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DEFORMATION QUANTIZATION

by **Alan WEINSTEIN**

1. INTRODUCTION

Quantum mechanics is often distinguished from classical mechanics by a statement to the effect that the observables in quantum mechanics, unlike those in classical mechanics, do not commute with one another. Yet classical mechanics is meant to give a description (with less precision) of the same physical world as is described by quantum mechanics. One mathematical transcription of this *correspondence principle* is the that there should be a family of (associative) algebras $\mathcal{A}_\hbar$ depending nicely in some sense upon a real parameter $\hbar$ such that $\mathcal{A}_0$ is the algebra of observables for classical mechanics, while $\mathcal{A}_\hbar$ is the algebra of observables for quantum mechanics. Here, $\hbar$ is the numerical value of Planck's constant when it is expressed in a unit of action characteristic of a class of systems under consideration. (This formulation avoids the paradox that we consider the limit $\hbar \to 0$ even though Planck's constant is a *fixed* physical magnitude.)

Although the terminology and much of the inspiration comes from physics, noncommutative deformations of commutative algebras have also played a role of increasing importance in mathematics itself, especially since the advent of quantum groups about 15 years ago.

In the theory of *formal deformation quantization*, the "family of algebras $\mathcal{A}_\hbar$" is in fact a family $\star_\hbar$ of associative multiplications on a fixed complex vector space $\mathcal{A}$. More precisely, this family is given by a sequence of bilinear mappings $B_j : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ for $j = 0, 1, \ldots$ so that

$$a \star_\hbar b = \sum B_j(a,b)\hbar^j.$$

The condition for associativity of the product is that

$$\sum_{j+k=n} B_j(a, B_k(b,c)) = \sum_{j+k=n} B_j(B_k(a,b), c)$$

for $n = 0, 1, 2, \cdots$.

The problem of formal deformation quantization is to classify such families up to equivalence, where an equivalence between formal deformations $\mathbf{B} = B_0, B_1, \ldots$ and $\mathbf{B}' = B'_0, B'_1, \ldots$ is, intuitively speaking, a formal family $G_{(\hbar)} : \mathcal{A} \to \mathcal{A}$ of maps such that

$G_{(\hbar)}(a \star_\hbar b) = G_{(\hbar)}(a) \star'_\hbar G_{(\hbar)}(b)$. More precisely, such a family is given by a sequence $\mathbf{G} = G_0, G_1, \ldots$ of linear maps from $\mathcal{A}$ to $\mathcal{A}$ which satisfy the conditions

$$\sum_{j+k+r=n} B_r(G_j(a), G_k(b)) = \sum_{r+s=n} G_s(B'_r(a,b))$$

for $n = 0, 1, 2, \ldots$.

It is often useful to think of the deformation quantization as giving an associative algebra structure on the space $\mathcal{A}[[\hbar]]$ of formal power series with coefficients in $\mathcal{A}$ and an equivalence as giving an isomorphism between such algebras.

In attempting to solve the existence problem recursively for the B_j's, one finds at each stage an equation of the form $\delta B_j = F_j$, where F is a quadratic expression in the terms determined previously; a similar equation arises for each G_j in the equivalence problem. The operator δ goes from bilinear to trilinear (or linear to bilinear) $\mathcal{A}$-valued functionals on $\mathcal{A}$ and is precisely the coboundary operator for Hochschild cohomology with values in $\mathcal{A}$ of the algebra $\mathcal{A}$ with multiplication given by B_0. (In the equivalence problem, one normally assumes the product $\star_0$ as given, so that $B_0 = B'_0$, and G_0 is assumed to be the identity.) This cohomological approach to the deformation of algebras was established in the 1960's by Gerstenhaber [Ge].

A program to apply the methods of Gerstenhaber to algebras of interest in classical and quantum mechanics was laid out in 1975 by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [BFFLS]. (Another survey of the current state of the art may be found in [FS].) Following general principles which are often attributed to Dirac [Di], the aim of this program has been to develop as much as possible of quantum mechanics in terms of the deformed algebra structures, without using the customary representations in Hilbert spaces. Here, $\mathcal{A}$ is taken to be the space $C^\infty(M)$ of smooth complex-valued functions on a manifold M which represents the classical phase space. The undeformed product $\star_0$ (i.e. B_0) is taken to be the usual pointwise multiplication, so that $(\mathcal{A}, \star_0)$ is the algebra of classical observables. Next, following Dirac, it is assumed that the "limit" $\lim_{\hbar\to 0}[(a \star_\hbar b - b \star_\hbar a)/i\hbar]$ (i.e. $B_1(a,b) - B_1(b,a)/i\hbar$ is equal to a given classical Poisson bracket $\{a,b\}$ on $\mathcal{A}$. This bracket should be a *Poisson structure* in the sense that it satisfies the axioms of a Lie algebra together with the Leibniz identity $\{ab,c\} = \{a,c\}b + a\{b,c\}$. In this context, a formal deformation $\mathbf{B} = B_0, B_1, \ldots$ is called a $\star$-product (or star-product) if each of the bilinear maps B_j is a differential operator in each of its arguments, annihilating the constant functions when $j \geq 1$. These conditions make the $\star$-product local and insure that the constant function 1 remains as the unit element. Occasionally, the parity condition $B_j(a,b) = (-1)^j \overline{B_j(b,a)}$ is also imposed.

From here on, we will use the terms "$\star$-product" and "(deformation) quantization" interchangeably.

Among the Poisson manifolds (manifolds equipped with Poisson structure), the symplectic manifolds are of particular interest. We recall that a symplectic manifold is a manifold M equipped with a closed non-degenerate 2-form. According to Darboux's Theorem, such a manifold is always locally isomorphic to $\mathbf{R}^{2n}$ equipped with the symplectic form expressed in coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ as $\sum_i dq_i \wedge dp_i$. The Poisson structure

$$\{a, b\} = \sum_j \left(\frac{\partial a}{\partial q_j} \frac{\partial b}{\partial p_j} - \frac{\partial a}{\partial p_j} \frac{\partial b}{\partial q_j} \right)$$

is invariant under all diffeomorphisms preserving the symplectic form, so there is a well-defined Poisson structure on any symplectic manifold. Non-symplectic manifolds arise for instance as quotients of symplectic manifolds by symmetry groups and as the classical limits of quantum groups.

The fundamental example of a $\star$-product is the *Moyal-Weyl product* on $\mathbf{R}^{2n}$ with the Poisson structure just described. It comes from the composition of operators on $C^\infty(\mathbf{R}^n)$ via Weyl's identification [Wy] of such operators with functions on $\mathbf{R}^{2n}$, and was used by Moyal [My] to study quantum statistical mechanics from the viewpoint of classical phase space. The term B_1 in the formal series for this product is just $i/2$ times the "Poisson operator" $(a, b) \mapsto \{a, b\}$, and the full series is essentially the exponential of B_1. We will define the "powers" of the Poisson operator which enter in this series in a slightly more general setting. Let V be a vector space, and let π be a skew-symmetric bilinear functional on V^*. The formula $\{a, b\} = \pi(da, db)$ defines a Poisson structure on V. Associated to the bilinear operator π is a unique differential operator $\Pi : C^\infty(V \times V) \to C^\infty(V \times V)$ with constant coefficients for which $\{a, b\} = \Delta^* \Pi(a \otimes b)$; here, $a \otimes b$ is the function $(y, z) \mapsto a(y)b(z)$, and $\Delta^* : C^\infty(V \times V) \to C^\infty(V)$ is restriction to the diagonal. Now we define the Moyal-Weyl product on V by

$$a \star_\hbar b = \Delta^* \exp(i\hbar\Pi/2)(a \otimes b).$$

The space $C^\infty(V)[[\hbar]]$ with this product will be called the *Weyl algebra* of V and denoted by $W(V)$.

If $(x_1, \ldots, x_m)$ are linear coordinates on V, then the Poisson brackets $\{x_r, x_s\}$ are constants π_{rs} (the components of π), and the operator B_j in the expansion of the Moyal-Weyl product is

$$(1) \qquad B_j(a, b)(x) = \frac{1}{j!} \left(\frac{i}{2} \sum_{r,s} \pi_{rs} \frac{\partial}{\partial y_r} \frac{\partial}{\partial z_s})^j (a(y)b(z)) \right)_{|y=z=x}.$$

On a general Poisson manifold, the Leibniz identity implies that the Poisson bracket is given by a skew-symmetric contravariant tensor (or "bivector") field π, called the Poisson tensor, via the formula $\{a,b\} = \pi(da, db)$. If the rank of the tensor π (i.e. the rank of the matrix function $\pi_{rs}(x) = \{x_r, x_s\}$ which represents it in local coordinates, or the rank of the corresponding mapping from 1-forms to vectors) is constant, then by a theorem of Lie [L] the Poisson manifold is locally isomorphic to a vector space with constant Poisson structure. Hence such Poisson manifolds, which are called *regular*, are always *locally* deformation quantizable; the problem is to patch together the local deformations to produce a global $\star$-product.

There is one case in which the patching together of local quantizations is easy. The Moyal-Weyl product on a vector space V with constant Poisson structure is invariant under all the affine automorphisms of V, since the notion of "operator with constant coefficients" used in defining the powers of the Poisson operator is invariant under such transformations. As a consequence, we can construct a global quantization of any Poisson manifold M covered by local isomorphisms with V for which the transition maps are affine. Such a covering exists when M admits a flat torsionless linear connection for which the covariant derivative $\nabla\pi$ is zero.

Torsionless Poisson connections already play an important role in the treatment of deformation quantization in [BFFLS]. Just as the term B_1 in the deformation is determined by the Poisson structure (up to equivalence, and exactly, if the parity condition is satisfied), so the term B_2 is essentially determined by a Poisson connection, which exists (but is not unique) on any regular Poisson manifold. (Note that existence of a connection with $\nabla\pi = 0$ implies that the Poisson structure must be regular.) The existence of a deformation quantization in the presence of a flat torsionless Poisson connection was first established in [BFFLS] by replacing the partial derivatives in (1) by covariant derivatives with respect to parallel vector fields.

When our Poisson manifold does not admit a flat torsionless Poisson connection, the hard work begins. [BFFLS] and [Gu] began a careful analysis of the Hochschild cohomology space $H^3(\mathcal{A}, \mathcal{A})$ which is home to the obstructions to successive construction of $B_3, B_4, \ldots$. It was soon found that the obstructions could be chased into the de Rham cohomology space $H^3_{\mathrm{deRham}}(M)$, so there is no obstruction to constructing a deformation quantization when the 3rd Betti number of M is zero. This step involved in an crucial way the much smaller Chevalley cohomology space $H^3_{\mathrm{Chev}}(\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$ considered as a Lie algebra via the Poisson bracket. (Deformations of this Lie algebra were studied in [V], a paper which was important for all these developments.) After further results in special cases (e.g. cotangent bundles), it was proven by de Wilde and Lecomte in [DeL1] that a deformation quantization exists on any symplectic manifold, so that at least in this case the obstructions in $H^3_{\mathrm{deRham}}(M)$ were only illusory. Their proof (as well as the version in [DeL2]) involved rather complicated calculations which made the result look rather "technical".

Some later versions of the existence proof still relied on patching together local Weyl algebras with nonlinear coordinate changes. In [KM2], Karasev and Maslov give further details of a proof, whose first outline was sketched in [KM1], which reduces the patching to rather standard sheaf-theoretic ideas. Their main idea is to realize the deformed algebra as operators on a sheaf of "wave-packets" built by gluing together standard sheaves over $\mathbf{R}^{2n}$ with the aid of operators of Fourier integral type. In fact, this sheaf of wave packets can be constructed only when a certain quantization condition (elucidated in [DaP] and equivalent to a standard condition in the theory of geometric quantization [Cz]) is satisfied, but the corresponding sheaf of operators always exists—it is only the "representation" which is missing when the condition is not satisfied.

Another proof of the existence of deformation quantization which uses patching ideas was given by Omori, Maeda, and Yoshioka [OMY1]. Although their proof still involves substantial computations, it uses a fundamental idea which is also basic in the proof of Fedosov (who discovered it independently). Each tangent space of a Poisson manifold M can be considered as an affine space with a constant Poisson structure, so it carries a natural Moyal-Weyl quantization. In this way, the tangent bundle TM becomes a Poisson manifold with the fibrewise Poisson bracket, and with a fibrewise quantization. To quantize M itself, we may try to identify a subalgebra of the quantized algebra $C^\infty(TM)[[\hbar]]$ with the vector space $C^\infty(M)[[\hbar]]$ in such a way that the induced multiplication on $C^\infty(M)[[\hbar]]$ gives a deformation quantization of M. Such an identification is called a *Weyl structure* in [OMY1]. Weyl structures are investigated from a classical viewpoint in [EW], where they are seen to be closely related to exponential mappings.

An affine space V with constant Poisson structure carries a Weyl structure defined as follows. Let $(x_1, \ldots, x_m)$ be affine coordinates, and let $(y_1, \ldots, y_m)$ be the corresponding linear coordinates on a typical tangent space, so that $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ are coordinates on TV, with Poisson structure defined by $\{y_i, y_j\} = \pi_{ij}$ and all other brackets between coordinate functions zero. The Weyl structure then consists of those "functions" $u(x, y, \hbar)$ which are invariant under the translations $(x, y, \hbar) \mapsto (x + c, y - c, \hbar)$. The restriction map $u(x, y, \hbar) \mapsto u(x, 0, \hbar)$ is then an isomorphism, with inverse $v(x, \hbar) \mapsto u(x, y, \hbar) = v(x+y, \hbar)$. Now the fibrewise Moyal-Weyl product (i.e. with the y_j as quantized variables, and x_j as parameters) goes over under this isomorphism to the usual Weyl-Moyal product on V (i.e. with the x_j as quantized variables). The existence proof in [OMY1] involves patching together the local Weyl structures arising from a covering of a symplectic manifold M by coordinate charts. The patching is rather complicated, and it uses at one point the gluing operators of [KM].

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2. FEDOSOV'S CONSTRUCTION

Fedosov overcomes the difficulty of patching together local Weyl structures by, in effect, making the canonical coordinate neighborhoods "infinitely small". To understand his idea, we should first think of elements of the deformed algebra $C^\infty(TM)[[\hbar]]$ as sections of the bundle $W(TM)$ over M whose fibre at $x \in M$ is $W(T_xM)$. When M is an affine space, the global Weyl structure described above can then be identified with a space of parallel sections of this bundle with respect to a certain flat connection on $W(TM)$. In a similar way we get a Weyl structure, and hence another construction of the deformation quantization, on any manifold with a flat torsionless Poisson connection.

Of course we are most interested in dealing with the case where M does not admit a flat Poisson connection, and this is where the most interesting part of Fedosov's proof comes in. In effect, he says that the tangent bundle of every symplectic (or regular Poisson) manifold *does* admit a flat Poisson connection, if one gives the appropriate extended meaning to that concept. Namely:

- The connection is constructed, not on the tangent bundle, but on the bundle $W(TM)$ of Weyl algebras. The "structure Lie algebra" of this connection, in which the connection forms take values, is $W(\mathbf{R}^{2n})$ acting on itself by the adjoint representation of its Lie algebra structure. Since the full Weyl algebra is used, and not just the quadratic functions which generate linear symplectic transformations, the structure group effectively allows nonlinear transformations of the (quantized) tangent spaces. Since even linear generating functions are included, the structure group even allows translations.
- In fact (this idea was also used in [OMY1]), it is not the full Weyl algebra of $\mathbf{R}^{2n}$ which serves as the typical fibre, but only a certain quotient: the *formal* Weyl algebra $FW(\mathbf{R}^{2n})$, consisting of formal Taylor expansions at the origin. Geometrically, one can think of this step as the replacement of the (quantized) tangent bundle by a formal neighborhood of the zero section. This step may appear to be inconsistent with the inclusion of translations in the structure group, since these do not leave the origin fixed. In fact, the effect is to force us to forget the group and to work only with the structure Lie algebra. A beneficial, and somewhat surprising, result of this effect is that a parallel section with respect to a flat connection is not determined by its value at a single point. This situation is very close to that in formal differential geometry, where the bundle of infinite jets of functions on a manifold M has a flat connection whose sections are the lifts of functions on M. (See Section 1 of [T] for a nice exposition, with references.)

Fedosov uses an iterative method for "flattening" a connection which is similar to that used in many differential geometric problems. (See [Mi-Ru] for an example, and [Ru] for a recent survey.) Given a local trivialization of a bundle over a manifold, a connection is given by a 1 form ϕ with values in the Lie algebra $\mathbf{g}$; the curvature of the connection is the Lie algebra valued 2 form $\Omega_\phi = d\phi + \frac{1}{2}[\phi, \phi]$. If the curvature is not zero, we may try to "improve" the connection by adding another Lie algebra valued 1 form ϵ. The curvature zero condition for $\phi + \epsilon$ is the quadratic equation $d\epsilon + [\phi, \epsilon] = -\Omega_\phi - \frac{1}{2}[\epsilon, \epsilon]$. Rather than trying to solve this equation exactly, we linearize it by dropping the term $-\frac{1}{2}[\epsilon, \epsilon]$. The operator $d + [\phi, \ \ ]$ is the covariant exterior derivative D_ϕ, so our linearized equation has the form

$$D_\phi \epsilon = -\Omega_\phi. \tag{2}$$

From the Bianchi identity, $D_\phi \Omega = 0$ it appears that the obstruction to solving (2) for ϵ lies in a cohomology space. This is not quite correct, since $D_\phi^2 = [\Omega_\phi, \ \ ]$, which is not zero because the connection ϕ is not yet flat.

Up to now, we have essentially been following Newton's method for solving nonlinear equations. At this point, we add an idea similar to one often attributed to Nash and Moser. (See Section III.6 of [S] for an exposition of this method with original references.) Since the linear differential equation (2) is only an approximation to the nonlinear one which we really want to solve, it is unnecessary to solve it precisely. Rather, it suffices to solve it approximately and to compensate for the error in the later iterations which will in any case be necessary to take care of the neglected quadratic term $-\frac{1}{2}[\epsilon, \epsilon]$. Such approximate solutions are constructed by some version of the Hodge decomposition. In the differential geometric applications mentioned above, the full story involves elliptic differential operators, Sobolev spaces, and so on, but in the case at hand, it turns out that the "Hodge theory" is purely algebraic and quite trivial.

2.1 On the formal Weyl algebra

The coefficients of the connection forms which we will use are sections of the bundle $FW(TM)$. Rather than measuring the size of these forms by the usual Sobolev norms involving derivatives, we shall use a pointwise algebraic measurement.

In the formal Weyl algebra $FW(V)$ of a Poisson vector space V, we assign the weight 2 to the variable $\hbar$ and the weight 1 to each linear function on V. We denote by $FW_r(V)$ the ideal generated by the monomials of weight r. Because the kth term in the expansion of the $\star$-product involves $2k$ derivatives and multiplication by $\hbar^k$, we obtain a filtration of the algebra $FW(V)$. We will also occasionally use the classical grading, compatible with the commutative multiplication but not with the $\star$-product, which assigns the weight 0 to $\hbar$ and 1 to each linear function on V.

The Lie algebra structure which we will use for the formal Weyl algebra is the quantum Poisson bracket [Di] defined by $[a, b] = (1/i\hbar)(a \star_\hbar b - b \star_\hbar a)$. The factor $(1/i\hbar)$ makes the quantum bracket reduce to the classical one (rather than to zero) when $h \to 0$. In addition, the quantum and classical brackets are equal when one of the entries contains only terms linear or quadratic in the variable on V, and they share the property $[FW_r(V), FW_s(V)] \subset FW_{r+s-2}(V)$, so that the adjoint action of any element of $FW_2(V)$ preserves the filtration.

Next we introduce the algebra $\mathcal{W}(V) = FW(V) \otimes \wedge^*(V)$, whose elements may be considered as differential forms on the "quantum space whose algebra of functions is $FW(V)$." $\mathcal{W}(V)$ inherits a filtration by subspaces $\mathcal{W}_r(V)$ from the formal Weyl algebra, and a grading from the exterior algebra. We can also consider $\mathcal{W}(V)$ as the algebra of infinite jets at the origin of differential forms on the classical space V, in which case we generally use the classical grading. In this way, $\mathcal{W}(V)$ inherits the exterior derivative operator, which we denote by δ. Remarkably, δ is also a derivation for the quantized algebra structure on $\mathcal{W}(V)$.

We may describe the operator δ in terms of linear coordinates $(x_1, \ldots, x_m)$ on V. With an eye toward the case where V is a tangent space, we denote the corresponding formal generators of $FW(V)$ by $(y_1, ..., y_m, \hbar)$ and the generators of $\wedge^*(V)$ by $(dx_1, \ldots, dx_m)$. Then $\mathcal{W}(V)$ is formally generated by the elements $y_i \otimes 1$, $\hbar \otimes 1$, and $1 \otimes dx_i$, and we have $\delta(y_i \otimes 1) = 1 \otimes dx_i$, $\delta(\hbar \otimes 1) = 0$, and $\delta(1 \otimes dx_i) = 0$. Notice that δ decreases the Weyl algebra filtration degree by 1 while it increases the exterior algebra grading by 1.

Since δ is essentially the de Rham operator on a contractible space, we expect the cohomology of the complex which it defines to be trivial. Fedosov makes this explicit by introducing the dual operator δ^* of contraction with the Euler vector field $\sum_i y_i \otimes \frac{\partial}{\partial x_i}$. More precisely, δ^* maps the monomial $y_{i_1} \cdots y_{i_p} \otimes dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ to

$$\sum_k (-1)^{k-1} y_{i_1} \cdots y_{i_p} y_{j_k} \otimes dx_{j_1} \wedge \cdots \wedge \widehat{dx_{j_k}} \wedge \cdots \wedge dx_{j_q}.$$

(This operator is *not* a derivation for the quantized algebra structure.) A simple computation (or the Cartan formula for the Lie derivative by the Euler vector field) shows that, on the monomial above, we have $\delta\delta^* + \delta^*\delta = (p+q)\mathrm{id}$, so that if we define the operator δ^{-1} to be $\frac{1}{p+q}\delta^*$ on the monomial above, and 0 on $1 \otimes 1$, we find that each element u of $\mathcal{W}(V)$ has the decomposition $u = \delta\delta^{-1}u + \delta^{-1}\delta u + \mathcal{H}u$, where the "harmonic" part $\mathcal{H}u$ of u is the part involving only powers of $\hbar$ and no y_i's or dx_i's, i.e. the pullback of u by the constant map from V to the origin. In other words, we have reproduced the usual proof of the Poincaré lemma via a homotopy operator from $\mathcal{H}$ to the identity.

When the Poisson vector space V is symplectic, the operator δ has another description. For any $a \in FW(V)$, $[y_i, a] = \{y_i, a\} = \sum_j \pi_{ij}(\partial a/\partial y_j)$. If (ω_{ij}) is the matrix of the symplectic structure, inverse to (π_{ij}), we get $\partial a/\partial y_i = [\sum_j \omega_{ij} y_j, a]$, and hence $\delta(a \otimes 1) = \sum_i (\partial a/\partial y_i) \otimes dx_i = [\sum_{ij} \omega_{ij} y_j \otimes dx_i, a \otimes 1]$. It follows from the derivation property that a similar equation holds for any element of $\mathcal{W}(V)$; i.e. the operator δ is equal to the adjoint action of the element $\sum_{ij} \omega_{ij} y_j \otimes dx_i$ (which is just the symplectic structure itself).

Of course, all the considerations above apply when V is replaced by a symplectic vector bundle E and $\mathcal{W}(V)$ by the space of sections of the associated bundle $\mathcal{W}(E) = FW(E) \otimes \wedge^*(E)$. In particular, when E is the tangent bundle of a symplectic manifold M, the operator δ and its relatives act on the algebra of differential forms on M with values in $FW(TM)$. Note that these operators are purely algebraic with respect to the variable in M, with δ being just the adjoint action of the symplectic structure considered as an $FW(TM)$-valued 1 form.

2.2 Flattening the connection

We are now ready for the iteration procedure to construct a flat connection on the bundle of Weyl algebras. For simplicity we describe the procedure in local canonical coordinates, but all the constructions are in fact intrinsic. We begin with an arbitrary (linear) Poisson connection on the tangent bundle of the symplectic manifold M. (In fact, one can even start with a connection with torsion–the torsion would be killed after the first iteration [Fe4].) This connection induces a covariant differentiation operator on the dual bundle, i.e. on the linear functions on fibres. In coordinates $(x_1, \ldots, x_m)$ on M and the corresponding basis $(y_1, \ldots, y_m)$ of linear functions, the connection form is a 1-form with values in the Lie algebra $\mathbf{sp}(m)$, whose elements may be identified with linear hamiltonian vector fields and hence with quadratic functions. Thus the connection form can be written as $\phi = \frac{1}{2}\sum \Gamma_{ijk} y_i y_j \otimes dx_k$. If we consider the same form (with the y_i's now interpreted as formal variables) as taking values in the bundle $FW(TM)$, it becomes the connection form for the associated connection on that bundle. Even if this connection were flat, it would not be the correct one to use for quantization, since its parallel sections would not be identifiable in any reasonable way with functions on M; instead we must use for our first approximation $\phi_0 = (\sum \omega_{kj} y_j + \frac{1}{2}\sum \Gamma_{ijk} y_i y_j) \otimes dx_k$.

To start the recursion, one calculates using the fact that the connection is symplectic and torsion free (see [Fe3]) that $\Omega_0 = -\frac{1}{2}\sum \omega_{ir} \otimes dx_i \wedge dx_r + \frac{1}{4}\sum R_{ijkl} y_i y_j \otimes dx_k \wedge dx_l = -1 \otimes \omega + R$, where ω is the symplectic form and R is the curvature of the original linear symplectic connection, considered as a 2 form with values in the Lie algebra of quadratic functions. The term $-1 \otimes \omega$ appears even when the linear connection is flat, but it causes no trouble because it is a central element of the Weyl Lie algebra and therefore acts trivially in the adjoint representation.

We will now try to construct a convergent (with respect to the filtration) sequence ϕ_n of connections whose curvatures Ω_n tend to the central element $-1 \otimes \omega$. Fedosov calls this central element the *Weyl curvature* of the limit connection; to simplify notation, we will write $\widehat{\Omega} = \Omega + 1 \otimes \omega$ for the form which should be zero, and we call this the *effective curvature* .

As suggested above, we let $\phi_{n+1} = \phi_n + \epsilon_{n+1}$, where ϵ_{n+1} is a section of $\mathcal{W}(TM)$

which is an approximate solution of the linearized equation for zero effective curvature $D_n\epsilon_{n+1} + \widehat{\Omega}_n = 0$. The operator $D_n = D_{\phi_n}$ will have the form $d + \delta + [c_n, \;\,]$, where c_n is an $FW(TM)$-valued 1 form. We will try to arrange for c_n to lie in $FW_2(TM)$ so that the operator $[c_n, \;\,]$, like d, is filtration preserving. Since δ *lowers* the filtration degree by 1, the principal part of the differential operator D_n will actually be the algebraic operator δ (and not d as it would be if we measured forms by the size of their derivatives.)

Instead of solving $D_n\epsilon_{n+1} + \widehat{\Omega}_n = 0$, then, we try to solve the simpler equation $\delta\epsilon_{n+1} + \widehat{\Omega}_n = 0$. In fact, we cannot solve even this equation exactly, because the Bianchi identity gives $D_n\widehat{\Omega}_n = 0$ instead of $\delta\widehat{\Omega}_n = 0$. (The term $1 \otimes \omega$ is killed by both operators.) Nevertheless, we do the best we can and let the errors take care of themselves later. Thus, we simply define $\epsilon_{n+1} = -\delta^{-1}(\widehat{\Omega}_n)$ and try to live with the consequences.

From the recursion relation $\Omega_{n+1} = \Omega_n + D_n\epsilon_{n+1} + \frac{1}{2}[\epsilon_{n+1}, \epsilon_{n+1}]$, we find after a straightforward calculation using the decompositions $D_n = d + \delta + [c_n, \;\,]$ and $u = \delta\delta^{-1}u + \delta^{-1}\delta u + \mathcal{H}u$ that

$$\widehat{\Omega}_{n+1} = \delta^{-1}\delta\widehat{\Omega}_n + \mathcal{H}\widehat{\Omega}_n + d\epsilon_{n+1} + [c_n, \epsilon_{n+1}] + \frac{1}{2}[\epsilon_{n+1}, \epsilon_{n+1}].$$

Using $D_n = d + \delta + [c_n, \;\,]$ again, we can rewrite this as

$$\widehat{\Omega}_{n+1} = \delta^{-1}D_n\widehat{\Omega}_n - \delta^{-1}d\widehat{\Omega}_n - \delta^{-1}[c_n, \widehat{\Omega}_n] + \mathcal{H}\widehat{\Omega}_n + d\epsilon_{n+1} + [c_n, \epsilon_{n+1}] + \frac{1}{2}[\epsilon_{n+1}, \epsilon_{n+1}].$$

By the Bianchi identity $D_n\Omega_n = 0$, we get

$$\widehat{\Omega}_{n+1} = \mathcal{H}\widehat{\Omega}_n - \delta^{-1}d\widehat{\Omega}_n - \delta^{-1}[c_n, \widehat{\Omega}_n] + d\epsilon_{n+1} + [c_n, \epsilon_{n+1}] + \frac{1}{2}[\epsilon_{n+1}, \epsilon_{n+1}]$$

Suppose now that $\widehat{\Omega}_n \in \mathcal{W}_r(TM)$ with $r \geq 1$. Then $\mathcal{H}\widehat{\Omega}_n = 0$ and $\epsilon_{n+1} \in \mathcal{W}_{r+1}(TM)$, so that $c_n \in \mathcal{W}_2(TM)$ and hence all the terms on the right hand side of the equation above belong to $\mathcal{W}_{r+1}(TM)$.

Since $\widehat{\Omega}_0 = R$ has filtration degree 2, we conclude that $\widehat{\Omega}_n$ has degree at least $n + 2$, and ϵ_{n+1} has degree at least $n + 3$, so the sequence ϕ_n converges to a connection form ϕ for which the curvature is $\Omega = -1 \otimes \omega$. This curvature is a central section, so the connection on $FW(TM)$ associated to ϕ by the adjoint representation $FW(TM)$ is flat. Since the adjoint action is by derivations of the multiplicative structure, the space of parallel sections is a subalgebra of the space of all sections.

The last step in Fedosov's construction is to show by a recursive construction similar to the one above that each element of $C^\infty(M)[[\hbar]]$ is the harmonic part of a unique parallel section of $FW(TM)$, so that $C^\infty(M)[[\hbar]]$ is identified with the space of parallel sections and thus inherits from it an algebra structure, which is easily shown to be a deformation quantization associated with the symplectic structure ω.

3. CLASSIFICATION OF $\star$-PRODUCTS

Using techniques similar to those in [Gu] and building on earlier work of Flato, Lichnerowicz, Sternheimer, and Vey, S. Gutt showed around 1980 that the construction of star products on a symplectic manifold involves a choice within an affine space of dimension $b_2(M)$ at each power of $\hbar$. The proof involved an analysis of Hochschild and Chevalley cohomology with differentiable cochains null on constants. We refer to [FS] for a more detailed discussion of the state of the theory through the early 1980's.

Fedosov [Fe4] showed that his iterative construction of a connection on $FW(TM)$ can be modified so that the curvature becomes $\sum \hbar^j \otimes \omega_j$, for any sequence of closed 2 forms ω_j such that ω_0 is the original symplectic structure ω. He also showed that the isomorphism class of the resulting $\star$-product depends precisely on the sequence of de Rham cohomology classes $[\omega_j] \in H^2(M, \mathbf{R})$ and in particular is independent of the initial choice of connection.

This left open the question of whether every $\star$-product is isomorphic to one obtained by Fedosov's construction. A positive answer to this question has given by Nest and Tsygan. Using a noncommutative version of Gelfand-Fuks cohomology, they construct in [NT1] for each deformation quantization a characteristic class in $H^2(M, \mathbf{R})[[\hbar]]$ with constant term ω. In [NT2], they show that this class determines the $\star$-product up to isomorphism and that it agrees with Fedosov's curvature for the $\star$-products constructed by his method. By Moser's classification [Ms] of nearby symplectic structures by their cohomology classes, the isomorphism classes of $\star$-products on a symplectic manifold are thus in 1-1 correspondence with isomorphism classes of formal deformations of the symplectic structure.

One consequence of this classification is that there is (up to isomorphism) a unique deformation quantization whose characteristic class is independent of $\hbar$. Although one might think that this special quantization is somehow the natural one, there is considerable evidence that the others are important as well. For instance, Fedosov [Fe5] shows that one needs to introduce $\star$-products with nonconstant characteristic class to make deformation quantization compatible with symplectic reduction. In addition, work in progress by Emmrich and the author suggests that $\star$-products with nonconstant characteristic classes may be related to geometric phases and deformations of symplectic forms which arise in the analysis of coupled wave equations [LtF].

4. TRACE AND INDEX

4.1. Strongly closed $\star$-products

A *trace* on a deformed algebra $C^\infty(M)[[\hbar]]$ is by definition a linear functional on the compactly supported functions $\tau : C_c^\infty(M) \to \hbar^{-m/2}\mathbf{C}[[\hbar]]$ whose formal extension to $C_c^\infty(M)[[\hbar]]$ satisfies the usual condition $\tau(a \star_\hbar b) = \tau(b \star_\hbar a)$. The negative powers of $\hbar$ (m is the dimension of M) are admitted because when $M = \mathbf{R}^{2n}$ the "natural" trace coming via the Weyl correspondence from the trace of operators is

$$a \mapsto (2\pi\hbar)^{-m/2} \int_M a\omega^{m/2}/(m/2)! \tag{3}$$

In [CoFS], a $\star$-product is called *strongly closed* if the functional (3) still defines a trace. It is shown there that in the obstruction theory for the classification of strongly closed $\star$-products Hochschild cohomology should be replaced by cyclic cohomology [Co]. The existence of a strongly closed $\star$-product on an arbitrary symplectic manifold was shown in [OMY2]. Fedosov [Fe4] constructed a trace for each of his $\star$-products, in which the linear functional (3) is applied not to a but to a series $\sum G_j(a)\hbar^j$, where the G_j are differential operators with G_0 the identity. By the classification of [NT2], it follows that every $\star$-product on a symplectic manifold is equivalent to a strongly closed $\star$-product. It is further shown in [NT2] that the set of traces for a $\star$-product on a symplectic manifold forms a 1-dimensional module over $\mathbf{C}[[\hbar]]$, so the trace is essentially unique.

4.2. Index theorems

The index of an elliptic operator is defined as the difference between the dimension of its kernel and that of its cokernel, which can also be interpreted as the difference between the traces of the projections on these two spaces. These two projections can be replaced by other pairs of operators more amenable to analysis, so that index theory comes down to a theory of computations of traces, which can then be applied in purely algebraic settings [Co]. In the context of deformation quantization, a setting for such theorems was given in [CoFS], while a detailed proof of an index theorem announced in [Fe2] may be found given in [Fe4]; other versions are in [NT1], and [NT2].

A basic ingredient in the algebraic formulation of index theorems is the appropriate counterpart of an elliptic operator. This can be defined in several ways; we present here one found in [Fe4].

Let $\mathcal{M}_N(\mathcal{A})$ denote the algebra of $N \times N$ matrices with entries in a $\mathbf{C}$-algebra $\mathcal{A}$. Any $\star$-product $\star_\hbar$ on the Poisson manifold M induces a deformation (which we also denote by $\star_\hbar$) of the product in the algebra $C^\infty(M, \mathcal{M}_N(C))$ of matrix valued functions; simply identify $C^\infty(M, \mathcal{M}_N(C))[[\hbar]]$ with $\mathcal{M}_N(C^\infty(M)[[\hbar]])$ and use the $\star$-product on the matrix elements. For convenience, we will denote this (deformed) algebra by $\mathcal{M}_N(M)$. A trace for the $\star$-product on $C^\infty(M)[[\hbar]]$ induces in the usual way a trace on $\mathcal{M}_N(M)$—take the sum of the traces of the diagonal elements. (We note that Fedosov [Fe4] constructs a deformation, with trace, of the algebra of endomorphisms of any complex vector bundle E over a symplectic manifold M; the starting data in this case are a linear connection on E and the usual symplectic connection on M. The index theorem extends to this more general setting.)

The domain and range of an ordinary elliptic operator are spaces of sections of vector bundles. Since any bundle is a subbundle of a trivial bundle, in the algebraic setting the bundles may be replaced by projections, i.e. elements p_1 and p_2 in $\mathcal{M}_N(M)$ such that $p_j \star_\hbar p_j = p_j$. In the "usual case", M is a cotangent bundle T^*X, and the projections are the pullbacks of matrix valued functions on X, for which the most commonly used $\star$-products coincide with ordinary matrix multiplication.)

An operator between sections of vector bundles is replaced in the algebraic setting by an element a of $\mathcal{M}_N(M)$ such that $a \star_\hbar p_1 = p_2 \star_\hbar a = a$. Ellipticity in the analytic case follows from the existence of an inverse modulo compact operators; in our algebraic setting this becomes an element $r \in \mathcal{M}_N(M)$ such that $p_1 - r \star_\hbar a$ and $p_2 - a \star_\hbar r$ have compact support.

The 4-tuple $\mathcal{E} = (p_1, p_2, a, r)$ is called an *elliptic element*. Its index is defined as

$$\operatorname{ind} \mathcal{E} = \operatorname{Tr}(p_1 - r \star_\hbar a) - \operatorname{Tr}(p_2 - a \star_\hbar r).$$

The classical, or geometric, limit of an elliptic element is the 4-tuple of symbols $\sigma(\mathcal{E}) = (\sigma(p_1), \sigma(p_2), \sigma(a), \sigma(r))$ obtained by setting $\hbar = 0$ in all the objects in $\mathcal{E}$. Here, $\sigma(p_1)$ and $\sigma(p_2)$ are projections onto vector bundles over M, and $\sigma(a)$ and $\sigma(r)$ are maps between these bundles which are inverse to one another outside a compact subset of M. These data define an element of K-theory with compact supports over M, which then has a Chern character $\operatorname{ch}(\sigma(\mathcal{E}))$ in $H^*_c(M, \mathbf{C})$.

As a symplectic manifold, M carries a compatible almost complex structure unique up to homotopy and hence a well defined Atiyah-Hirzebruch class $\widehat{A}(M) \in H^*(M, \mathbf{C})$. Also determined by the symplectic structure is its de Rham cohomology class $[\omega]$. This completes the data needed for the statement of the index theorem, which is the following formula (in which evaluation on the fundamental homology class of M is written as integration):

$$\operatorname{ind} \mathcal{E} = \int_M \operatorname{ch}(\sigma(\mathcal{E})) e^{[\omega]/2\pi\hbar} \widehat{A}(M).$$

Remarks

The left hand side of the index formula is by definition an element of $\hbar^{-m/2}\mathbf{C}[[\hbar]]$, while the right hand side is a polynomial of degree at most $m/2$ in $\hbar^{-1}$.

When M is compact, the conditions to be satisfied by a and r are vacuous; we can even take a and r to be zero. In this case, the index just depends on the element of K-theory defined by the projections p_1 and p_2. If $N = 1$, $p_1 = 1$, and $p_2 = 0$, we get the formula:

$$\operatorname{Tr}(1) = \int_M e^{[\omega]/2\pi\hbar} \widehat{A}(M).$$

This is consistent with the idea (see for instance [BoG]) that functions on M may often be identified with operators on a space whose dimension is given, via the Riemann-Roch theorem, by the right hand side of the formula above.

5. SOME QUESTIONS

A fundamental question remains. *Is every Poisson manifold deformation quantizable?* This question may be broken into the following two parts. Is every Poisson manifold locally deformation quantizable? Is every locally deformation quantizable Poisson manifold globally deformation quantizable? (It is not hard to show that deformation quantizations of two open subsets which are equivalent on the intersection of the sets can be "patched" to produce a deformation quantization of the union.) Donin [Do] has given an algebraic reformulation of Fedosov's method which shows, in particular, that there is a deformation quantization of the field of *rational* functions on any Poisson algebraic variety.

The algebra $C^\infty(M)$ is the Lie algebra of (a 1-dimensional central extension of) the group of symplectic transformations of M. Is there a version of deformation quantization which applies to this group? This problem is related to, but not totally solved by, the $\star$-exponentials of [BFFLS]. What is the corresponding index theorem? For homogeneous symplectic transformations of cotangent bundles, this would be an index theorem for elliptic Fourier integral operators. (See [Wi1].) In the cotangent bundle case, it is hard to find examples of homogeneous symplectic transformations which are not isotopic to cotangent lifts of diffeomorphisms (for which the index problem reduces to the pseudodifferential case). For general symplectic manifolds, on the other hand, there are plenty of examples, so this index theorem would be of real interest.

We have completely ignored in this paper the problem of *strict deformation quantization*, where one seeks a deformation which is not merely a formal power series in $\hbar$ but actually exists as an algebra for sufficiently small $\hbar$. For example, the reader may consult [Ri] and [Wi2] for some of the analytical and geometric aspects of this problem and [CaGR] for the related problem of convergence of the series in $\hbar$ which define the $\star$-product.

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Alan WEINSTEIN

Centre Emile Borel
Institut Henri Poincaré
11 r. Pierre et Marie Curie
F-75231 PARIS CEDEX 05

and

Department of Mathematics
University of California
Berkeley, CA 94720, USA