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SELF-SIMILARITY OF POISSON STRUCTURES ON TORI

KENTARO MIKAMI
Department of Computer Science and Engineering
Faculty of Engineering and Resource Science
Akita University, Akita, 010–8502, Japan
E-mail: mikami@math.akita-u.ac.jp

ALAN WEINSTEIN
Department of Mathematics, University of California
Berkeley, CA 94720, USA
E-mail: alanw@math.berkeley.edu

Abstract. We study the group of diffeomorphisms of a 3-dimensional Poisson torus which preserve the Poisson structure up to a constant multiplier, and the group of similarity ratios.

1. Introduction. A *scaling vector field* on a Poisson manifold (M, π) is a vector field X such that the Lie derivative $L_X\pi$ is a constant multiple of π. The set of all these vector fields forms a Lie algebra $\mathcal{L} = \mathcal{L}(M, \pi)$ containing the Poisson vector fields $\mathcal{L}_0$ as an ideal. The quotient $\mathcal{L}/\mathcal{L}_0$ has dimension 1 if π is exact and dimension 0 otherwise. If we think of $\mathcal{L}$ and $\mathcal{L}_0$ as Lie algebras of infinitesimal transformations, the corresponding groups of finite transformations are the *self-similarities* $\mathcal{G}$, i.e. the diffeomorphisms of M which preserve π up to a constant multiplier, and the normal subgroup $\mathcal{G}_0$ of Poisson automorphisms. The quotient $\mathcal{G}/\mathcal{G}_0$ is naturally isomorphic to the *scaling group* $\mathcal{S}(M, \pi)$, the subgroup of the nonzero real numbers consisting of the possible similarity ratios (i.e. multipliers). By convention, the similarity ratio for any self-similarity of $\pi = 0$ is declared to be 1.

The scaling group of $\mathbf{R}^{2n}$ with the standard symplectic structure is all of $\mathbf{R}^\times$. For a compact symplectic manifold, the scaling group is constrained by the finiteness of volume to be either $\{1\}$ or $\{-1, 1\}$. Iglésias [1] showed that the scaling groups are countable

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for nonexact symplectic manifolds. For general Poisson manifolds, scaling groups were introduced and discussed, with somewhat different names and notation, in Section 10 of [7] and Section 5 of [8] in connection with related quantum constructions. They are also related, though somewhat distantly, to the "von Neumann fundamental groups" of operator algebras (see the discussion in [5]).

In this paper, we study constant (i.e. translation-invariant) Poisson structures on the 3-dimensional torus and show that their scaling groups are finitely generated subgroups of $\mathbf{R}^\times$ having rank 0, 1, or 2.

As in Iglésias and Lachaud's study of singular "irrational tori" [2], we use homology theory to reduce the problem from geometry to linear algebra. We prove a result (Lemma 1 below) in elementary linear algebra (replacing some of the considerations in Section 2 of [2]), and then apply the Dirichlet unit theorem. We check and refine the results by direct calculation in several examples.

An alternative approach to finding centralizers in $SL_3(\mathbf{Z})$ would be to use results of Prasad and Raghunathan [4] on Cartan subgroups of algebraic groups. We do not pursue this approach here.

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2. Invariant geometry on the 3-torus. As a basis for the constant vector fields on the torus $\mathbf{T}^3$ we take the coordinate vector fields ∂_1, ∂_2, and ∂_3; the naturally associated basis for the constant bivector fields is $\partial_i^* = \partial_{i+1} \wedge \partial_{i+2}$ (indices modulo 3). For a constant Poisson structure $\pi = \sum a_i \partial_i^*$, the symplectic leaves are the integral manifolds of the 1-form $\sum a_i dx_i$. They are 2-tori, cylinders, or planes according to whether the rational vector space $\mathbf{Q}a_1 + \mathbf{Q}a_2 + \mathbf{Q}a_3$ has dimension 1, 2, or 3. We will refer to this dimension as the *class* of the Poisson structure π.

For a structure of class 1, the symplectic leaves all have the same finite volume, from which it follows that $\mathcal{S}$ is contained in $\{-1, 1\}$. In fact, when the class is 1, some real multiple of (a_1, a_2, a_3) has relatively prime integer entries. This 3-vector can be taken as one column of a matrix in $GL_3(\mathbf{Z})$, which defines an automorphism of the torus transforming π into a multiple of $\partial_2 \wedge \partial_3$. The map $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ is then a self-similarity with similarity ratio -1. Thus the scaling group $\mathcal{S}$ is equal to $\{-1, 1\}$ for any structure of class 1. In particular, any such structure is *reversible* in the sense that -1 belongs to the scaling group. The group $\mathcal{G}$ itself is quite large; it is the semidirect product of the diffeomorphisms of the circle of symplectic leaves (all contained in $\mathcal{G}_0$) by the group of maps from the circle to the area preserving diffeomorphisms of a 2-torus (containing in particular the constant map to $(x_2, x_3) \mapsto (x_2, -x_3)$). It is an extension of $\mathcal{G}_0$ by $\mathbf{Z}_2$.

Having disposed of the class 1 case, we will henceforth restrict our attention to structures of classes 2 and 3, which are much more rigid by virtue of the density of their symplectic leaves. In fact, each such structure is *uniquely unimodular* in the sense that it admits an invariant measure which is unique up to a constant multiple. According to Section 9 of [6], a uniquely unimodular Poisson manifold (M, π) with finite invariant

volume admits a well defined *normalized fundamental cycle* $[\pi]$ in $H_2(M, \mathbf{R})$. If ϕ is a self-similarity of (M, π) with similarity ratio ρ, then the induced map $[\phi]_2$ on the second homology satisfies $[\phi]_2([\pi]) = \rho[\pi]$, i.e. $[\pi]$ is an eigenvector of $[\phi]_2$ with eigenvalue ρ.

To apply the general ideas above to the case of the 3-torus, we will use the natural identifications (valid for any torus)

$$H_2 \cong (H^2)^* \cong (\text{constant 2-forms})^* \cong \text{constant bivector fields},$$

under which the fundamental class $[\pi]$ becomes identified with π itself. Also, $[\phi]_2$ is the second exterior power $\wedge^2[\phi]_1$ of the induced map on $H_1 \cong$ constant vector fields. Finally, we identify H_1 and H_2 with $\mathbf{R}^3$, so that $[\phi]_1$ and $[\phi]_2$ are represented by 3×3 matrices B and A respectively, which belong to $GL_3(\mathbf{Z})$. In fact, in dimension 3, $A = \wedge^2 B$ is simply the classical adjoint $(\det A)({}^tA)^{-1}$, which belongs to $SL_3(\mathbf{Z})$. (In other words, the second exterior power of a 3-dimensional vector space carries a natural orientation!) We also note that the map $A \mapsto (\det A)({}^tA)^{-1}$ is surjective with kernel $\{-I, I\}$.

As a consequence of this analysis, we have the following reduction of our problem for classes 2 and 3.

Definition 1. *The* projective stabilizer *of a vector* $\mathbf{a} \in \mathbf{R}^n$ *is the subgroup of* $SL_n(\mathbf{Z})$ *having* $\mathbf{a}$ *as an eigenvector; i.e. it is the stabilizer* $SL_n(\mathbf{Z})^{<\mathbf{a}>}$ *of the line* $<\mathbf{a}>$ *through* $\mathbf{a}$ *for the action of* $SL_n(\mathbf{Z})$ *on* $\mathbf{RP}^{n-1}$. *The group of possible eigenvalues (naturally isomorphic to the quotient of the projective stabilizer by the stabilizer* $SL_n(\mathbf{Z})^{\mathbf{a}}$ *of* $\mathbf{a}$ *itself) is the* scaling group $\mathcal{S}(\mathbf{a}) \subseteq \mathbf{R}^\times$ *of* $\mathbf{a}$.

Proposition 1. *Let* $\pi = \sum a_i \partial_i^*$ *be a Poisson structure of class 2 or 3 on* $\mathbf{T}^3$. *Then* $\mathcal{S}(\mathbf{T}^3, \pi) = \mathcal{S}(a_1, a_2, a_3)$.

The question of reversibility is rather easily disposed of.

Proposition 2. *A Poisson structure is reversible if and only if it is of class 1 or 2.*

Proof. We have already seen that a structure of class 1 is reversible. Supposing now that the structure is of class 2 or 3, we will apply Proposition 1. If -1 is an eigenvalue of $A \in SL_3(\mathbf{Z})$ with eigenvalue -1, then the kernel of the integer matrix $A + I$ has a basis of rational eigenvectors. If -1 is a simple eigenvalue, any eigenvector is a multiple of a rational vector, and the corresponding Poisson structure is of class 1. If -1 is a double eigenvalue, then the third eigenvalue must be 1 since A is unimodular. There is an eigenvector for the eigenvalue 1 with rational entries, and hence one with relatively prime integer entries. It follows that A is conjugate in $SL_3(\mathbf{Z})$ to a matrix of the form

$$A = \begin{bmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{bmatrix}.$$

Since -1 is still a double eigenvalue, the matrix must in fact have the form

$$A = \begin{bmatrix} 1 & a & b \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now we may assume (using the torus automorphism corresponding to the conjugating matrix) that A had this triangular form to begin with. If (x, y, z) is an eigenvector with

eigenvalue -1, then $x = -\frac{1}{2}(ay + bz)$; since a and b are integers, the corresponding Poisson structure is of class 1 or 2.

For the converse, assuming that the Poisson structure has class 2, its cylindrical symplectic leaf contains a circle which can be transformed by an element of $SL_3(\mathbf{Z})$ to the standard "first coordinate circle." This means that $\mathbf{a} = (0, a_2, a_3)$, which is reversed by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

■

3. An algebraic lemma. The analysis of class 2 structures in the proof above implies that the scaling group of such a structure is equal to the scaling group of a vector in $\mathbf{R}^2$ for the action of $GL_2(\mathbf{Z})$. Thus the structures of class 2 and class 3 can be studied in a very similar way. The following lemma will be useful in both cases. Although it could be derived from the results in Section 2 of [2], we prefer to give a direct, elementary proof.

LEMMA 1. Assume that the characteristic polynomial of $A \in GL_n(\mathbf{Z})$ is irreducible over $\mathbf{Q}$. If $B \in GL_n(\mathbf{Z})$ has at least one eigenvector in common with A, then B has the same eigenvectors as A and is a rational polynomial in A, in particular, $AB = BA$.

Furthermore, if B has a rational eigenvalue, then B is a scalar matrix, and hence, $B = kI$ for some $k \in \{\pm 1\}$.

Proof. If λ is an eigenvalue of A, the last row of $A - \lambda I$ is a linear combination of the others, so to find an eigenvector

$${}^t[x_1 \cdots x_n]$$

it suffices to solve the system

$$\begin{bmatrix} a_{11} - \lambda & \cdots & a_{1n'} \\ \vdots & \ddots & \vdots \\ a_{n'1} & \cdots & a_{n'n'} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n'} \end{bmatrix} = -x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{n'n} \end{bmatrix} \tag{1}$$

where $n' = n - 1$.

Since the characteristic polynomial of A is irreducible, λ is not a root of any polynomial of degree n', so

$$D := \begin{vmatrix} a_{11} - \lambda & \cdots & a_{1n'} \\ \vdots & \ddots & \vdots \\ a_{n'1} & \cdots & a_{n'n'} - \lambda \end{vmatrix} \neq 0 .$$

Taking $x_n = D$ in (1), we get by Cramer's rule a unique solution vector whose entries are polynomials of degree $n - 1$ in λ and the matrix entries a_{ij}. (The last entry of the solution is $x_n = D$.) It follows that we get an eigenvector of the form

$${}^t[p_1(\lambda) \cdots p_{n'}(\lambda) p_n(\lambda)]$$

where the same polynomials $p_j \in \mathbf{Z}[x]$ work for all the eigenvalues λ_i of A. Thus, an eigenvector for each λ_i is given by

$$\mathbf{e}_i = {}^t[p_1(\lambda_i) \cdots p_n(\lambda_i)] .$$

Now let $B \in GL_n(\mathbf{Z})$ have $\mathbf{e}_i$ as an eigenvector for some i. The eigenvector condition can be written as $(B\mathbf{e}_i) \wedge \mathbf{e}_i = 0$, which when written in terms of the p_j's is a set of $\binom{n}{2}$ polynomial equations with integer coefficients, of degree $(2n-2)$, all satisfied by λ_i. Since any polynomial with integer coefficients having λ_i as a root is divisible by the (irreducible) characteristic polynomial of A, such a polynomial has *all* of the λ_j's as roots; hence $B\mathbf{e}_j \wedge \mathbf{e}_j = 0$ for each j, and each $\mathbf{e}_j$ is an eigenvector for B. Thus B commutes with A.

To show that $B \in \mathbf{Q}[A]$, we note first that at least one p_j is not identically zero and hence, since its degree is less than n, has no λ_i as a root. The eigenvalue equation $B\mathbf{e}_j = \rho_j \mathbf{e}_j$ implies that

$$\sum B_{jk} p_k(\lambda_i) = \rho_i p_j(\lambda_i) \ ,$$

so $\rho_i = \dfrac{\sum B_{jk} p_k(\lambda_i)}{p_j(\lambda_i)}$ is a rational function $\dfrac{q}{r}$ of λ_i with integer coefficients independent of i. Hence $B = \dfrac{q(A)}{r(A)}$, and since $\mathbf{Q}[A]$ is a field (the minimal polynomial of A being irreducible), $B \in \mathbf{Q}[A]$.

Finally, if some ρ_i is rational, the polynomial $q - \rho_i r$ has rational coefficients and degree $n-1$, and has λ_i as a root, so $q - \rho_i r$ is identically zero, and hence $\dfrac{q}{r}$ is constant, so all ρ_i are equal, and B is a scalar matrix. ∎

4. Structures of class 3. Let $\pi = \sum a_i \partial_i^*$ be a Poisson structure of class 3, and suppose that $\mathcal{S}(a_1, a_2, a_3)$ is nontrivial; i.e. $A\mathbf{a} = \rho\mathbf{a}$ for some $\mathbf{a} \in SL_3(\mathbf{Z})$, $\rho \neq 1$. If A had a rational eigenvalue, it would have to be ± 1, and then A would have class 1 or 2, so we can assume that the characteristic polynomial of A is irreducible over $\mathbf{Q}$. By Lemma 1, the projective stabilizer of A consists of those matrices in $SL_3(\mathbf{Z})$ which are rational polynomials in A. That is, if $B \in SL_3(\mathbf{Z})$ shares one of the eigenvectors of A, Lemma 1 implies that

$$B = (r_0 I + r_1 A + r_2 A^2)/r_3$$

for some integers r_0, r_1, r_2 and $r_3 \neq 0$, so that B belongs to the field $\mathbf{Q}[A]$ generated by A, which is an algebraic number field. Since we assume that $B \in M_3\mathbf{Z}$, its characteristic polynomial has integer coefficients, and hence B is an algebraic integer. From the definition (see [3]) of the norm operator N of the number field $\mathbf{Q}[A]$ as a product of conjugates, we have

$$N(B) = \prod_{j=1}^{3} (r_0 + r_1\lambda_j + r_2{\lambda_j}^2)/r_3 \ .$$

Since A and B are simultaneously diagonalizable, $(r_0 + r_1\lambda_j + r_2{\lambda_j}^2)/r_3$ $(j = 1, 2, 3)$ are the eigenvalues of B, so $\prod_{j=1}^{3} (r_0 + r_1\lambda_j + r_2{\lambda_j}^2)/r_3 = \det(B)$, and we get $N(B) = \det(B)$. Thus, our $B \in SL_3(\mathbf{Z})$ is a unit with norm 1.

According to Dirichlet's Theorem on the units in number fields [3], the group of units is $\mathcal{U} \times \mathbf{Z}^{r+s-1}$ where

r = the number of real roots of χ_A, $\quad 2s$ = the number of complex roots of of χ_A

and $\mathcal{U}$ ={ roots of 1 in the number field }. Since the degree of the characteristic polynomial is 3 (and so odd), the finite cyclic group $\mathcal{U}$ is just $\{\pm 1\}$. If we assume that the eigenvalues of A are all real, the rank of the group of units is 2. If a generator g of the free abelian group has norm -1, then $-g$ is a generator of norm 1. Thus, we can get a set of generators of norm 1. Since norm 1 is equivalent to determinant 1, the arrive at the following description of the scaling group of a Poisson structure of class 3.

Proposition 3. Let $A \in SL_3(\mathbf{Z})$ be diagonalizable with (real) irrational eigenvalues. The subgroup of $B \in SL_3(\mathbf{Z})$ which shares one fixed eigenvector $\mathbf{e}$ of A is written as $\{P^m Q^n \mid m, n \in \mathbf{Z}\}$, where $P, Q \in SL_3(\mathbf{Z})$ are given by rational polynomials of A as

$$P = \frac{p_0 I + p_1 A + p_2 A^2}{p_3},\ Q = \frac{q_0 I + q_1 A + q_2 A^2}{q_3}\ .$$

The set $\mathcal{S}$ of eigenvalues is equal to $\{{\rho_1}^m {\rho_2}^n \mid m, n \in \mathbf{Z}\}$ where $\rho_1 = \dfrac{p_0\lambda + p_1\lambda + p_2\lambda^2}{p_3}$, $\rho_2 = \dfrac{q_0\lambda + q_1\lambda + q_2\lambda^2}{q_3}$.

Remark 1. We know of no general way to specify P and Q. The problem is close to that of determining which algebraic integers in $\mathbf{Q}[A]$ are matrices with integer entries. Here are two examples.

(1) Take the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. As generators of the group of units, we can take $g_1 = -3I + A$ and $g_2 = 2I - A$. Then we may set $P = g_1$ and $Q = g_2$. We see that $A = P^{-2}$.

(2) $A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & -1 & -1 \\ -2 & -2 & -3 \end{bmatrix}$. As generators of the group of units, we can take $(A - I)/2$ and $(-A - I)/2$. They are not integer matrices, and we have to set

$P = ((A - I)/2)\,((-A - I)/2)^2 = A$ and

$Q = ((A - I)/2)^4\,((-A - I)/2) = (3A^2 - 8A - I)/2$.

We consider next the case where two eigenvalues of A are non-real. In this case, the group of units is free of rank !, and we get a result similar to that in Proposition 3.

Proposition 4. Let $A \in SL_3(\mathbf{Z})$ have one irrational and two non-real eigenvalues. Let $\mathbf{e}$ be a eigenvector for a non-real eigenvalue, and consider the Poisson structure π given by $\pi = \sqrt{-1}\mathbf{e} \wedge \overline{\mathbf{e}}$. The subgroup $\mathcal{G}$ of $B \in SL_3(\mathbf{Z})$ with the property

$$\wedge^2(B)(\pi) = \rho\pi$$

is written as $\{P^m \mid m \in \mathbf{Z}\}$, where $P \in SL_3(\mathbf{Z})$ is a rational polynomial in A:

$$P = \frac{p_0 I + p_1 A + p_2 A^2}{p_3}\ .$$

The set $\mathcal{S}$ of eigenvalues is equal to $\{\rho^m \mid m \in \mathbf{Z}\}$ where $\rho = \dfrac{p_0\lambda + p_1\lambda + p_2\lambda^2}{p_3}$ and λ is the real eigenvalue of A.

Remark 2. (1) Take the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$. As generator of the group of units, we can take A itself and $\mathcal{G} = \{A^m \mid m \in \mathbf{Z}\}$.

(2) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -8 \\ 0 & 1 & 0 \end{bmatrix}$. As generator of the group of units, we can take $(A^2 - 2A + 2I)/5$, which is not a $\mathbf{Z}$-matrix, and $A = \big((A^2 - 2A + 2I)/5\big)^2$. Now we see that

$$\mathcal{G} = \{A^m \mid m \in \mathbf{Z}\} \ .$$

(3) If we start with A^2 for A in (2), then $\mathcal{G}$ is generated by A, not by A^2.

5. Structures of class 2. The analysis of structures of class 3 is similar to that for class 3, except that we only need to consider 2×2 matrices. The relevant number field is now quadratic, and our conclusion is the following.

Proposition 5. *If a Poisson structure of class 2 is a nontrivial eigenvector for a matrix $A \in SL_3(\mathbf{Z})$, then the scaling group of this structure has the form $\{\pm(\rho^k)^m \mid m \in \mathbf{Z}\}$ where $\rho = \dfrac{r_0 + r_1\lambda}{r_2}$ is the fundamental unit of the number field $\mathbf{Q}[\lambda]$ of $\lambda^2 - (\mathrm{traceA} - \varepsilon)\lambda + \varepsilon = 0$ and k is some integer.*

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