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Near Equality in the Hausdorff-Young inequality

For any locally compact Abelian group, the Hausdorff-Young inequality states that the Fourier transform maps $L^p$ to $L^q$, where the two exponents are conjugate and $p \in [1, 2]$. For $d$, the optimal constant was found Babenko (1961) for $q$ an even integer, and by Beckner (1975) for general exponents. Lieb (1990) showed that all extremizers are Gaussian functions.

We establish a stabler uniqueness theorem, showing that if a function $f$ nearly achieves the optimal constant in the inequality, then $f$ must be close in norm to a Gaussian. This can equivalently be formulated as a precompactness theorem: If $f_\nu = 1$, and if $\hat{f}_\nu \to A_p$ where $A_p$ is the optimal constant in the inequality, then there exists a sequence of norm-preserving symmetries $\psi_\nu$ of the inequality such that the sequence $(\hat{\psi}_\nu(f_\nu))$ is precompact in $L^p$. Such a result can also be viewed as a strengthening of the inequality.

The proof relies on ingredients taken from from additive combinatorics. Central to the reasoning are arithmetic progressions; more exactly, arithmetic progressions of arbitrarily high rank.