking of the energy associated to the eigenvalue to the continuous spectrum by a *fully nonlinear* mechanism. (Soffer–Weinstein, Sigal–Zhou, Zhou, Cuccagna, Cuccagna–Mizumachi, Bambusi–Cuccagna etc.) Would require verification of so-called "nonlinear Fermi golden rule."

Motivation behind the proof of Theorem 3: existence of a gap eigenvalue

Pass to half-line formulation: The goal is to understand the spectral properties of the linearized operator about Q_λ. For convenience, we can conjugate the problem to $L^2([0,\infty))$ via

$$\phi := \sinh^{\frac{1}{2}} r(\psi - Q_\lambda).$$

Then the linearized operator takes the form

$$\mathcal{L}_\lambda \phi = -\phi'' + \frac{3}{4}\frac{1}{\sinh^2}\phi + \frac{1}{4}\phi + V_\lambda \phi \tag{6}$$

where

$$V_\lambda(r) = \frac{cos 2Q_\lambda - 1}{\sinh^2 r} = \frac{-2\lambda^2}{(\cosh^2(r/2) + \lambda^2 \sinh^2(r/2))^2}$$

existence of a gap eigenvalue, continued

Observation: Change variables. Introduce "renormalized" coordinates

$$\rho := \lambda r, \quad \widetilde{\phi}(\rho) := \phi(r), \quad \widetilde{\mathcal{L}}_\lambda \widetilde{\phi} = \frac{1}{\lambda^2}(\mathcal{L}_\lambda \phi)(\cdot/\lambda)$$

Obtain

$$\widetilde{\mathcal{L}}_\lambda := -\partial_r^2 + \frac{3}{4\lambda^2 \sinh^2(\rho/\lambda)} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} V_\lambda(\rho/\lambda)$$
$$\frac{1}{\lambda^2} V_\lambda(\rho/\lambda) = \frac{-2}{(\cosh^2(\rho/2\lambda) + \lambda^2 \sinh^2(\rho/2\lambda))^2}$$

Formal convergence to Euc. linearized operator: Note that for each fixed $r > 0$, "$\widetilde{\mathcal{L}}_\lambda \to \mathcal{L}_{\mathrm{euc}}$" as $\lambda \to \infty$, where

$$\mathcal{L}_{euc}\phi = -\phi'' + \frac{3}{4\rho^2}\phi + V_{\mathrm{euc}}\phi, \quad V_{\mathrm{euc}}(\rho) = \frac{-2}{(1+(\rho/2)^2)^2}$$

is obtained by linearizing the Euc. WM eq. about Q_{euc}. Note that $\mathcal{L}_{\mathrm{euc}}$ has a threshold resonance, $\mathcal{L}_{\mathrm{euc}}\varphi_{\mathrm{euc}} = 0$,

$$\varphi_{euc}(\rho) = \frac{\rho^{\frac{3}{2}}}{1+(\rho/2)^2}$$

Existence of gap e-val: Sketch of the proof: I

Step I: Sturm oscillation theory. Existence of an eigenvalue for $\mathcal{L}_\lambda$ below $\frac{1}{4}$ is equivalent to the following statement:

Proposition 1

Any solution to

$$\widetilde{\mathcal{L}}_\lambda \widetilde{\phi}_0 = \frac{1}{4\lambda^2}\widetilde{\phi}_0, \quad \widetilde{\phi}_0 \in L^2[0,c], \ \forall\ 0 < c < \infty$$

must change sign.

Roughly speaking: Sign change of $\widetilde{\phi}_0$ is a consequence of the interplay between

1. The existence of the Euclidean threshold resonance φ_{euc}, which is a positive solution to $\mathcal{L}_{\mathrm{euc}}\varphi_{\mathrm{euc}} = 0$.
2. The formal convergence "$\mathcal{L}_\lambda \to \mathcal{L}_{\mathrm{euc}}$", (note that these are half-line operators with strongly singular potentials. Nonetheless this formal limit is useful on compact intervals in ρ which can then be extended to large intervals of size $[0, \varepsilon\lambda]$ via a contradiction hypothesis.)
3. The spectral gap $(0, \frac{1}{4\lambda^2})$ for $\widetilde{\mathcal{L}}_\lambda$.

Existence of gap e-val: Sketch of the proof: II

Step II: To exploit the formal convergence "$\mathcal{L}_\lambda \to \mathcal{L}_{\rm euc}$", we renormalize about the threshold resonance $\varphi_{\rm euc}$. Define

$$f(\rho) := \widetilde{\phi}_0(\rho)/\varphi_{\rm euc}(\rho)$$

Then f solves

$$(f'\varphi_{\rm euc}^2)' = \varphi_{\rm euc}^2 W_\lambda f$$

$$f(\rho) = 1 + \int_0^\rho \int_0^\tau \frac{\varphi_{\rm euc}^2(\sigma)}{\varphi_{\rm euc}^2(\tau)} W_\lambda(\sigma) f(\sigma) \mathrm{d}\sigma \mathrm{d}\tau, \quad f(0) = 1, f'(0) = 0$$

We call $W_\lambda = \mathcal{L}_\lambda - \mathcal{L}_{\rm euc}$ the renormalized potential – gains extra smallness factor of λ^{-2}. In particular, we can find $\varepsilon > 0$, p_0, c_1 indept. of λ, so that

$$W_\lambda(\rho) \le -\frac{c_1}{\lambda^2}, \quad \rho_0 \le \rho \le \varepsilon\lambda \tag{7}$$

Contradiction assumption: Suppose that $f(\rho)$ is always positive.

Existence of gap e-val: Sketch of the proof: II continued

Cheat for a moment: Suppose we could in fact assume that

$$f(\rho) \geq c_2 > 0 \tag{8}$$

Recalling (7) (actually, let's cheat a bit more and assume $W_\lambda(\rho) \leq -\frac{c_1}{\lambda^2}$ on $[0, \varepsilon\lambda)$), we then have

$$\begin{aligned} f(\rho) &= 1 + \int_0^\rho \int_0^\tau \frac{\varphi_{\mathrm{euc}}^2(\sigma)}{\varphi_{\mathrm{euc}}^2(\tau)} W_\lambda(\sigma) f(\sigma) \mathrm{d}\sigma \mathrm{d}\tau \\ &\leq 1 - \frac{c_1 c_2}{\lambda^2} \int_0^\rho \frac{1}{\varphi_{\mathrm{euc}}^2(\tau)} \mathrm{d}\tau \int_0^\tau \varphi_{\mathrm{euc}}^2(\sigma) \mathrm{d}\sigma \\ &\lesssim 1 - \frac{c\rho^2}{\lambda^2} \log(2+\rho) \end{aligned}$$

Setting $\rho = \varepsilon\lambda$ (recall that ε is fixed) yields

$$f(\varepsilon\lambda) \lesssim 1 - c\varepsilon^2 \log(2 + \varepsilon\lambda) < 0$$

which gives a contradiction by taking λ large enough.

Existence of gap e-val: Sketch of the proof: III

Step III It remains to establish (8). In fact, we show that

Lemma 1

Either $f(\rho)$ changes sign or there exists $\varepsilon_1 > 0, c_2 > 0$ independent of λ so that

$$f(\rho) \geq c_2 > 0, \quad \forall\ 0 \leq \rho \leq \varepsilon_1 \lambda$$

This can be rephrased in terms of the original threshold function ϕ_0 which is our L^2_{loc} solution to $\mathcal{L}_\lambda \phi = \frac{1}{4}\phi$.

- To prove Lemma 1 we need to bring in information from $r = \infty$.
- Use explicit solution ζ_0^λ to $\mathcal{L}_\lambda \zeta_0^\lambda = 0$ obtained by differentiating $\partial_\lambda Q_\lambda =: \zeta_0^\lambda$, and its conjugate ζ_∞^λ which behaves like

$$\zeta_\infty^\lambda \sim e^{-r/2} \quad \text{as} \quad r \to \infty$$

- Show that if $\phi_0(r) > 0, \ \forall r$, then

$$\phi_0'/\phi \geq (\zeta_\infty^\lambda)'/\zeta_\infty^\lambda$$

  which yields explicit lower bound on log derivative of ϕ_0, which can be parlayed into an explicit lower bound for $f(\rho)$.

Thank you for listening!