Some comments on the history, theory, and applications of symplectic reduction

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Abstract. We make some brief remarks about the history, theory and applications of symplectic reduction. We concentrate on developments surrounding our paper Marsden and Weinstein (1974) and the closely related work of Meyer (1973).

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1. Reduction of Symplectic Manifolds

Most readers of this volume presumably know how symplectic reduction goes: given a hamiltonian action of a Lie group on a symplectic manifold, one divides a level set of a momentum map by the action of a suitable subgroup to form a new symplectic manifold. Before the division step, one has a manifold (possibly singular, an occurrence without which this volume would not exist) carrying a degenerate closed 2-form. Removing such a degeneracy by passing to a quotient space was a well-known differential-geometric operation promoted by Élie Cartan (1922). The "suitable subgroup" related to a momentum mapping was identified by Steven Smale (1970) in a special case, without the symplectic trappings. It was Smale's work that inspired the general symplectic construction by Meyer and ourselves.

More should be said about momentum maps themselves. The idea that an action of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold P should be accompanied by a map $J : P \to \mathfrak{g}^*$ which is equivariant with respect to the coadjoint action, and the fact that the orbits of this action are themselves symplectic manifolds both occur already in Lie (1890); the links with mechanics also rely on the work of Lagrange, Poisson, Jacobi and Noether. In modern form, the momentum map and its equivariance were rediscovered by Kostant (1966) and Souriau (1966, 1970) in the general symplectic case and by Smale (1970) for the case of the lifted action from a manifold Q to its cotangent bundle $P = T^*Q$.

As for terminology, neither Lie nor Kostant gave the map J a special name. Smale referred to it as the "angular momentum" by generalization from the special case $G = SO(3)$, while Souriau called it by the French word "moment". In our paper Marsden and Weinstein (1974), following usage emerging at that time, we used the English word "moment" for J, but we were soon set straight by Richard Cushman and Hans Duistermaat, who convinced us that the proper English translation of Souriau's French word was "momentum," which had the added benefit of meshing with Smale's designation and standard usage in mechanics. Since 1976 or so, we have referred to J as a momentum map (or mapping); for example, this term is used in Abraham and Marsden (1978). On the other hand, Guillemin and Sternberg popularized the continuing use of "moment" in English, and both words coexist today. (See the footnote on page 133 of Mikami and Weinstein (1988) for a semi-serious attempt to bridge the gap.) It is a curious twist, as comes out in work on collective nuclear motion (Guillemin and Sternberg, (1980)) and plasma physics (Marsden and Weinstein (1982) and Marsden, Weinstein, Ratiu and Schmid (1983)), that moments of inertia and moments of probability distributions can actually be the values of momentum maps! See Marsden and Ratiu (1999) for more on the history of the momentum map.

Passing to reduction itself, we find many precursors in the case where G is abelian, the components of the momentum map then forming a system of functions in involution (i.e. the Poisson bracket of any two is zero). The use of k such functions to reduce a phase space to one having $2k$ fewer dimensions may be found already in the work of Lagrange, Poisson, Jacobi, and Routh; it is well described in, for example, Whittaker (1907). Smale (1970) noted that Jacobi's "elimination of the node" in $SO(3)$ symmetric problems is best understood as division of a nonzero angular momentum level by the $SO(2)$ subgroup that fixes the momentum value. In his setting of cotangent bundles, Smale clearly stated that the coadjoint isotropy group G_μ of $\mu \in \mathfrak{g}^*$ leaves $J^{-1}(\mu)$ invariant (Smale (1970), Corollary 4.5), but he only divided by G_μ after fixing the total energy as well, in order to obtain the "minimal" manifold on which to analyze the reduced dynamics. The goal of his "topology and mechanics" program was to use topology, and specifically Morse theory, to study relative equilibria.

In Marsden and Weinstein (1974), we combined Souriau's momentum map for general symplectic actions, Smale's idea of dividing the momentum level by the coadjoint isotropy group, and Cartan's idea of removing the degeneracy of a 2-form by passing to the leaf space of the form's null foliation. The key observation was that the leaves of the null foliation are precisely the (connected components of the) orbits of the coadjoint isotropy group. The same observation was made by Meyer (1973), except that Meyer worked in terms of a basis for the Lie algebra $\mathfrak{g}$ and identified the subgroup G_μ as the group which left the momentum level set $J^{-1}(\mu)$ invariant. In this way, he did not have to deal with the equivariance properties of the coadjoint representation.

Perhaps our favorite example in Marsden and Weinstein (1974) was the construction of the coadjoint orbits in $\mathfrak{g}^*$ by reduction of the cotangent bundle T^*G with its canonical symplectic structure. This example, which "explained" Kostant and Souriau's formula for this structure, is typical of many of the ensuing applications of reduction, in which the procedure is applied to a "trivial" symplectic manifold to produce something interesting. When G is the group of (volume preserving) diffeomorphisms of a compact manifold (possibly with boundary), one obtains the Euler equations for (incompressible) fluids by reduction from the lagrangian formulation of the equations of motion, an idea exploited by Arnold (1966,a) and Ebin and Marsden (1970). This sort of description of a fluid goes back to Poincaré (using the Euler–Poincaré equations) and to the thesis of Ehrenfest (as geodesics on the diffeomorphism group), written under the direction of Boltzmann. We shall comment on this in a little more detail below.

Another example in Marsden and Weinstein (1974) came from general relativity, namely the reduction of the cotangent bundle of the space of riemannian metrics on a manifold M by the action of the group of diffeomorphisms of M. In this case, restriction to the zero momentum level is the so called divergence constraint of general relativity, and one is led to a construction of a symplectic structure on a space of isometry classes of Einstein manifolds. Here one sees a precursor of an idea of Atiyah and Bott (1982), which has led to some of the most spectacular applications of reduction in mathematical physics and related areas of pure mathematics, especially low-dimensional topology.

Atiyah and Bott start with the space $\mathcal{A}$ of connections on a principal bundle with compact structure group K over a closed oriented surface Σ. For simplicity of description, assume that this bundle is trivial. Using a bi-invariant inner product on its Lie algebra $\mathfrak{k}$ and integration over Σ, they define a skew-symmetric pairing on $\mathfrak{k}$-valued 1-forms on Σ which gives a symplectic structure on $\mathcal{A}$. This structure is invariant under the action of the gauge group G of bundle automorphisms. The dual of the Lie algebra of G may be identified with $\mathfrak{k}$-valued 2-forms on Σ, and the curvature map from connections to 2-forms turns out to be an equivariant momentum map for the G action. Reducing at the momentum level zero therefore amounts to taking the space of flat connections and passing to the moduli space of their gauge equivalence classes. This moduli space $\mathcal{M}$ thus inherits a symplectic structure. But the holonomy construction allows one to identify $\mathcal{M}$ with the space of homomorphisms from the fundamental group of Σ to K, modulo conjugation by elements of K. The latter space is also identifiable with a space of isomorphism classes of holomorphic vector bundles when Σ is equipped with a complex structure. One thus obtains a symplectic structure on these other moduli spaces as well.

In the paragraphs above, we have blithely been assuming that the momentum levels and their quotients are smooth manifolds. Of course, this is not always the case, as was already noted in Smale (1970) and analyzed (even in the infinite-dimensional case) in Arms, Marsden and Moncrief (1981). We will make just a few more comments about singular reduction below, leaving the reader to learn much more from the contents of this volume.

The rest of this preface will consist of further remarks about reduction, most of them historical.

2. History before 1960

So far, we have presented reduction as a mathematical construction, but this construction is actually rooted in classical work on mechanical systems with symmetry by such masters as Euler, Lagrange, Hamilton, Jacobi, Routh, Riemann, Liouville, Lie, and Poincaré. The aim of their work was to eliminate variables associated with symmetries in order to simplify calculations in concrete examples. Much of this work was done with coordinates, although the deep connection between mechanics and geometry was already evident. Whittaker (1907) gives a good picture of the theory as it existed before about 1910.

A highlight of this early theory was Routh (1860, 1884), on reduction of systems with cyclic variables, introducing the amended potential for the reduced system. Routh's work was closely related to the reduction of systems with integrals in involution studied by Jacobi and Liouville around 1870; it corresponds to the modern theory of Lagrangian reduction for the action of Abelian groups.

The rigid body, whose equations were discovered by Euler around 1740, was a key example of reduction–what we would call today either coadjoint orbit reduction or Euler–Poincaré reduction, depending on one's point of view. Lagrange (1788) already understood reduction of the rigid body equation by a method not so far from what we do today with the symmetry group SO(3). Later authors, unfortunately, relied so much on coordinates (especially Euler angles) that there is little mention of SO(3) in classical mechanics books written before 1990! In addition, there seemed to be little appreciation until recently for the role of topological notions; for example, the fact that one cannot globally split off cyclic variables for the S^1 action on the configuration space of the heavy top. The Hopf fibration was sitting, waiting to be discovered, in the reduction theory for the classical rigid body, but it was not explicitly found by H. Hopf until around 1940. Hopf was apparently unaware that this example is of great mechanical interest; the gap between workers in mechanics and geometers seems to have been particularly wide at that time.

Another noteworthy instance of reduction is Jacobi's elimination of the node for reducing the gravitational (or electrostatic) n-body problem by means of the group SE(3) of Euclidean motions, around 1860 or so. This example has been of course been a mainstay of celestial mechanics. It is related to the work done by Riemann, Jacobi, Poincaré and others on rotating fluid masses held together by gravitational forces, such as stars. Hidden in these examples is much of the beauty of modern reduction, stability and bifurcation theory for mechanical systems with symmetry.

Both symplectic and Poisson geometry have their roots in the work of Lagrange and Jacobi and matured considerably at the hands of Lie, who discovered many remarkably modern concepts such as the Lie–Poisson bracket on the dual of a Lie algebra (see Weinstein (1983) and Marsden and Ratiu (1999) for more details). How Lie could have viewed it so divorced from its roots in mechanics is a bit of a mystery. We can only guess that he was inspired by Jacobi, Lagrange and Riemann and then quickly abstracted the ideas. As we have already hinted at, it was the famous paper Poincaré (1901) where we find what we call today the Euler–Poincaré equations–a generalization of the Euler equations for both fluids and the rigid body to general Lie algebras. It is perhaps curious that Poincaré seemed not to stress the symplectic ideas of Lie, and it is not clear to what extent he understood what we would call today Euler–Poincaré reduction, a theme picked up later by Arnold (1966,a).

It was only with the development and physical application of the notion of a manifold, pioneered by Lie, Poincaré, Weyl, Cartan, Reeb, Synge and many others, that a more general and intrinsic view of mechanics emerged.

3. 1960-1972

Beginning in the 1960's, the subject of geometric mechanics exploded with the basic contributions of people such as (alphabetically and nonexhaustively) Abraham, Arnold, Kirillov, Kostant, Mackey, MacLane, Segal, Sternberg, Smale, and Souriau. Kirillov and Kostant found deep connections between mechanics and pure mathematics in their work on the orbit method in group representations, while Arnold, Smale, and Souriau were in closer touch with mechanics.

The modern vision of mechanics combines strong links to important questions in pure mathematics with the traditional classical mechanics of particles, rigid bodies, fields, fluids, plasmas, and elastic solids, as well as quantum and relativistic theories. Symmetries in these theories vary from obvious translational and rotational symmetries to less obvious particle relabeling symmetries in fluids and plasmas, to the "hidden" symmetries underlying integrable systems. As we have already mentioned, reduction theory concerns the removal of variables using symmetries and their associated conservation laws. Variational principles, in addition to symplectic and Poisson geometry, provide fundamental tools for this endeavor. In fact, conservation of the momentum map associated with a symmetry group action is a geometric expression of the classical Noether theorem (discovered by variational, not symplectic methods).

For us, the modern era of reduction theory began with the fundamental papers of Arnold (1966,a) and Smale (1970). Arnold focused on systems whose configuration manifold is a Lie group, while Smale focused on bifurcations of relative equilibria. Both linked their theory strongly with examples. For Arnold, they were the same examples as for Poincaré, namely the rigid body and fluids, for which he went on to develop powerful stability methods, as in Arnold (1966,b).

For Smale, the motivating example was celestial mechanics, especially the study of the number and stability of relative equilibria by a topological study of the energy-momentum mapping. He gave an intrinsic geometric account of the amended potential and in doing so, discovered what later became known as the mechanical connection. (Smale seems not to have recognized that the interesting object he called α is a principal connection; this was first noted by Kummer (1981).) One of Smale's key ideas in studying relative equilibria was to link mechanics with topology via the fact that relative equilibria are critical points of the amended potential. Besides giving a beautiful exposition of the momentum map, he also emphasized the connection between singularities and symmetry, observing that the symmetry group of a phase space point has positive dimension iff that point is not a regular point of the momentum map restricted to a fibre of the cotangent bundle (Smale (1970), Proposition 6.2). He went on from here to develop his topology and mechanics program and to apply it to the planar n-body problem. The topology and mechanics program definitely involved reduction ideas, as in Smale's construction of the quotients of integral manifolds, as in $I_{c,p}/S^1$ (Smale (1970), page 320). He also understood Jacobi's elimination of the node in this context, although he did not attempt to give any general theory of reduction along these lines. In summary, Smale set the stage for symplectic reduction: he realized the importance of the momentum map and of quotient constructions, and he worked out explicit examples like the planar n-body problem with its S^1 symmetry group. (Interestingly, he pointed out that one should really use the nonabelian group SE(2); his feeling of unease with fixing the center of mass of an n-body system is remarkably perceptive.)

To synthesize the Lie algebra reduction methods of Arnold (1966,a) with the techniques of Smale (1970) on the reduction of cotangent bundles by Abelian groups, we were led in our paper (Marsden and Weinstein (1974)) to develop reduction theory in the general context of symplectic manifolds and equivariant momentum maps. This takes us up to about 1972.

4. Post 1972

An important contribution was made by Marle (1976), who divides the inverse image of an orbit by its characteristic foliation to obtain the product of an orbit and a reduced manifold. In particular, one finds that P_μ is symplectically diffeomorphic to an "orbit-reduced" space $P_\mu \cong J^{-1}(\mathcal{O}_\mu)/G$, where $\mathcal{O}_\mu$ is a coadjoint orbit of G. From this it follows that the P_μ are symplectic leaves in the Poisson space P/G. The related paper of Kazhdan, Kostant and Sternberg (1978) was one of the first to notice deep links between reduction and integrable systems. In particular, they found that the Calogero–Moser systems could be obtained by reducing a system that was trivially integrable; in this way, reduction provided a method of producing an interesting integrable system from a simple one. This point of view was used again by, for example, Bobenko, Reyman and Semenov-Tian-Shansky (1989) in their spectacular group theoretic explanation of the integrability of the Kowalewski top.

4.1. Noncanonical Poisson Brackets

The Hamiltonian description of many physical systems, such as rigid bodies and fluids in Eulerian variables, requires noncanonical Poisson brackets and constrained variational principles of the sort studied by Lie and Poincaré. An example of a noncanonical Poisson bracket is the Lie–Poisson bracket on $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$. These Poisson structures, including the coadjoint orbits as their symplectic leaves, were known to Lie around 1890, although Lie does not seem to have recognized their importance in mechanics.

In mechanics, the remarkably modern (but rather out of touch with the corresponding mathematical developments) book by Sudarshan and Mukunda (1974) showed via explicit examples how systems like rigid bodies could be written in terms of noncanonical brackets. See also Nambu (1973). Others in the physics community, such as Morrison and Greene (1980) also discovered noncanonical bracket formalisms for fluid and magnetohydrodynamic systems. In the 1980's, many fluid and plasma systems were shown to have a noncanonical Poisson formulation. It was Marsden and Weinstein (1982, 1983) who first applied reduction techniques to these systems. The philosophy was that any mechanical system has its roots somewhere as a cotangent bundle and that one can recover noncanonical brackets by reduction. This ran contrary to the point of view, taken by some physicists, that one should guess at what a Poisson structure might be and then to try to limit the guesses by the constraint of Jacobi's identity. In the simplest Poisson reduction process, one starts with a Poisson manifold P on which a group G acts by Poisson maps and then forms the quotient space P/G, which, if not singular, inherits a natural Poisson structure itself. Of course, the Lie–Poisson structure on $\mathfrak{g}^*$ is inherited in exactly this way from the canonical symplectic structure on T^*G. One of the attractions of this Poisson bracket formalism was its use in stability theory. This literature is now very large, but Holm, Marsden, Ratiu and Weinstein (1985) is representative.

The way in which the Poisson structure on P_μ is related to that on P/G was clarified in a generalization of Poisson reduction due to Marsden and Ratiu (1986), a technique that has also proven useful in integrable systems (see, e.g., Pedroni (1995) and Vanhaecke (1996)).

Reduction theory for mechanical systems with symmetry has proven to be a powerful tool enabling advances in stability theory (from the Arnold method to the energy-momentum method) as well as in bifurcation theory of mechanical systems, geometric phases via reconstruction—the inverse of reduction—as well as uses in control theory from stabilization results to a deeper understanding of locomotion. For a general introduction to some of these ideas and for further references, see Marsden, Montgomery and Ratiu (1990), Marsden and Ostrowski (1998), and Marsden and Ratiu (1999).

4.2. Lagrangian Reduction

Routh reduction for Lagrangian systems is classically associated with systems having cyclic variables (this is almost synonymous with having an Abelian symmetry group); modern expositions of this theory can be found in Arnold, Kozlov and Neishtadt (1988) and in Marsden and Ratiu (1999), §8.9. A key feature of Routh reduction is that when one drops the Euler–Lagrange equations to the quotient space associated with the symmetry, and when the momentum map is constrained to a specified value (i.e., when the cyclic variables and their velocities are eliminated using the given value of the momentum), then the resulting equations are in Euler–Lagrange form not with respect to the Lagrangian itself, but with respect to a modified function called the Routhian. Routh (1877) applied his method to stability theory; this was a precursor to the energy-momentum method for stability that synthesizes Arnold's and Routh's methods (Simo, Lewis and Marsden (1991); see Marsden (1992) for an exposition and references). Routh's stability method is still widely used in mechanics.

Another key ingredient in Lagrangian reduction is the classical work of Poincaré (1901) in which the Euler–Poincaré equations were introduced. Poincaré realized that the equations of fluids, free rigid bodies, and heavy tops could all be described in Lie algebraic terms in a beautiful way. The importance of these equations was realized by Hamel (1904, 1949) and Chetayev (1941).

4.3. Tangent and Cotangent Bundle Reduction

The simplest case of cotangent bundle reduction is reduction of $P = T^*Q$ at $\mu = 0$, giving $P_0 = T^*(Q/G)$ with the canonical symplectic form. Another basic case is when G is Abelian. Here, $(T^*Q)_\mu \cong T^*(Q/G)$, but the latter has a symplectic structure modified by magnetic terms, that is, by the curvature of the mechanical connection.

The Abelian version of cotangent bundle reduction was developed by Smale (1970) and Satzer (1977) and was generalized to the nonabelian case in Abraham and Marsden (1978). Kummer (1981) introduced the interpretations of these results in terms of a connection, now called the mechanical connection. The geometry of this situation was used to great effect in, for example, Guichardet (1984), Iwai(1987, 1990), and Montgomery (1984, 1990, 1991). Routh reduction may be viewed as the Lagrangian analogue of cotangent bundle reduction.

Tangent and cotangent bundle reduction evolved into what we now term as the "bundle picture" or the "gauge theory of mechanics". This picture was first developed by Montgomery, Marsden and Ratiu (1984) and Montgomery (1984, 1986). That work was motivated and influenced by the work of Sternberg (1977) and Weinstein (1978) on a "Yang-Mills construction" which is, in turn, motivated by Wong's equations, i.e. the equations for a particle moving in a Yang-Mills field. The main result of the bundle picture gives a structure to the quotient spaces $(T^*Q)/G$ and $(TQ)/G$ when G acts by the cotangent and tangent lifted actions.

4.4. Semidirect Product Reduction

Recall that in the simplest case of a semidirect product, one has a Lie group G that acts on a vector space V (and hence on its dual V^*) and then one forms the semidirect product $S = G \circledS V$, generalizing the semidirect product structure of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \circledS \mathbb{R}^3$.

Consider the isotropy group G_{a_0} for some $a_0 \in V^*$. The semidirect product reduction theorem states that each of the symplectic reduced spaces for the action of G_{a_0} on T^*G is symplectically diffeomorphic to a coadjoint orbit in $(\mathfrak{g} \circledS V)^*$, the dual of the Lie algebra of the semi-direct product. This semidirect product theory was developed by Guillemin and Sternberg (1978, 1980), Ratiu (1980, 1981, 1982), and Marsden, Ratiu and Weinstein (1984,a,b). The Lagrangian reduction version of this theory was developed by Holm, Marsden and Ratiu (1998,a). This construction is used in applications where one has advected quantities (such as the direction of gravity in the heavy top, density in compressible flow and the magnetic field in MHD). Its Lagrangian counterpart was developed in Holm, Marsden and Ratiu (1998,b), along with applications to continuum mechanics. Cendra, Holm, Hoyle and Marsden (1998) applied this idea to the Maxwell–Vlasov equations of plasma physics. Cendra, Holm, Marsden, and Ratiu (1998) showed how Lagrangian semidirect product theory fits into the general framework of Lagrangian reduction.

4.5. Nonabelian Routh Reduction

The papers by Marsden and Scheurle (1993,a,b) showed how to generalize the Routh theory to the nonabelian case and how to get the Euler–Poincaré equations for matrix groups by the important technique of reducing variational principles. This approach was motivated by earlier work of Cendra and Marsden (1987) and Cendra, Ibort and Marsden (1987). Related ideas stressing the groupoid point of view were given in Weinstein (1996). The work of Bloch, Krishnaprasad, Marsden and Murray (1996) generalized the Euler–Poincaré variational structure to general Lie groups, and Cendra, Marsden and Ratiu (2000,a) carried out a Lagrangian reduction theory that extends the Euler–Poincaré case to arbitrary configuration manifolds. This work is the Lagrangian analogue of Poisson reduction, in the sense that no momentum map constraint is imposed.

Until recently, the Lagrangian side of the reduction story has lacked a general category that is the Lagrangian analogue of Poisson manifolds. One candidate is the category of Lie algebroids, as explained in Weinstein (1996). Another is that of Lagrange–Poincaré bundles, developed in Cendra, Marsden and Ratiu (2000,a). Both have tangent bundles and Lie algebras as basic examples. The latter work also develops the Lagrangian analogue of reduction for central extensions and, as in the case of symplectic reduction by stages (see Marsden, Misiolek, Perlmutter and Ratiu (2000)), cocycles and curvatures enter in a natural way.

The Lagrangian analogue of the symplectic bundle picture is the bundle $(TQ)/G$, which is a vector bundle over Q/G; In particular, the equations and variational principles live on this space. For $Q = G$ this reduces to Euler–Poincaré reduction and for G Abelian, it reduces to the classical Routh procedure. A G-invariant Lagrangian L on TQ induces a Lagrangian l on $(TQ)/G$. The resulting equations inherited on this space are the Lagrange–Poincaré equations (or the reduced Euler–Lagrange equations).

Lagrangian reduction has proven very useful in optimal control problems. It was used in Koon and Marsden (1997) to extend the falling cat theorem of Montgomery (1990) to the case of nonholonomic systems as well as to non-zero values of the momentum map.

4.6. Reduction by Stages and Group Extensions

There are many precursors to the general theory of reduction by stages. A simple version for the product of two groups was given in Marsden and Weinstein (1974). Other versions are due to Sjamaar and Lerman (1991) and Landsman (1995, 1998).

The semidirect product reduction theorem can be very nicely viewed using reduction by stages: one reduces T^*S by the action of the semidirect product group $S = G Ⓢ V$ in two stages, first by the action of V at a point a_0 and then by the action of G_{a_0}. Semidirect product reduction by stages for actions of semidirect products on general symplectic manifolds was developed and applied to underwater vehicle dynamics in Leonard and Marsden (1997). Motivated partly by semidirect product reduction, Marsden, Misiolek, Perlmutter and Ratiu (2000) gave a generalization of semidirect product theory in which one has a group M with a normal subgroup $N \subset M$ (so M is a group extension of N) and M acts on a symplectic manifold P. One wants to reduce P in two stages, first by N and then by M/N. On the Poisson level this is easy: $P/M \cong (P/N)/(M/N)$, but on the symplectic level it is quite subtle.

An interesting extension which is not a semidirect product is the Bott–Virasoro group, where the Gelfand-Fuchs cocycle may be interpreted as the curvature of a mechanical connection. The work of Cendra, Marsden and Ratiu (2000,a) briefly described above, contains a Lagrangian analogue of reduction for group extensions and reduction by stages.

4.7. Singular Reduction

Singular reduction starts with the observation of Smale (1970) that we have already mentioned: $z \in P$ is a regular point of a momentum map J iff z has no continuous isotropy. Motivated by this, Arms, Marsden and Moncrief (1981, 1982) showed that (under hypotheses which include the ellipticity of certain operators and which reduce more or less to a properness assumption on the group action in the finite dimensional case) the level sets $J^{-1}(0)$ of an equivariant momentum map J have quadratic singularities at points with continuous symmetry. While such a result is easy to prove for compact group actions on finite dimensional manifolds (using the equivariant Darboux theorem), the main examples of Arms, Marsden and Moncrief (1981) were, in fact, infinite dimensional—both the phase space and the group. Singular points in the level sets of the momentum map are related to convexity properties of the momentum map in that the singular points in phase space map to corresponding singular points in the the image polytope.

The paper of Otto (1987) showed that if G is a compact Lie group acting freely, $J^{-1}(0)/G$ is an orbifold. The detailed structure of $J^{-1}(0)/G$ for compact Lie groups acting on finite dimensional manifolds was determined by Sjamaar and Lerman (1991); their work was extended to proper Lie group actions and to $J^{-1}(\mathcal{O}_\mu)/G$ by Bates and Lerman (1997), with the assumption that $\mathcal{O}_\mu$ be locally closed in $\mathfrak{g}^*$. Ortega (1998) and Ortega and Ratiu (2001) redid the entire singular reduction theory for proper Lie group actions starting with the point reduced spaces $J^{-1}(\mu)/G_\mu$, and also connected it to the more algebraic approach of Arms, Cushman and Gotay (1991). Specific examples of singular reduction, with further references, may be found in Cushman and Bates (1997). Huebschmann (1998) has made an unusually careful study of the singularities of moduli spaces of flat connections; also see other papers cited therein, as well as his paper in this volume.

4.8. The Method of Invariants

This method seeks to parametrize quotient spaces by group invariant functions. It has a rich history going back to Hilbert's invariant theory. It has been of great use in bifurcation with symmetry (see Golubitsky, Stewart and Schaeffer (1988), for instance). In mechanics, the method was developed by Kummer, Cushman, Rod and coworkers in the 1980's. We will not attempt to give a literature survey here, other than to refer to Kummer (1990), Kirk, Marsden and Silber (1996), Alber, Luther, Marsden and Robbins (1998) and Cushman and Bates (1997) for more details and references.

4.9. Nonholonomic Systems

Nonholonomic mechanical systems (such as systems with rolling constraints) provide a very interesting class of systems where the reduction procedure has to be modified. In fact this provides a class of systems that gives rise to an almost Poisson structure, i.e. a bracket which does not necessarily satisfy the Jacobi identity. Reduction theory for nonholonomic systems has made a lot of progress, but many interesting questions still remain. A few key references are Koiller (1992), Bates and Sniatycki (1993), Bloch, Krishnaprasad, Marsden and Murray (1996), and Koon and Marsden (1998). We refer to Cendra, Marsden and Ratiu (2000,b) for a more detailed historical review.

4.10. Quantum Mechanics

Of course geometric mechanics has a lot to say about quantum mechanics. One popular topic (perhaps more among mathematicians than physicists) is the issue of quantization. This large subject is considered in detail in many works, such as Guillemin and Sternberg (1977), Abraham and Marsden (1978), Woodhouse (1992), Bates and Weinstein (1997), and Landsman (1998). This is intimately connected with important topics such as the geometric phase. In our own attempt to understand some of this (Marsden and Weinstein (1979); see also Eckmann and Seneor (1976)), we found much to be gained by studying simple examples.

A whole industry has grown up (led by Guillemin and Sternberg (1982)) around the question of reduction and quantization and the issue of whether or not these operations commute; the answer is generally "yes." While the subject has matured very much mathematically, there is a surprising lack of attention to examples. For instance, it is hard to find references that even treat the classical and supposedly well understood example of the rigid body, which, by the way, was the topic of Casimir's (1931) thesis..

Another interesting issue, which is directly the subject of this volume, is that of the role of singular reduction in quantum problems. As far as we know, one of the first papers in this topic and still one of the interesting ones is that of Emmrich and Römer (1990), which was written when most of the literature on singular reduction was just getting started. This paper indicates that wave functions often 'congregate' near singular points, which goes counter to the sometimes quoted statement that singular points in quantum problems are a set of measure zero so cannot possibly be important. It is also noteworthy that some of the most fundamental and important field theories have singularities in their solution space at some of the most interesting and physically relevant solutions, namely the symmetric ones in Einstein's gravitational theory; it is still not understood what role these singularities might have on quantum gravity. See for example, Moncrief (1978), Fischer, Marsden and Moncrief (1980), Arms, Marsden and Moncrief (1982), and Fischer and Moncrief (1997) and references therein and for other interesting links with Teichmüller and Thurston theory.

The uses of geometric mechanics in quantum mechanics goes much beyond the issues already mentioned, especially in the physics and chemistry communities, where one is interested in topics such as separating rotational and vibrational motions; the ideas of reduction are central here (see Marsden (1992) for some of the classical aspects of this subject). We mention only the recent papers of Littlejohn and Reinsch (1997) and Tanimura and Iwai (2000) as examples of the wonderful things one can do with geometric mechanics in quantum theory.

4.11. Multisymplectic Reduction and Discrete Mechanical Systems

Reduction theory is by no means completed. For example, for PDE's, the multisymplectic (as opposed to symplectic) framework seems appropriate, both for relativistic and nonrelativistic systems. In fact, this approach has experienced somewhat of a revival since it has been realized that it is rather useful for numerical computation (see Marsden, Patrick and Shkoller (1998)). Only a few instances and examples of multisymplectic reduction are really well understood (see Marsden, Montgomery, Morrison and Thompson (1986) and Castrillon Lopez, Ratiu and Shkoller (2000)), so one can expect to see more activity in this area as well.

Another emerging area, also motivated by numerical analysis, is that of discrete mechanics. Here the idea is to replace the velocity phase space TQ by $Q \times Q$, with the role of a velocity vector played by a pair of nearby points. This has been a powerful tool for numerical analysis, reproducing standard symplectic integration algorithms and much more. See, for example, Kane, Marsden, Ortiz and West (2000) for a recent article. This subject, too, has its own reduction theory. See Marsden, Pekarsky and Shkoller (1999), Bobenko and Suris (1999), and Jalnapurkar, Leok, Marsden and West (2001). Discrete mechanics also has some intriguing links with quantization, since Feynman himself first defined path integrals through a limiting process using the sort of discretization used in the discrete action principle; see Feynman and Hibbs (1965).

4.12. Conclusions

As we hope this brief history shows, reduction theory remains very much an active and exciting area of research with many significant applications to problems of interest in both engineering and physics. This is rather remarkable since this theory in its modern form is nearly 30 years old. The future still looks bright with many exciting things remaining to be accomplished.

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