The symplectic structure on moduli space
(in memory of Andreas Floer)

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1 Introduction

The "moduli space" referred to in the title of this paper is the space $\mathcal{M}$ of gauge equivalence classes of flat connections on a principal G bundle over a compact Riemann surface Σ. Atiyah and Bott [1] constructed a symplectic structure $\omega_{\mathcal{M}}$ on $\mathcal{M}$ by symplectic reduction from the infinite dimensional symplectic manifold of *all* connections. Since $\mathcal{M}$ is a finite-dimensional object, it seems desirable to have a finite-dimensional construction of its symplectic form. In fact, $\mathcal{M}$ can also be realized as the representation space $\mathrm{Hom}(\pi, G)/G$, where π is the fundamental group of Σ and G acts by conjugation. Using the resulting identification of the tangent spaces of $\mathcal{M}$ with cohomology spaces of π with suitable coefficients, Goldman [9] gave a direct construction of $\omega_{\mathcal{M}}$ as a nondegenerate 2-form, but he was unable to prove that this form is closed without recourse to the infinite-dimensional picture. This gap in Goldman's approach was filled recently by Karshon [10], who showed that $d\omega_{\mathcal{M}} = 0$ by methods of group cohomology, without using the space of connections. [1]

The purpose of this paper is to place Karshon's computations in an appropriate general setting, namely the double complex of forms on cartesian powers of G [5] [14], or more precisely the corresponding complex of equivariant forms [2] [3] with respect to the action of G by conjugation.

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[1] L. Guillou and J. Huebschmann also inform me that they have found finite-dimensional proofs that the form is closed.

Andreas Floer's refinement of the Casson invariant for homology 3-spheres was constructed in terms of the gradient "flow" of the Chern-Simons invariant on a space of connections. Floer himself remarked that there should be an alternative construction of his invariant which, like Casson's own, takes place on the finite-dimensional space $\mathcal{M}$. [2] Such a construction would use the symplectic structure of $\mathcal{M}$ in an essential way. We hope that the new description of the symplectic structure here may be of some assistance in the realization of Floer's hope.

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2 The de Rham–bar bicomplex

The standard "bar" complex for computing the cohomology of a group G [6] involves functions on cartesian powers of the group. When G is a Lie group, the functions can be replaced by differential forms, and one obtains a double complex $C^{**}(G)$, with $C^{p,q}(G)$ defined to be the space $\Omega^q(G^p)$ of q forms on G^p. The second coboundary operator is just the exterior differential d, while the first, δ, is defined by the usual formula from group cohomology, with some reinterpretation of the notation:

$$\delta\alpha(a_0, \ldots, a_p) =$$

$$\alpha(a_1, \ldots, a_p) + \sum_{i=1}^{p}(-1)^i\alpha(a_0, \ldots, a_{i-1}a_i, \ldots, a_p) + (-1)^{p+1}\alpha(a_0, \ldots, a_{p-1}).$$

Here, when ϕ is a differential form on Y and $y = f(x)$ is the notation "in variables" for a mapping $f : X \to Y$, the expression $\phi(f(x))$ is interpreted to mean the pullback of ϕ to X by f, just as in the case of functions.

The importance of the de Rham-bar bicomplex revolves around the fact that the cohomology of the corresponding total complex (with differential $(-1)^p d + \delta$) is naturally isomorphic to the real cohomology $H^*(BG)$ of the universal classifying space of G. Furthermore, there is a natural mapping Φ from the ring of invariant polynomials on the Lie algebra $\mathfrak{g}$ to the total cocycles in $C^{**}(G)$ (see [14]).

Application of the de Rham–bar bicomplex $C^{**}(G)$ to the moduli space comes about through the following construction. Let π be a discrete group. We consider the space $\mathrm{Hom}(\pi, G)$ as a singular "manifold" as is usual in this subject [9]. The evaluation mappings $E : \pi^p \times \mathrm{Hom}(\pi, G) \to G^p$ induce

[2] Some progress in this direction is reported in [7].

pullbacks $E^* : C^{p,q}(G) \to \Omega^q(\pi^p \times \mathrm{Hom}(\pi, G))$. Since π is discrete, the last space can be identified with $C^p(\pi) \otimes \Omega^q(\mathrm{Hom}(\pi, G))$, where $C^p(\pi)$ is the usual space of (real-valued) p-cochains on π, i.e. the functions on π^p.

Our key observation is that E^* is a mapping of bicomplexes, where $C^*(\pi) \otimes \Omega^*(\mathrm{Hom}(\pi, G))$ is considered as the tensor product of the bar complex of π and the de Rham complex of $\mathrm{Hom}(\pi, G)$. Composing E^* with Φ, we get a map from the invariant polynomials on $\mathfrak{g}$ to the tensor product of the cocycles in $C^*(\pi)$ and the closed forms on $\mathrm{Hom}(\pi, G)$. Applying this map to the quadratic polynomial on $\mathfrak{g}$ given by an invariant symmetric bilinear form, and pairing the result with the "fundamental cycle" for π, when π is the fundamental group of a Riemann surface, we obtain a closed 2-form $\tilde{\omega}$ on $\mathrm{Hom}(\pi, G)$.

It turns out that $\tilde{\omega}$ is just the pullback of the symplectic structure on moduli space. It is clear from our discussion so far that $\tilde{\omega}$ is invariant under the adjoint action of G, but it is not so clear that $\tilde{\omega}$ is horizontal. Horizontality is, however, analogous to closedness where the operator d is replaced by the operators of interior product with the infinitesimal generators of the adjoint action. The sum of these two operators is essentially the equivariant differential [2] [3], so it will be natural to extend all of our bicomplexes and complexes to their equivariant versions.

3 Maurer-Cartan calculus

Given a Lie algebra $\mathfrak{g}$ and a manifold M, the $\mathfrak{g}$ valued differential forms on M form a super-Lie algebra; i.e. the algebraic identities defining a Lie algebra are satisfied by $\mathfrak{g}$ valued forms with the appropriate inclusion of signs [11]. In particular, $[\eta, \eta]$ is not necessarily zero when η is a $\mathfrak{g}$-valued 1-form, but the super Jacobi identity implies that $[\eta, [\eta, \eta]] = 0$. For further properties of vector valued differential forms, we refer to [4].

On any Lie group G, we denote by ω the $\mathfrak{g}$-valued left invariant 1-form which maps each tangent vector to the left invariant vector field having that value. The corresponding right invariant form will be denoted by $\overline{\omega}$. They are related by the equation

$$\overline{\omega} = \phi\omega,$$

where ϕ is the adjoint representation, considered as a 0-form on G with values in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$.

The forms ω and $\overline{\omega}$ satisfy the structure equations

$$d\omega = -\frac{1}{2}[\omega, \omega]$$

and

$$d\overline{\omega} = \frac{1}{2}[\overline{\omega}, \overline{\omega}].$$

Suppose now that $\mathfrak{g}$ has an Ad-invariant inner product (not necessarily positive definite), which we denote simply by $\cdot$. Ad-invariance means that $[x, y] \cdot z = -y \cdot [x, z]$ for any three elements of $\mathfrak{g}$, so that the triple product

$$[x, y] \cdot z \tag{1}$$

is completely antisymmetric. For $\mathfrak{g}$-valued 1-forms, it is completely symmetric.

We consider the 3-form

$$\lambda = \frac{1}{6}\omega \cdot [\omega, \omega].$$

Lemma 3.1 $d\lambda = 0$.

Proof. $[\omega, \omega]$ is closed by the structure equation, so

$$d\lambda = -\frac{1}{12}[\omega, \omega] \cdot [\omega, \omega].$$

By Ad-invariance, the last expression equals $-\frac{1}{12}\omega \cdot [\omega, [\omega, \omega]]$, which vanishes by the super Jacobi identity.

□

We will use the following notation for forms and vector fields on $G \times G$. If α is any differential form on G, we denote by α_i the pullback of α to $G \times G$ by the projection p_i onto the i'th component. If X is a vector field on G, we denote by X_i the vector field on $G \times G$ which projects to X under p_i and to zero under the other projection. $m : G \times G \to G$ is group multiplication.

In terms of this notation, we have the following lemma, whose proof is left to the reader:

Lemma 3.2 $m^*\omega = \phi_2^{-1}\omega_1 + \omega_2$.

The object which will eventually give us the symplectic structure on moduli space is the 2-form

$$\Omega = \omega_1 \cdot \overline{\omega}_2$$

on $G \times G$. Using the structure equations and the relation between ω and $\overline{\omega}$, we compute the exterior derivative

$$d\Omega = -\frac{1}{2}([\omega_1, \omega_1] \cdot \phi_2\omega_2 + \omega_1 \cdot [\phi_2\omega_2, \phi_2\omega_2]).$$

The form $d\Omega$ is evidently nonzero (unless G is abelian). However, we have:

Lemma 3.3

$$d\Omega = \delta\lambda.$$

Proof.

$$-\delta\lambda = m^*\lambda - \lambda_1 - \lambda_2 =$$

$$\frac{1}{6}(\phi_2^{-1}\omega_1 + \omega_2)\cdot[\phi_2^{-1}\omega_1 + \omega_2, \phi_2^{-1}\omega_1 + \omega_2] - \frac{1}{6}\omega_1\cdot[\omega_1,\omega_1] - \frac{1}{6}\omega_2\cdot[\omega_2,\omega_2].$$

Using the fact that the values of ϕ are automorphisms of both the bracket and inner product structures on $\mathfrak{g}$, we may simplify the expression above to:

$$\frac{1}{2}([\omega_1,\omega_1]\cdot\phi_2\omega_2 + \omega_1\cdot[\phi_2\omega_2,\phi_2\omega_2]).$$

$\square$

Another simple computation, which we leave to the reader, shows:

Lemma 3.4 $\delta\Omega = 0$.

Combining Lemmas 3.1, 3.3, and 3.4, we conclude:

Corollary 3.5 *$\Omega - \lambda$ is a cocycle of degree 4 in the total complex of the double complex $C^{**}(G)$.*

Remark The forms Ω and λ were originally found by analysis of objects in [10], but in fact $\Omega - \lambda$ is precisely the result of applying the Bott-Shulman map Φ to the invariant polynomial $X\cdot X$ on $\mathfrak{g}$.

4 The equivariant theory

The equivariant cohomology of a G-manifold M can be computed using the complex of *equivariant differential forms* on M [2] [3]. By definition, these are polynomial maps from $\mathfrak{g}$ to $\Omega^*(M)$ which are equivariant with respect to the adjoint representation. The equivariant differential d_G is defined by $(d_G\alpha)(X) = (d - i_{X_M})(\alpha(X))$. Here, X_M denotes the vector field given by the Lie algebra element X acting on M.

In the setting of the previous section, if a group H acts on G by automorphisms, it is natural to extend the double complex $C^{**}(G)$ to its equivariant version $C_H^{**}(G)$, where $C_H^{p,q}(G)$ consists now of the H-equivariant q-forms on G^p, and the second differential is replaced by d_H. [3] In what follows, we will restrict our attention to the special case where $H = G$, acting by inner automorphisms.

The form $\Omega - \lambda$, which is closed for the operator $\delta + (-1)^p d$, is *not* equivariantly closed. The extra part of the equivariant differential is calculated in the next two lemmas, in which the X on the right hand side is to be interpreted as a constant $\mathfrak{g}$-valued 0-form on G.

Lemma 4.1 $i_{X_G}\lambda = -d(X \cdot (\omega + \overline{\omega}))$.

Proof. We begin by noting that X_G for the adjoint representation is just $X - \overline{X}$, where $\overline{X}$ is the right-invariant vector field which agrees at the identity with the left invariant vector field X. Then we have

$$6 i_{X_G}\lambda = i_{X-\overline{X}}(\omega \cdot [\omega, \omega]).$$

Since λ is also equal to $\overline{\omega} \cdot [\overline{\omega}, \overline{\omega}]$, the last expression can be rewritten as

$$i_X(\omega \cdot [\omega, \omega]) - i_{\overline{X}}(\overline{\omega} \cdot [\overline{\omega}, \overline{\omega}]).$$

Using the equations $i_X\omega = X$ and $i_{\overline{X}}\overline{\omega} = X$ and the symmetry properties of the triple product (1), the last expression simplifies to

$$3X \cdot ([\omega, \omega] - [\overline{\omega}, \overline{\omega}]),$$

which, by the structure equations, is just

$$-d(6X \cdot (\omega + \overline{\omega})).$$

□

Lemma 4.2 $i_{X_{G\times G}}\Omega = X \cdot (\omega_1 - \phi_2^{-1}\omega_1 + \overline{\omega}_2 - \phi_1\overline{\omega}_2)$.

Proof. Since $X_{G\times G} = X_1 - \overline{X}_1 + X_2 - \overline{X}_2$, we have

$$i_{X_{G\times G}}\Omega = i_{X_1 - \overline{X}_1 + X_2 - \overline{X}_2}(\omega_1 \cdot \overline{\omega}_2)$$

Using the fact that $\overline{X} = \phi^{-1}X$, one can easily transform the last expression into the right hand side of the statement of the lemma.

[3] We expect that the cohomology of the corresponding total complex should be isomorphic to the H-equivariant cohomology of BG, which is in turn isomorphic to the cohomology of BK, where K is the semidirect product of H with G.

□

Lemmas 4.1 and 4.2 suggest that a piece of the puzzle is missing, namely the equivariant form

$$\theta(X) = X \cdot (\omega + \overline{\omega}).$$

The calculations above show that

$$(\delta + (-1)^p d_G)(\Omega - \lambda + \theta)(X) = -i_{X_{G\times G}}\Omega + \delta\theta(X) + i_{X_G}\theta(X).$$

The next two lemmas now complete the picture.

Lemma 4.3 $i_{X_G}\theta(X) = 0.$

Proof.

$$i_{X_G}\theta(X) = i_{X-\overline{X}}(X \cdot (\omega + \overline{\omega})) = X \cdot (i_X\omega - i_{\overline{X}}\overline{\omega}) = 0.$$

□

Lemma 4.4 $\delta\theta(X) = i_{X_{G\times G}}\Omega$

Proof.

$$\begin{aligned} \delta\theta(X) &= \theta(X)_1 + \theta(X)_2 - m^*\theta(X) \\ &= X \cdot (\omega_1 + \overline{\omega}_1 + \omega_2 + \overline{\omega}_2 - (\phi_2^{-1}\omega_1 + \omega_2 + \overline{\omega}_1 + \phi_1\overline{\omega}_2)) \\ &= X \cdot (\omega_1 + \overline{\omega}_2 - (\phi_2^{-1}\omega_1 + \phi_1\overline{\omega}_2)). \end{aligned}$$

Now compare with Lemma 4.2.

□

As a consequence of the preceding calculations, we have

$$(\delta + (-1)^p d_G)(\Omega - \lambda + \theta) = 0.$$

We state our main results below.

Theorem 4.5 *$-\lambda + \theta$ is an equivariantly closed form of degree 3 on G, and $Q_4 = \Omega - \lambda + \theta$ is an equivariantly closed element of degree 4 in the total complex of the bicomplex $C_G^{**}(G)$.*

5 Forms on representation spaces

To complete our work, we have little to do but to pull back the 4-form Q_4 of Theorem 4.5 by the evaluation maps $E : \pi^p \times \mathrm{Hom}(\pi, G) \to G^p$. Since each E is obviously equivariant, E^*Q_4 is a cocycle in the total complex of the bicomplex $C^*(\pi) \otimes C^*_G(\mathrm{Hom}(\pi, G))$. By pairing with cycles in $C_*(\pi)$, we obtain equivariant forms on $\mathrm{Hom}(\pi, G)$.

If $c \in C_2(\pi)$ is a 2-cycle, then

$$d_G\langle E^*\Omega, c\rangle = d_G\langle E^*Q_4, c\rangle = \langle E^* d_G Q_4, c\rangle$$

$$= E^*\langle -\delta Q_4, c\rangle = -E^*\langle Q_4, \partial c\rangle = 0,$$

so $\langle E^*\Omega, c\rangle$ is an equivariantly closed 2-form on $\mathrm{Hom}(\pi, G)$. A similar computation using $\Omega - \lambda$ and the ordinary differential shows that $\langle E^*\Omega, c\rangle$ is closed. It follows that $\langle E^*\Omega, c\rangle$ is a G-invariant closed form on $\mathrm{Hom}(\pi, G)$ which is annihilated by $i_{X_{\mathrm{Hom}(\pi,\mathrm{G})}}$ for every $X \in \mathfrak{g}$, so that it is the pullback of a closed 2-form $\omega_{\mathcal{M}}$ on (at least the smooth part of) the moduli space $\mathcal{M}$.

If we pair with a 2-boundary ∂b, then we have

$$\langle E^*\Omega, \partial b\rangle = \langle \delta E^*\Omega, b\rangle = \langle E^*\delta\Omega, b\rangle,$$

which is zero by Lemma 3.4. Thus we have proven:

Theorem 5.1 *Pairing $E^*\Omega$ with 2-cycles defines a natural homomorphism from $H_2(\pi)$ to the space of closed 2-forms on $\mathcal{M}$.*

Following [10], one may show easily that if π is the fundamental group of a closed Riemann surface and c is the fundamental cycle of that surface, then $\omega_{\mathcal{M}}$ is the usual symplectic structure on moduli space.

It is also possible to pair $E^*(\lambda)$ with a class of degree 1 in $H_*(\pi)$ (i.e. a conjugacy class in π). The resulting object is a closed, invariant, 3-form on $\mathrm{Hom}(\pi, G)$. This form does not, however, push down to $\mathcal{M}$. In fact, if we subtract $E^*\theta$ from $E^*\lambda$, the result is an equivariantly closed form on $\mathrm{Hom}(\pi, G)$, but it can be pushed down as a class to $\mathcal{M}$ only where the G action is free, and as a form only with the aid of a connection for that action.

A reader familiar with other work on the cohomology of moduli spaces may recognize that we have found, in a new guise, the generating classes of Newstead [12]. (Actually, these classes only apply directly to the "twisted" moduli spaces; see [1].) One of Newstead's generators is, however, missing, namely the class of degree 4 obtained by pairing with the fundamental 0-cycle of π. In fact, we can obtain this class as well by considering the element of $C_G^{0,4}(G)$ given simply by the invariant quadratic polynomial $\Phi(X) = X \cdot X$. This pulls back to $\Omega_G^*(\mathrm{Hom}(\pi, G))$ as the same polynomial, which is an equivariant form of degree 4. The limitations on pushing down to $\mathcal{M}$ are the same as those just discussed for the classes of degree 3.

It is perhaps useful to describe the symplectic form on $\mathcal{M}$, as we have constructed it, in more concrete terms. A 2-cycle for π is just a (finite) formal linear combination of elements of $\pi \times \pi$. Pairing $E^*(\Omega)$ with such a cycle gives a linear combination of pullbacks of Ω from G^2 to $\mathrm{Hom}(\pi, G)$ by evaluation on pairs of elements of π. Now choose a finite subset γ of π which contains a generating set and is large enough so that our cycle is supported in $\gamma \times \gamma$. We consider the singular space $\mathrm{Hom}(\pi, G)$ as embedded into the smooth manifold G^γ of maps from γ to G. The evaluation map E extends in an obvious way to a map from $\pi^2 \times G^\gamma$ to G^2, so when we pair with the cycle we get a smooth 2-form on G^γ which extends our form on $\mathrm{Hom}(\pi, G)$. It is important to note, though, that the extended evaluation map is no longer compatible with the two bicomplex structures, so *the extended 2-form is not necessarily closed*. We do not know whether a smooth closed extension can be found; nevertheless, it is reassuring to see the form on the singular space $\mathrm{Hom}(\pi, G)$ extended to a smooth form on a smooth manifold.

6 Discussion

Our construction of the symplectic structures on moduli spaces raises several questions.

First of all, it would be nice to prove with our formalism that the closed 2-form on moduli space is nondegenerate. One way to do so would be to construct the corresponding Poisson structure by an analogous procedure. This approach leads to the idea of applying our method to the moduli spaces of flat connections on manifolds with boundary, which are Poisson manifolds in which the symplectic leaves are given by specifying the conjugacy class of the holonomy on each boundary circle.

Secondly, we have dealt with only the characteristic class of degree 4 for the group G. It turns out that this leads to several equivariant cohomology classes on $\mathrm{Hom}(\pi, G)$, only one of which passes to a canonical form on $\mathcal{M}$. It would be interesting to carry out a similar analysis for the entire characteristic ring of G.

Although the forms Ω and λ were seen to arise directly by the Bott-Shulman construction, the equivariant form $\theta(X)$ was put in "by hand". It should be possible to find a "grand unified theory" encompassing all three of these forms, as well as the degree 4 class mentioned near the end of Section 5, by using a suitable de Rham model for $H^*_G(BG)$.

The geometric quantization of $\mathcal{M}$, which is so important for topological quantum field theory, begins with the construction of a line bundle whose curvature is $\omega_{\mathcal{M}}$. The construction of this line bundle in [13] relies on the Chern-Simons functional. Is it possible to construct the line bundle from an object analogous to the de Rham–bar complex?

We hope to return to these points in a sequel to this paper.

References

[1] Atiyah, M.F. and Bott, R., The Yang-Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond.* **A 308** (1982), 523-615.

[2] Atiyah, M.F. and Bott, R., The moment map and equivariant cohomology, *Topology* **23** (1984), 1-28.

[3] Berline, N., Getzler, E., and Vergne, M., *Heat Kernels and Dirac Operators*, Berlin/New York, Springer-Verlag, 1992.

[4] Bishop, R.L., and Crittenden, R.J., *Geometry of Manifolds*, Academic Press, New York, 1964.

[5] Bott, R., On the Chern-Weil homomorphism and the continuous cohomology of Lie groups, *Advances in Math.* **11** (1973), 289-303.

[6] Brown, K.S., *Cohomology of groups*, Springer-Verlag, New York, 1982.

[7] Cappell, S.E., Lee, R., and Miller, E.Y., A symplectic geometry approach to generalized Casson invariants of 3-manifolds, *Bull. Amer. Math. Soc.* **22** (1990), 269-275.

[8] Floer, A., An instanton invariant for three manifolds, *Comm. Math. Phys.* **118** (1988), 215-240.

[9] Goldman, W., The symplectic nature of fundamental groups of surfaces, *Adv. in Math.* **54** (1984), 200-225.

[10] Karshon, Y., An algebraic proof for the symplectic structure of moduli space, *Proc. Amer. Math. Soc.* (to appear).

[11] Manin, Y.I., Gauge field theory and complex geometry, Berlin/New York, Springer-Verlag, 1988.

[12] Newstead, P.E., Characteristic classes of stable bundles over an algebraic curve, *Trans. Amer. Math. Soc.* **169** (1972), 337-345.

[13] Ramadas, T.R., Singer, I.M., and Weitsman, J., Some coments on Chern-Simons gauge theory, *Commun. Math. Phys.* **126** (1989), 409-420.

[14] Shulman, H.B., Characteristic classes and foliations, Ph.D. Thesis, University of California, Berkeley (1972).