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Beyond Poisson structures

The object of this paper is to present a unified approach to the geometry of hamiltonian vector fields and the underlying closed 2-forms or Poisson brackets. The approach is based on concepts introduced in [3] for symmetric bilinear forms, applied here to skew forms.

The idea of a Poisson bracket on a subalgebra of $C^\infty(X)$ (X a smooth manifold) goes back to Dirac [2]; see also Hermann [4] and Sniatycki [10]. Dirac showed how a bracket on functions on a manifold induces a bracket on a subalgebra of functions on any submanifold; this is the constrained or Dirac bracket.

Another approach to brackets on subalgebras of functions is to look at the algebra of hamiltonian vector fields for a degenerate closed 2-form; this has been done by Pnevmatikos [7,8,9], Lichnerowicz [5], and others. Functions constant on the characteristic foliation of the 2-form generate hamiltonian vector fields, and hence have Poisson brackets defined in the usual way. The class of such functions may be smaller than expected, since the characteristic distribution may increase dimension on a closed set of measure zero. Singularities of closed 2-forms have also been studied by Martinet [6].

In this paper we define tensorial objects which correspond to brackets on subalgebras of functions, generalizing the way in which bi-vector fields correspond to Poisson brackets defined on all functions. These objects are subbundles $L \subset TX \oplus T^*X$, and in the cases of Poisson structures and 2-forms are the graphs of the maps $B: T^*X \to TX$ and $\Omega: TX \to T^*X$ respectively. In each of these two cases, integrability is defined as the vanishing of a 3-tensor, namely $[B,B]$ (the Schouten bracket of B with itself) or $d\Omega$. In general, we get a bi-vector on the quotient $TX/L \cap TX$, which gives us a bracket on the algebra of functions "constant along $L \cap TX$." Our integrability condition is then the vanishing of a 3-tensor on L which implies that this is actually a Poisson bracket; the condition also implies the integrability of $L \cap TX$.

The flip side of this picture is provided by the distribution $\rho(L) \subset TX$, (ρ is the projection of $TX \oplus T^*X$ onto TX) on which we define a 2-form $\Omega_L: \rho(L) \to \rho(L)^*$; this form has characteristic subbundle $L \cap TX \subset \rho(L)$. The vanishing of the integrability 3-tensor on L implies that $\rho(L)$ is integrable and that Ω_L is a closed 2-form. Thus, we get an algebra of hamiltonian vector fields generated by functions constant on $L \cap TX$, which in turn gives a Poisson bracket on these functions; this is the same as the bracket discussed above.

The subsets $\rho(L)$ and $L \cap TX$ are not distributions in the usual sense, since their dimensions do not have to be everywhere constant; $\rho(L)$ is maximal on an open dense set, and $L \cap TX$ is minimal on an open dense set (not necessarily the same open set). At best they may be integrable in the sense of Sussman [1];[11]: there is a maximal integral manifold through every point (if the dimensions are constant, the distributions are called regular). However, as with closed 2-forms, this does not always have to be the case: $\ker\Omega$ may not satisfy this maximal integral manifold property.

It is interesting to note that regular foliations are special cases of integrable Dirac structures; one simply requires that Ω_L be everywhere zero. This forces the dimension of the distribution $\rho(L)$ to be constant.

Vector Spaces

Let V be a vector space and let $L \subset V \oplus V^*$ be a maximally isotropic subspace under the pairing $\langle v \oplus v^*, w \oplus w^* \rangle_+ = \langle v^*, w \rangle + \langle w^*, v \rangle$; then L has the dimension of V. We call such an L a Dirac structure on V. We will show that each Dirac structure corresponds to a subspace of V, with a skew-symmetric bilinear form on it (compare [3]).

Let ρ and ρ^* be the projections from $V \oplus V^*$ onto V and V^* respectively. Then $\ker \rho |_L = L \cap V^*$ and $\ker \rho^* |_L = L \cap V$, so that

$$\rho(L)^\circ = L \cap V^* \text{ and } \rho^*(L) = (L \cap V)^\circ$$

(note that $L \cap V$ may be thought of as a subspace of either $V \oplus V^*$ or V, as suits the circumstance; similarly for $L \cap V^*$).

Now consider the subspace $E = \rho(L) \subset V$. Define: $\Omega(\rho(x)) = -\rho^*(x)|_E$; this gives a map $\Omega: E \to E^*$ which is skew symmetric since $\langle \rho^*(x), \rho(y) \rangle + \langle \rho^*(y), \rho(x) \rangle = 0$ for all $x, y \in L$. To see that Ω is well defined, suppose we have $x, x' \in L$ such that $\rho(x) = \rho(x')$; we will show that $\rho^*(x)|_E = \rho^*(x')|_E$. In fact, since $\rho(x) = \rho(x')$, $x - x' \in \ker \rho|_L$, so $x - x' \in L \cap V^*$; therefore $\rho^*(x - x') \in \rho(L)^\circ = E^\circ$, which says exactly that $\rho^*(x)|_E = \rho^*(x')|_E$. Notice that $L \cap V \subset E$ is the kernel of Ω.

Along the same lines we also get a subspace $\rho^*(L) \subset V^*$, and a skew symmetric map $\Pi: \rho^*(L) \to \rho^*(L)^*$ whose kernel is $L \cap V^*$. We have:

$$\rho^*(L)^* = V/\rho^*(L)^\circ = V/L \cap V, \text{ or } \rho^*(L) = (V/L \cap V)^*$$

so this gives us $\Pi: (V/L \cap V)^* \to V/L \cap V$. Thus if we consider Ω to be a two-form on E, Π is a bi-vector on the quotient $V/L \cap V = V/\ker\Omega$.

Choosing a basis for L is the same as giving maps $a: R^n \to V$ and $b: R^n \to V^*$, so that the basis becomes: $(ae_1, be_1), \ldots, (ae_n, be_n)$; notice that for these to span an n-dimensional space, we must have: $\ker a \cap \ker b = \{0\}$.

Now the isotropy of L tells us that $a^*b+b^*a=0$, i.e. the map $a^*b:R^n \to R^{n*}$ is skew symmetric. Notice that if a is invertible, we may identify V with R^n so that b becomes a map $\tilde{b}:V\to V^*$; thus L is the graph of $\tilde{b}$. Similarly, if b is invertible, L is the graph of $\tilde{a}:V^*\to V$.

For now let us suppose that $V \approx V^*$, say via a choice of metric, so that L is given by a pair of maps $a,b: R^n \to V$ such that a^*b is skew and $\ker a \cap \ker b = \{0\}$. We will see that $a-b$ and $a+b$ are invertible. Suppose $x \in \ker a-b$; then $ax=bx$. Now $\langle a^*bx,x\rangle + \langle b^*ax,x\rangle = 0$, so $\langle a^*ax,x\rangle + \langle b^*bx,x\rangle = 0$. But this says that $\|ax\|^2+\|bx\|^2=0$, so $ax=0$ and $bx=0$. Therefore $x \in \ker a \cap \ker b$, so $x=0$, and $a-b$ is invertible; similarly for $a+b$. Now let $U=(a+b)(a-b)^{-1}$; then UU^* equals

$$\begin{aligned}(a+b)(a-b)^{-1}(a^*-b^*)^{-1}(a^*+b^*) &= (a+b)[(a^*-b^*)(a-b)]^{-1}(a^*+b^*)\\ &= (a+b)[(a^*a-a^*b-b^*a+b^*b)]^{-1}(a^*+b^*)\\ &= (a+b)[(a^*a+b^*a+a^*b+b^*b)]^{-1}(a^*+b^*)\\ &= (a+b)[(a^*+b^*)(a+b)]^{-1}(a^*+b^*)\\ &= I\end{aligned}$$

Therefore $U=(a+b)(a-b)^{-1}$ is orthogonal; the map $a,b \to U$ will be called the generalized Cayley transform. If a is invertible, it becomes the Cayley transform $ba^{-1} \to (I+ba^{-1})(I-ba^{-1})^{-1}$, since ba^{-1} is skew symmetric; a similar argument holds if b is invertible.

The action of GL(n) on (a,b) given by $(a,b)\times\gamma=(a\gamma,b\gamma)$ amounts to a change of basis in our "reference space", and so $(a\gamma,b\gamma)$ still represents the same Dirac structure L. Notice that the map $(a,b)\to U$ is invariant under this action. Therefore, the space of Dirac structures on V is in one-to-one correspondence with the group O(n). There is also an action of GL(V) given by $(a,b)\times\delta=(\delta^{-1}a,\delta^*b)$ whose orbits are the isomorphism classes of Dirac structures.

Dirac Bundles

We now extend the previous notion of Dirac structures to tangent bundles. A Dirac structure on TX is a maximally isotropic subbundle of the Whitney sum bundle $TX \oplus T^*X$ under the pairing $\langle\ ,\ \rangle_+$, just as in the linear case.

$\rho(L)$ is now a "distribution" in TX, $\Omega:\rho(L)\to\rho(L)^*$ is a 2-form on this distribution, and $L\cap TX \subset \rho(L)$ is the kernel of this 2-form. Our integrability condition will tell us that $\rho(L)$ and $L\cap TX$ are Frobenius integrable, and that $d\Omega$ (the exterior derivative in the leaves) is zero. Thus an integrable Dirac structure on TX (which we will call a Dirac structure on the manifold X) gives a singular "foliation" of X (since dimensions of leaves can jump) with a closed 2-form on each "leaf".

Two important examples of Dirac structures on a manifold X are pre-symplectic structures and Poisson structures, which are the graphs of maps $TX\to T^*X$ and $T^*X\to TX$ respectively; the skew symmetry of the maps makes their graphs isotropic, and their further structure (closedness or vanishing of the Schouten bracket) is the condition of integrability. We wish to determine a general integrability condition for Dirac structures which contains both of these as special cases.

We define a map $H:L\to L^*$ by $H(X,\omega)((Y,\mu))=\mu(X)=-\omega(Y)$, which we may think of as a bilinear form on L; it is skew symmetric because L is isotropic under $\langle\ ,\ \rangle_+$, and it is just the pullback of the natural symplectic form on $TL\oplus T^*L$. Recall that the 2-form $\Omega:\rho(L)\to\rho(L)^*$ is given by $\langle\Omega(X),Y\rangle=-\omega(Y)$; thus we clearly have the relation $\rho_L{}^*\Omega=H$, where $\rho_L=\rho\circ\iota_L$.

For the moment let us assume that $\rho(L)$ is integrable and of constant rank; this will allow us to define and hence compute $d\Omega$ on each integral manifold. Let $(X_1,\omega_1),(X_2,\omega_2)$, and (X_3,ω_3) be local sections of L, and define

$$H_{ij}\equiv H(X_i,\omega_i)((X_j,\omega_j))=\omega_j(X_i) \text{ and } \Omega_{ij}=\omega_j(X_i).$$

Then we have: $$d\Omega(X_1,X_2,X_3) = X_1\cdot\Omega_{23} - X_2\cdot\Omega_{13} + X_3\cdot\Omega_{12}$$
$$+\,\Omega(X_1,[X_2,X_3]) - \Omega(X_2,[X_1,X_3]) + \Omega(X_3,[X_1,X_2])$$
$$= X_1\cdot H_{23} + X_2\cdot H_{31} - X_3\cdot H_{12}$$
$$-\,\omega_1[X_2,X_3] - \omega_2[X_3,X_1] - \omega_3[X_1,X_2].$$

Notice that the last expression is defined on all sections of $TX\oplus T^*X$ and not just of L. Now using the formula

$$d\omega_1(X_2,X_3) = X_2\cdot\omega_1(X_3) - X_3\cdot\omega_1(X_2) - \omega_1[X_2,X_3]$$
$$= X_2\cdot H_{31} - X_3\cdot H_{21} - \omega_1[X_2,X_3]$$ we find that

$$d\Omega(X_1,X_2,X_3) = d\omega_1(X_2,X_3) + d\omega_2(X_3,X_1) + d\omega_3(X_1,X_2) - X_1\cdot H_{23} - X_3\cdot H_{12} - X_2\cdot H_{31}$$

Thus we get a totally skew symmetric expression on sections of L:

$$T((X_1,\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3))$$
$$= d\omega_1(X_2,X_3) - d\omega_2(X_1,X_3) + d\omega_3(X_1,X_2) - X_1\cdot H_{23} + X_2\cdot H_{13} - X_3\cdot H_{12}$$

This is the restriction to L of

$$\tilde{T}((X,\omega)\otimes(Y,\mu)\otimes(Z,\nu)) = d\omega(Y,Z) + d\mu(Z,X) + d\nu(X,Y)$$
$$+\,{}^1/_2(X\cdot\langle(Y,\mu),(Z,\nu)\rangle_- + Y\cdot\langle(Z,\nu),(X,\omega)\rangle_- + Z\cdot\langle(X,\omega),(Y,\mu)\rangle_-),$$

where $\langle\ ,\ \rangle_-$ is the skew symmetric pairing on all sections of $TX\oplus T^*X$ given by:
$\langle(X,\omega),(Y,\mu)\rangle_- = \omega(Y) - \mu(X)$.

We will now see that T is a tensor on L.

$$T((fX_1,f\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3))$$
$$= d(f\omega_1)(X_2,X_3) - d\omega_2(fX_1,X_3) + d\omega_3(fX_1,X_2) - fX_1\cdot H_{23} + X_2\cdot fH_{13} - X_3\cdot fH_{12}$$
$$= (df\wedge\omega_1)(X_2,X_3) + f(d\omega_1(X_2,X_3) - d\omega_2(X_1,X_3) + d\omega_3(X_1,X_2))$$
$$+\,(X_2\cdot f)H_{13} - (X_3\cdot f)H_{12} - f(X_1\cdot H_{23} - X_2\cdot H_{13} + X_3\cdot H_{12})$$
$$= (X_2\cdot f)\omega_1(X_3) - (X_3\cdot f)\omega_1(X_2) + (X_2\cdot f)H_{13} - (X_3\cdot f)H_{12}$$
$$+\,f\,T((X_1,\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3)).$$

Thus $$T((fX_1,f\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3)) - f\,T((X_1,\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3))$$
$$= (X_2\cdot f)(\omega_1(X_3)+\omega_3(X_1)) - (X_3\cdot f)(\omega_1(X_2)+\omega_2(X_1)).$$

If we let $e_i = (X_i,\omega_i)$ then the last expression may be rewritten:

$$T(fe_1\otimes e_2\otimes e_3) - f\,T(e_1\otimes e_2\otimes e_3) = \rho(e_2)\cdot f\langle e_1,e_3\rangle_+ - \rho(e_3)\cdot f\langle e_1,e_2\rangle_+$$

which is clearly zero when restricted to any subbundle isotropic under the pairing $\langle\ ,\ \rangle_+$. Therefore T is a 3-tensor on L.

Note that there are two expressions for T on L:

$$T((X_1,\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3))$$
$$= d\omega_1(X_2,X_3) + d\omega_3(X_1,X_2) + d\omega_2(X_3,X_1) - X_1\cdot H_{23} - X_3\cdot H_{12} - X_2\cdot H_{31}$$

and $$T((X_1,\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3))$$
$$= X_1\cdot H_{23} + X_3\cdot H_{12} + X_2\cdot H_{31} - \omega_1([X_2,X_3]) - \omega_3([X_1,X_2]) - \omega_2([X_3,X_1])$$

Since T is a 3-tensor on L, we may think of it as giving us two maps: $T_2 : L\otimes L\to L^*$ and $T_1 : L\to L^*\otimes L^*$; notice that $T_1 = T_2{}^*$. Of course, the vanishing of T implies the vanishing of both of these maps. We will now examine the consequences of this fact.

Let $C = L\cap TX$; the assumption of constant rank of C implies that it has local sections. Consider $T_2|_{C\times C}$: Let (Z,ν) be a local section of L and let $(X,0)$ and $(Y,0)$ be local sections of C.

Then $\langle T_2((X,0)\otimes(Y,0)),(Z,\nu)\rangle = -\nu([X,Y])$
by the second expression for T.
Now since ν may take each value in $\rho^*(L) = (L\cap TX)^\circ$, $\nu([X,Y]) = 0$ for all ν if, and only if, $[X,Y]\in L\cap TX$. Therefore $T_2|_{C\times C} = 0 \Leftrightarrow C$ is integrable.

Now consider $T_1|_{L\cap T^*X}$: $\langle T_1(0,\nu),(X,\omega)\otimes(Y,\mu)\rangle = -\nu([X,Y])$;
once again, this vanishes for all $\nu\in L\cap T^*X$ exactly when $[X,Y]\in\rho(L)$. So $T_1|_{L\cap T^*X} = 0 \Leftrightarrow \rho(L)$ is integrable (again modulo a condition of constant rank).

Thus $T = 0$ implies that $\rho(L)$ and $L\cap TX$ are both integrable. If $\rho(L)$ is integrable (and of constant dimension), then $T = \rho_L{}^*d\Omega$, and so the vanishing of T also says that $d\Omega = 0$.

Admissible functions and hamiltonian vector fields

Since $(L\cap TX)^\circ = \rho^*(L)$, the integrability of the distribution $L\cap TX$ implies that a basis (X_i,ω_i) may be chosen such that the ω_i are closed (this may be arrived at by taking linear combinations of an arbitrary basis). Thus, the ω_i are locally differentials of admissible functions, where we call a function admissible if (X,df) is a section of L for some vector field X. We call X a hamiltonian vector field for f.

We may define a bracket on admissible functions: $\{f_1,f_2\} = X_2\cdot f_1 = df_1(X_2)$, where X_i is a hamiltonian vector field for f_i. We will see later that the bracket of two admissible functions is again admissible.

If $\varphi : X \to X$ is a diffeomorphism, then φ acts on L as follows: $\varphi\cdot(X,\omega) = (\varphi^*X,\varphi^*\omega)$ (where $\varphi^*X = (\varphi_*{}^{-1})X$). The infinitesimal version of this is an action on L by vector fields:

$$\xi\cdot(X,\omega) = (\mathcal{L}_\xi X, \mathcal{L}_\xi \omega) = ([\xi,X], d\omega(\xi,\cdot) + d(\omega(\xi)))$$

Theorem. Let L be an integrable Dirac strucure and let $(X_1,\omega_1),(X_2,\omega_2)$, and (X_3,ω_3) be sections of L. Then: $\langle \mathcal{L}_{X_1}(X_2,\omega_2),(X_3,\omega_3)\rangle_+ = d\omega_1(X_2,X_3)$.

Corollary. If $\rho(L)$ and $L\cap TX$ have constant dimension, then

$$X\cdot L \subset L \Leftrightarrow d\omega\,\big|_{\rho(L)} = 0.$$

Proof of theorem. By the formula preceding the theorem we have

$$X_1\cdot(X_2,\omega_2) = ([X_1,X_2], d\omega_2(X_1,\cdot) + d(\omega_2(X_1)))$$
$$= ([X_1,X_2], d\omega_2(X_1,\cdot) + dH_{12}),$$

so
$$\langle X_1\cdot(X_2,\omega_2),(X_3,\omega_3)\rangle_+ = \omega_3\cdot[X_1,X_2] + d\omega_2(X_1,X_3) + X_3\cdot H_{12}$$
$$= X_1\cdot\omega_3(X_2) - X_2\cdot\omega_3(X_1) - d\omega_3(X_1,X_2) - d\omega_2(X_3,X_1) + X_3\cdot H_{12}$$
$$= -\ T((X_1,\omega_1)\otimes(X_2,\omega_2)\otimes(X_3,\omega_3)) + d\omega_1(X_2,X_3)$$
$= d\omega_1(X_2,X_3)$ since T vanishes on L by hypothesis. QED

Proof of Corollary. Use the fact that $X\cdot L\subset L \Leftrightarrow \langle X\cdot L, L\rangle_+ = 0$.

Dirac vector fields

By the previous corollary, locally hamiltonian vector fields are Dirac vector fields, i.e. infinitesimal automorphisms of the Dirac structure on X. In particular, if L has a basis (X_i,ω_i) with $d\omega_i = 0$, $\rho(L)$ is spanned by infinitesimal automorphisms.

Now let X be an arbitrary vector field, and $(X_1,\omega_1),(X_2,\omega_2)$ sections of L:

$$\langle X\cdot(X_1,\omega_1),(X_2,\omega_2)\rangle_+ = \langle([X,X_1], d\omega_1(X,\cdot) + d(\omega_1(X))),(X_2,\omega_2)\rangle_+$$
$$= \omega_2([X,X_1]) + X_2\cdot\omega_1(X) + d\omega_1(X,X_2)$$
$$= \omega_2([X,X_1]) + X_2\cdot\omega_1(X) + X\cdot\omega_1(X_2) - X_2\cdot\omega_1(X) - \omega_1([X,X_2])$$
$$= X\cdot\omega_1(X_2) + \omega_2([X,X_1]) - \omega_1([X,X_2])$$

Thus the condition that L be invariant under X is: $X\cdot H_{12} = \omega_2([X,X_1]) - \omega_1([X,X_2])$ for all sections $(X_1,\omega_1),(X_2,\omega_2)$ of L.

If L is the graph of a Poisson bundle map, and if $\omega_1 = df$ and $\omega_2 = dg$, then this says:

$$X\cdot\{f,g\} = dg([X,X_f]) - df([X,X_g])$$
$$= X(X_f(g)) - X_f(X(g)) - X(X_g(f)) + X_g(X(f))$$
$$= 2X\cdot\{f,g\} - \{f,X(g)\} - \{X(f),g\}$$

or
$$X\cdot\{f,g\} = \{X(f),g\} + \{f,X(g)\},$$

which is just the condition that X be a Poisson vector field.

If L is the graph of a pre-symplectic structure, then

$$\omega_1 = \Omega(X_1,\cdot),\ \text{so}\ H_{12} = \omega_2(X_1) = \Omega(X_2,X_1) = -\Omega(X_1,X_2).$$

Therefore $X\cdot H_{12} = -X\cdot\Omega(X_1,X_2) = -\mathcal{L}_X\Omega(X_1,X_2) - \Omega([X,X_1],X_2) - \Omega(X_1,[X,X_2])$.
Also $\omega_2([X,X_1]) - \omega_1([X,X_2]) = \Omega(X_2,[X,X_1]) - \Omega(X_1,[X,X_2])$, so the equation for a Dirac vector field says that $\mathcal{L}_X\Omega(X_1,X_2) = 0$ for all X_1,X_2 in L; in this case $\rho(L) = TX$, so this actually says $\mathcal{L}_X\Omega = 0$ which is the condition for a locally hamiltonian vector field, i.e. the only vector fields leaving a pre-symplectic structure invariant are the locally hamiltonian ones.

Now let (Y,μ) be a section of an integrable Dirac structure L such that $d\mu = 0$; then $X\cdot(Y,\mu) = ([X,Y], d\mu(X,\cdot) + d(\mu(X))) = ([X,Y], d(\mu(X)))$.

Suppose that $(X_f, df), (X_g, dg), (X_h, dh)$ are all sections of L. Recall that we get a bracket on admissible functions, given by $\{f,g\} = X_f \cdot g$, so the action of the hamiltonian vector fields is $X_f \cdot (X_g, dg) = ([X_f, X_g], d\{f,g\})$; we know that this is again in L, so we find that $\{f,g\}$ is admissible with hamiltonian vector field $[X_f, X_g]$. In addition,

$$\{f,\{g,h\}\} = X_f \cdot \{g,h\} = -[X_g, X_h] \cdot f = -X_g \cdot \{h,f\} + X_h \cdot \{g,f\} = -\{g,\{h,f\}\} + \{h,\{g,f\}\},$$

or after rearranging: $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$. Thus the bracket on admissible functions satisfies the Jacobi identity.

The Leibniz identity also holds, as follows: if f and g are admissible functions then the identity $d(fg) = f\,dg + g\,df$ implies that fg has hamiltonian vector field $fX_g + gX_f$, and so fg is also admissible. Therefore

$$\{fg,h\} = X_{fg} \cdot h = fX_g \cdot h + gX_f \cdot h = f\{g,h\} + g\{f,h\}.$$

Hence the admissible functions form a Poisson algebra.

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