A Survey of Singular Reduction

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1 Introduction

Reduction of a classical system with symmetries was known even at the times of Euler and Jacobi. However, it had not been formalized before the works of Meyer [12], Marsden and Weinstein [10]. In [10], Marsden and Weinstein considered a symplectic manifold (M, ω) with a Hamiltonian action of G and moment map $\mu : M \to \mathfrak{g}$. If 0 is a regular value of μ, then $\mu^{-1}(0)$ is a manifold and furthermore if the action of the group is free and proper on $\mu^{-1}(0)$, then $\mu^{-1}(0)/G$ naturally becomes a symplectic manifold. However, two bad things may happen:

First, $\mu^{-1}(0)$ may be a manifold but the action may not be proper or free. Then, one may possibly obtain orbifolds but this is not our topic here.

Second, 0 may not be a regular value. Then, even the set $\mu^{-1}(0)$ can be quite bad. Unfortunately, this is not a pathological case and it comes in examples in physics. Indeed, most interesting cases happen when preimage of 0 or more generally $\xi \in \mathfrak{g}^*$ is singular as there are more symmetries. Many examples coming from physics like classical field theories, homogenous Yang-Mills equations have singular solution spaces. For more examples see [1], [3], [4].

Therefore, it is important to describe the structure of singularities and singular reduced spaces. The aim of this survey is to describe some results in that direction. We will first give an overview of some approaches, then give Sjamaar's description of singular reduction by stratifications. Then we will mention some results on the geometry of them. In the last section, we mention quantization of singular reduction.

2 First Attempts

One of the first examinations of the singularities of moment map occurred in [3] and [4]. In these papers, it is shown that for a point in zero level being singular is equivalent to having infinitesimal symmetries. To be more precise:

Let (M, ω) be a symplectic manifold, G be a group with an Hamiltonian action on M and with moment map $\mu : M \to \mathfrak{g}^*$. Let $x_0 \in \mu^{-1}(0)$ and $\mathfrak{g}_{x_0}$ be the Lie algebra of the stabilizer of x_0. Then the elements of $\mathfrak{g}_{x_0} \backslash \{0\}$ are called infinitesimal symmetries of x_0. It is easy to see that $x_0 \in \mu^{-1}(0)$ is a singular point of μ if and only if $\mathfrak{g}_{x_0} \backslash \{0\}$ is non-empty, where $\mathfrak{g}$ is the Lie algebra of stabilizer of x_0.

In general the local structure around a singularity of a smooth map can be bad. However, provided that some conditions are satisfied, the singularities of $\mu^{-1}(0)$ are conical.

To show this for the case μ is quadratic at $x_0 \in \mu^{-1}(0)$ when M is linear, Arms, Marsden and Moncrief, work as follows:
They first found a local diffeomorphism mapping $\mu^{-1}(0) \cap S_{x_0}$ to a cone C_{x_0}, where S_{x_0} is a slice around x_0 for the action of G on M. Then, this was used to show $\mu^{-1}(0)$ is locally diffeomorphic to $C_{x_0} \times G/G.x_0$. With some further work the same thing can be shown without quadraticity assumption on μ.

Some conditions should be satisfied to have the above situation, but there are many important cases where the above hold for example, when M and G are finite dimensional and G is compact. Also, when $M = T^*\mathbb{R}^3$ and $G = SO_3(\mathbb{R})$, where the action is the one coming from standard G action on $\mathbb{R}^3$. A moment map $T^*\mathbb{R}^3 \cong \mathbb{R}^6 \to \mathfrak{so}_3(\mathbb{R}) \cong \mathbb{R}^3$ is given by $(q, p) \mapsto q \times p$. Then $\mu^{-1}(0) = \{(q, p) : q \times p = 0\}$ is a cone over a three manifold.

In their paper [3], they give infinite dimensional examples as well. For instance, they show that the set of solutions for the Yang-Mills equations over a Lorentzian manifold has conical singularities. Similarly, in [4], they show that the space of solutions of Einstein Vacuum Equations at certain spacetimes has conical singularities.

Also, one example of singular reduction is given in [3]. Let $x_0 \in \mu^{-1}(0)$ be a singular point. Then there is a neighborhood V of x_0 such that for all $x \in V$ the identity component of the stabilizer G_x is contained in a conjugate of G_{x_0}. Denote the points of V with the same orbit type as x_0 by N_{x_0}. Let $\mathscr{N}_{x_0} = N_{x_0} \cap S_{x_0}$. Intersecting with a slice is roughly same as taking the quotient. Then it could be shown that $\mu^{-1}(0) \cap \mathscr{N}_{x_0}$ is a symplectic submanifold of M around x_0. $\mu^{-1}(0) \cap \mathscr{N}_{x_0}$ actually corresponds to the main stratum for the orbit type stratifications, which we will discuss later.

As mentioned in the previous paragraph, in the next section, we will describe a nice geometric structure on symplectic quotients, called stratifications, in which case the quotient will become a union of symplectic manifolds. But, the examples of stratifications had already started to come. For instance, the space of geometrically equivalent solutions for Einstein vacuum equations, and Einstein-Yang-Mills equations can be shown to be stratified spaces. See [4] for details.

To have a nice reduction, one asks for a geometric object and an extra structure on it, possibly a "Poisson structure". However, in case we do not look for the geometric object, we may still have an "algebraic reduction", as defined by Weinstein and Śniatycki in [18]. It can be described as follows:

Let G, M, μ be as before. $\mu^X \in C^\infty(M)$ is defined by $\mu^X(x) = \langle \mu(x), X\rangle$ for $X \in \mathfrak{g}$ and let I be the ideal of $C^\infty(M)$ generated by μ^X, for $X \in \mathfrak{g}$. Then, it can be shown that I is a G-invariant, Poisson ideal, and we have a Poisson algebra $\mathscr{A} = (C^\infty(M)/I)^G$. In case 0 is a regular value of μ we have $\mathscr{A} = C^\infty(\mu^{-1}(0)/G)$. Otherwise, we just take $\mathscr{A}$ above to be our "reduced algebra". This is called the algebraic reduction. By above, this definition extends Marsden-Weinstein reduction in the regular case.

However,in this definition we lack a geometrical object, and it is in general hard to deal with algebraic reduction of a manifold. Hence, it is natural to look for more geometric ways of reduction. We will describe two ways to associate a geometric object to our algebraic reduction following [2]. The first one is the Dirac reduction:

Let $Z \subset M$ be closed. For reduction we will take $Z = \mu^{-1}(0)$. Let $I(Z)$ be the set of smooth functions vanishing on Z and $F(Z)$ be the set of of first class ones, namely $F(Z) = \{f \in I(Z) : \{f, I(Z)\} \subset I(Z)\}$. Then define the set of observables, $OB(Z) = \{f \in C^\infty(M) : \{f, F(Z)\} \subset I(Z)\}$. Note that $OB(Z) = \{f \in C^\infty(M) : \{f, \mu^X\} = 0 \ on\ Z\ for\ all\ X \in \mathfrak{g}\}$ when $Z = \mu^{-1}(0)$ and G is compact and connected. We will identify points that are indistinguishable under observables,i.e. for $x, y \in Z$ say $x \sim y$ if $f(x) = f(y)$ for all $f \in OB(Z)$ and define $\hat{Z} = Z/\sim$.

We also need to put an appropriate function algebra on $\hat{Z}$. Put $\hat{W}^\infty(\hat{Z})$ to be functions induced from the ones in $OB(Z)$. Then $\hat{W}^\infty(\hat{Z}) \cong OB(Z)/I(Z)$. And, we want a Poisson structure on $\hat{W}^\infty(\hat{Z})$. The first natural idea is inducing a Poisson structure from $C^\infty(M)$, namely given $f, g \in \hat{W}^\infty(\hat{Z})$, take $\bar{f}, \bar{g} \in OB(Z)$ inducing f and g. Then define $\{f, g\}$ to be function induced by $\{\bar{f}, \bar{g}\}$.

However, this does not always work. First, $\{\bar{f}, \bar{g}\}$ may not be in $OB(Z)$. Also, even when $OB(Z)$ is a Poisson subalgebra, $I(Z)$ may not be a Poisson ideal, in which case we do not get a well defined bracket. Indeed, assuming that $OB(Z)$ is a Poisson subalgebra, it is proven in [2] that above definition gives a well defined Poisson structure if and only if Z is "first order" i.e. $I(Z) = F(Z)$. When this happens, we have the Dirac reduction.

Other approach described in [2] is geometric reduction. For this, let Z and M be as in the previous paragraph. Recall that the tangent cone C_qZ at q to Z is the set of velocity vectors of paths in Z passing through q. Define T_qZ to be $span(C_qZ)$. Then, at each $q \in Z$, ω restricts to an alternating form on T_qZ and it has a nullspace there, denote by $ker(\omega)$, and we have "conical kernel", with the terms of [2] , which is defined to be $cer(\omega) = ker(\omega) \cap C_qZ$.

Now, for $p, q \in Z$ say $p \simeq q$ if p and q can be connected by a path in Z whose velocity vector always lie in $cer(\omega)$. Then define $\hat{\hat{Z}} = Z/\simeq$. We are considering the leaf space, in some sense.

Define the function algebra on it, $\hat{\hat{W}}^{\infty}(\hat{\hat{Z}})$, to be the set of functions that are induced by a Whitney smooth function on Z that has derivative 0 in the direction of vectors in $cer(\omega)$. Then, again the most natural attempt to define $\{f, g\}$, for $f, g \in \hat{\hat{W}}^{\infty}(\hat{\hat{Z}})$, is taking functions $\bar{f}, \bar{g}$ inducing f and g respectively and looking the function induced by $\{\bar{f}, \bar{g}\}$. However, in [2], the authors define the bracket by using "Hamiltonian vector fields" on Z, which are defined in different way than the mere restrictions of Hamiltonian fields on M. Then, provided that some conditions are satisfied it is shown this defines a bracket, which could be obtained as above. Luckily, in case when $Z = \mu^{-1}(0)$ and G are compact and connected and Dirac reduction exists, this turns out to happen.

We will take $Z = \mu^{-1}(0)$ in the above constructions and assume G is compact and connected. Then, one can show the two equivalence relations above are the same, so geometric and Dirac reductions give the same topological spaces. Moreover,the Poisson bracket for geometric reduction always exists and in case $\mu^{-1}(0)$ is first class, i.e. when bracket for the Dirac reduction exist, the Poisson algebras of the two reductions are the same, and this is the same as Marsden-Weinstein reduction, if 0 is a regular value of μ. Also, in this case this Poisson algebra is the same as in the algebraic reduction, providing a geometric model for it, provided that the G-invariant elements of the ideal generated by μ^X, for $X \in \mathfrak{g}$ and those in the vanishing ideal of $\mu^{-1}(0)$ coincide. In particular, when $\mu^{-1}(0)$ is first class, then this happens,and Poisson structure becomes non-degenerate.

Another very natural way to define singular reduction came in [1]. Their simple observation was that M/G is a Poisson variety and induces a Poisson structure on Whitney smooth functions on $\mu^{-1}(0)/G \hookrightarrow M/G$. For this, the construction actually works for any group and any action. However, when the action is not proper there might be non-trivial Casimirs in $W^\infty(\mu^{-1}(0)/G)$, i.e. an $f \neq 0$ such that $\{f, g\} = 0$ for all $g \in W^\infty(\mu^{-1}(0)/G)$. This does not happen when the action is proper, which is shown in the same paper. To show this they use the fact that $C^\infty(M)^G$ seperates orbits, which follows from properness of the action, and the local connectedness of $\mu^{-1}(0)$. The latter follows from the conical structure of the singularities, proven in [3]. Note that Arms, Cushman and Gotay argue for general level sets and not only $\mu^{-1}(0)$.

Also, they show that in the regular case this reduction procedure agrees with Marsden-Weinstein reduction for $\mu^{-1}(0)$, provided that the action is proper. Besides it agrees with geometric and Dirac reductions mentioned above in case the group is compact.

There is also a more recent approach to reduction from derived algebraic geometry, which takes care of the singularities or bad group actions. For details see [14].

3 Symplectic Stratifications on $\mu^{-1}(0)$

To have a "nice" symplectic quotient we both need a geometrical object and a function algebra on it. For nice enough Hamiltonian actions we have both an algebra and a topological space $\mu^{-1}(0)/G$. However, we still do not know much about $\mu^{-1}(0)/G$, which is a quotient of a possibly non-manifold, and which could be quite bad. We already mentioned it is a union of symplectic manifolds. In this section, we will see that those manifolds "patch together in a nice way", i.e. they form symplectic stratified spaces. We will also mention some results on their geometry, dynamics and cohomology.

3.1 Stratified Spaces

As we pointed earlier, the "singular reduction" tend to be a union of manifolds, patched in a nice way. In this section, we will make sense of "patched nicely", following [9] and [16].

Definition 3.1 *A paracompact, Hausdorff space X is called* decomposed *if $X = \bigsqcup_j S_j$ for a locally finite family of manifolds $\{S_j\}$ satisfying*

$$S_i \cap \overline{S_j} \neq \emptyset \text{ implies } S_i \subset \overline{S_j}$$

Then S_j are called the pieces of X.

Definition 3.2 *Let $X = \bigsqcup_j S_j$ be a decomposed space. Then it is called* stratified *if given $x \in S_j$ there is a neighborhood U of x in X and a ball B around x in S_j and "a stratified space L of lower depth" such that U and $B \times \mathring{C}L$ are homeomorphic via a decomposition preserving map, where $\mathring{C}L$ is the cone over L. Observe that this definition is inductive.*

Example 3.1 *The simplest non trivial example of a decomposed space and a stratified space is a cone over a manifold, namely*

$$\mathring{C}M = (M \times I)/(M \times \{0\})$$

It is the union of $M \times (0, 1]$ and a point. Instead of M we can take any stratified space.

Example 3.2 *Let M be a manifold, G be a compact Lie group acting on M. For a closed subgroup H let $M_{(H)}$ denote the union of orbits of elements with stabilizer H, or equivalently the set of elements with stabilizer conjugate to H. Here (H) denotes the set of conjugates of H. Then, the family $\{M_{(H)}\}$ gives a stratification of M. Note that we may need to refine or take some unions before finding actual S_j as before.*

Example 3.3 *Assume $X \subset \mathbb{R}^n$ is a decomposed space, where pieces are smooth submanifolds of $\mathbb{R}^n$. Assume the following is satisfied:*

> *(Whitney condition B) Given $S_i \subset \overline{S_j}$, $\{x_k\}$, $\{y_k\}$ are sequences in S_i and S_j resp. converging to x, if the sequence of lines through x_k and y_k converges to l in $\mathbb{P}^{n-1}$ and if $T_{y_k}S_j$ converges to $T \in Gr_{dimS_j}(\mathbb{R}^n)$, then we have $l \subset T$.*

Then it can be shown that this decomposition is indeed a stratification. See [6] or [16].

Definition 3.3 *Let* $X = \bigsqcup_j S_j$ *be a stratified space.* X *is called* symplectic *if it is "smooth", the pieces are symplectic manifolds,* $C^\infty(X)$ *has a Poisson structure and* $S_j \hookrightarrow X$ *is a Poisson embedding for each* j*. Note that in this case the "dynamics" on* X *determines the pieces uniquely. (If we assume the pieces are connected)*
A map $\phi : X \to Y$ *between two symplectic stratified spaces is a* morphism *if it pulls* $C^\infty(Y)$ *back to* $C^\infty(X)$ *and* $\phi^* : C^\infty(Y) \to C^\infty(X)$ *is Poisson. Note in this case* ϕ *necessarily sends pieces into pieces.*

Assume G is compact and connected, and M and μ are as before. Then one can prove that $\mu^{-1}(0) \cap M_{(H)}$ is a submanifold. Furthermore, when we divide it by the action of G we again obtain a manifold, which we denote by $(M_0)_{(H)}$.
$\mu^{-1}(0) \cap M_{(H)}$ is Poisson, and the quotient becomes a symplectic manifold. Besides, clearly, $M_0 = \mu^{-1}(0)/G = \bigsqcup_{(H)} (M_0)_{(H)}$ and we obtain a decomposition. That it is locally finite can be shown using a local normal form theorem for μ. It can also be shown that among $\mu^{-1}(0) \cap M_{(H)}$ for all H, one is open and dense which we denote by Z_{prin}. Accordingly, one of the pieces of M_0 is open and dense denoted by $(M_0)_{prin}$.

Moreover, we can find a G-equivariant, proper embedding of M into a finite dimensional real representation V of G. Hence, we have an embedding $M/G \hookrightarrow V/G$ and we can embed the latter into some $\mathbb{R}^n$ using the G-invariant polynomials. Thus, we have an embedding of M/G into $\mathbb{R}^n$ and by restriction we obtain a proper embedding of M_0 into $\mathbb{R}^n$.

It can be shown that the image of above embedding satisfies Whitney Condition B. Hence, it is a stratified space. Moreover, the map preserves pieces therefore, M_0 is stratified. That the pieces are symplectic, $C^\infty(M_0)$ is Poisson and embeddings are Poisson embeddings are easy to show; thus, we now know that M_0 is a symplectic stratified space.

In the above argument we assumed compactness of the group, as in [16]. However, this assumption can be replaced by the properness of the action, as in [5]. Then, one can still prove that $\mu^{-1}(0)/G$ is a stratified space, basically using similar tecniques to those in [16]. Actually, there they consider an arbitrary coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ and reduction at that orbit. However, in this case we at least need local closedness of $\mathcal{O}$. See [5] for more details. For more general treatment of this subject, where there is no assumption on $\mathcal{O}$ as well, see [13].

3.2 Dynamics and Geometry on Singular Reduced Spaces

In this section we briefly mention the dynamics and the geometry of singular quotients. Let M, G, μ as before.Assume G is compact. Let $M_0 = \mu^{-1}(0)/G$, which is a stratified symplectic space.

First, Hamiltonian dynamics makes sense on symplectic stratified spaces. Recall that for each smooth function H on a symplectic manifold (M, ω), one can define X_H to be the unique vector field satisfying $i_{X_H}\omega = dH$. This vector field generates a flow, say $\{\phi_t\}$ on M. Unfortunately, differential equations does not make sense on stratified spaces. However, observe that $\{\phi_t\}$ is uniquely determined by the property:

$$\frac{d(f \circ \phi_t)}{dt} = \{f, H\} \ for\ all\ smooth\ f \tag{1}$$

The equation (1) makes sense on stratified spaces as well. In our case any H comes from a G-invariant function $\tilde{H}$ on M. $\tilde{H}$ generates a flow, which is G-equivariant, preserving $\mu^{-1}(0)$. Thus, its restriction induces to a flow on M_0. Besides, the equation (1) is satisfied on M and thus on M_0. However, this is actually a way to define Poisson bracket on M_0. Still, the dynamics makes sense on singular reduced space.

It is easy to show that the flow satisfying (1) on M_0 is unique. However, enough smooth functions to seperate points, for instance if we drop Hausdorff assumption, this may not always hold on "symplectic stratified spaces". See [5].

Besides, it is easy to show that the Hamiltonian flow preserves leaves, and the action of all smooth functions is transitive. Hence, the dynamics, or the Poisson structure, determines the pieces of the stratification, as we mentioned earlier.

Above, we have commuting actions of G and the one parameter group generated by $X_{\tilde{H}}$. Instead, if we had commuting Hamiltonian actions of G_1 and G_2, both are taken to be compact, with moment maps μ_1 and μ_2 then we would have an "Hamiltonian action" of G_2 on $M_0 = \mu^{-1}(0)/G_1$ with moment map $\bar{\mu_2}$ induced by μ_2. Then, we can reduce it again,i.e. take $\bar{\mu_2}^{-1}(0)/G_2$, and we get a new symplectic stratified space. Indeed, we can first reduce with respect to G_2 and then G_1 or with respect to $G_1 \times G_2$ and get the same symplectic stratified space. This is the analogue of reduction in stages in singular case. There is a further generalization to arbitrary extensions rather then just products. For details see [16].

Note also that for a given Hamiltonian action of G on M, as before, and K on M_0, with comoment maps $\mathfrak{g} \to C^\infty(M)$ and $\mathfrak{k} \to C^\infty(M_0)$, where $\mathfrak{g}$ and $\mathfrak{k}$ are Lie algebras of G and K respectively, finding a compact extension $\tilde{G}$ of G by K and a comoment map $\tilde{\mathfrak{g}} \to C^\infty(M)$, which extends the one on $\mathfrak{g}$ and induces $\mathfrak{k} \to C^\infty(M_0)$, the one above, would let us apply the theorem mentioned at the end of previous paragraph. Hence, by our main theorem, when we apply reduction to M_0, we would get a stratified space, which could be obtained by reduction for $\tilde{G}$ action.

At the beginning we started by making sense of "smooth" functions, i.e. 0-forms, on M_0. However, as we did not define a tangent space, what a differential form means is not clear. We can define a form by adapting the way we defined $C^\infty(M_0)$, namely:

Definition 3.4 $\Omega^j(M_0)$ *is the set of* j*-forms on* $(M_0)_{prin}$ *that pulls back to restriction of a* G*-invariant form on* M *to* Z_{prin}*.*

Example 3.4 *We make the pieces symplectic just as in the regular reduction; hence, by definition* ω_{prin}*, on* $(M_0)_{prin}$*, pulls back to* $\omega \mid_{Z_{prin}}$ *so it gives a 2-form in the above sense.*

Then, 0-forms just become the smooth functions in the sense we defined before. Note also that in the regular reduction the above definition gives the usual smooth forms on M_0.

$\{\Omega^j(M_0)\}$, a subcomplex of $\{\Omega^j((M_0)_{prin})\}$, is closed under differentials and wedge product. Thus, we can speak about their cohomology. Then, "de Rham Theorem" holds, i.e. this cohomology is equivalent to the singular cohomology with real coefficients. To show this, one obtains sheaves of differentials on M_0, namely for each open subset U we take the set of differentials on $U \cap (M_0)_{prin}$ pulling back to restrictions of a G-invariant forms on $\pi^{-1}(U)$,where π is the quotient map. We denote these sheaves by $\Omega^j(M_0)$, as well. Then, they turn out to be acyclic and

$$\mathbb{R} \to \Omega^0(M_0) \to \Omega^1(M_0) \to ...$$

becomes an acyclic resolution of constant sheaf on M_0. This is shown by proving a Poincaré Lemma on our stratified space. Thus, we get the equivalence of our new de Rham and singular cohomologies. For more details about this cohomology see [15].

4 Quantization

Quantization of a classical system roughly means replacing observables by operators on a certain Hilbert space. In more mathematical language, we start with a symplectic manifold (M, ω) that has an integral symplectic form. Then, we have a line bundle with a connection (L, ∇) over M that has curvature with cohomology class equal to (a non-zero multiple of) $[\omega]$. Then we consider (pre)-Hilbert space of sections, where the function algebra acts by

$$\nabla_{X_f} + 2\pi i f$$

where X_f is the Hamiltonian vector field associated to f. Using polarizations (i.e. integrable Lagrangian distributions), we reduce the dimension of the representation and get a geometric quantization of our manifold. Note also that if we have an Hamiltonian action of a (simply connected) group G, then we get a representation of it when we quantize.

In the regular reduction, we can quantize before and then consider the G-invariant sections, or we can first apply reduction then quantize. In [7], it is shown that they give the same vector space, when G and M are compact and the polarization is Kähler. In particular, one concludes that trivial representation occurs in the representation corresponding to M if and only if 0 is in the image of the moment map. Thus, this result, that "quantization commutes with reduction", can be stated in terms of multiplicities.

So far we have not made sense of quantization for singular spaces. It is clear that we need to associate representations to our Poisson algebras. Also, what is desirable is to have "quantization commutes with reduction".

Recall that in algebraic reduction we had a Poisson algebra $\mathscr{A} = (C^\infty(M)/I)^G$, where G, μ are as before and I is generated by μ^X, $X \in \mathfrak{g}$. We do not have a geometric object here but we actually do not need it to quantize as long as we can make sense of above things. Indeed, there are ways to quantize the Poisson algebras but this is not our topic here.

In [18], there is an example for quantization of algebraic reduction, using algebraic versions of the concepts for geometric quantization. The authors consider $\mathbb{R}^4$ with standard symplectic structure and with a Hamiltonian $\mathbb{R}$ action on it. Then, instead of sections of a line bundle, which would be trivial if existed, they consider $S = \mathbb{C} \otimes \mathscr{A}$, and instead of a connection they consider a map $Der\mathscr{A} \times S \to S$, which is $\mathscr{A}$-bilinear in the first variable and satisfies Leibniz rule in the second. They define polarizations in an algebraic way, as the maximal commuting subalgebras of $\mathscr{A}$ and take a specific one. Then, they quantize as usual and show quantization commutes with reduction for this example.

In [17], Śniatycki gives a general way to quantize algebraic reduction, In the regular case, one can quantize the reduction by first getting a prequantum line bundle (L, ∇) on M, restricting to $\mu^{-1}(0)$ and pushing forward to M_0 by considering G-invariant sections. In his paper, Śniatycki applies the algebraic analogue of this to algebraic reduction. More precisely: Let S be the space of sections of a prequantization line bundle on M. Then S is a $C^\infty(M)$ module. Consider S/IS, where I is as above, which would correspond to restricting L to $\mu^{-1}(0)$ in regular case. Then consider $(S/IS)^G$, which would correspond to pushing forward to M_0. Then $(S/IS)^G$ is a natural $\mathscr{A} = (C^\infty(M)/I)^G$ module.

Let P be a G-invariant polarization on M and consider $C^\infty_P(M)$, the set of P-invariant functions, whose Hamiltonian vector fields preserve P, and consider the set of P-invariant sections of L (with respect to ∇), denoted by S_P. S_P is a $C^\infty_P(M)$ module, and G naturally acts on S_P. Then, the image of S_P in S/IS, denote by S^0_P is a $C^\infty_P(M)/I$ module and $(S^0_P)^G$ is a $\mathscr{A}_P := (C^\infty_P(M)/I)^G$ module. This is our quantization. Note that in the same paper Śniatycki shows how to quantize the reduction on orbits other than $\mu^{-1}(0)$ as well. Furthermore, when the polarization is "Kähler" and some other conditions are satisfied, they prove the space of G-invariant sections of the quantization of M is isomorphic to quantization of algebraic reduction, i.e.

$$(S_P)^G \cong (S^0_P)^G$$

Thus, quantization commutes with reduction.

There are also ways to make sense of quantization for singular symplectic quotients, one of which was described by Sjamaar and Meinrenken in [11]. There they defined quantization in a different way, as the G-equivariant index of a certain operator on a prequantum bundle L. This -denoted by $RR(M, L)$- "Riemann-Roch number" lies in the character ring of G. Then they define Riemann-Roch numbers for the singular quotients as well. They show that the G-invariant part of above character is equal to $RR(\mu^{-1}(0)/G, L_0)$, where L_0 is an "orbibundle" on $\mu^{-1}(0)/G$ obtained from L by restricting to $\mu^{-1}(0)$ and then taking the quotient by the action, similar to [7]. Hence, quantization commutes with reduction.

Above, we only mentioned geometric quantization. For different approaches to quantization of singular reduction, in particular for deformation quantization see [8].

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