Instructions. BE SURE TO WRITE YOUR NAME AND YOUR GSI’S NAME ON YOUR BLUE BOOK. Read the problems very carefully to be sure that you understand the statements. All work should be shown in the blue book; writing should be legible and clear, and there should be enough work shown to justify your answers. Indicate the final answers to problems by circling them. [Point values of problems are in square brackets. There are eight problems, with a total point value is 120, for 40% of your course grade.]

PLEASE HAND IN YOUR PREPARED NOTES ALONG WITH YOUR BLUE BOOK. YOU SHOULD NOT HAND IN THIS EXAM SHEET.

1. [18 points] The vector functions \( \mathbf{r}_1(t) = (\cos t, \sin t, t^2) \) and \( \mathbf{r}_2 = (\cos t, -\sin t, 0) \) describe two curves which lie on the cylinder \( x^2 + y^2 = 1 \).
   (a) The two curves go through the same point \( P \) at \( t = 0 \). Find that point, and find the tangent vectors of the two curves at \( P \).
   (b) For \( t \neq 0 \), there is a unique plane \( S_t \) through the points \( \mathbf{r}_1(t), \mathbf{r}_2(t) \), and \( P \). Among all the normal vectors to the plane \( S_t \), there is one which has the form \( \mathbf{u}(t) = (1, a(t), b(t)) \), where \( a(t) \) and \( b(t) \) are functions of \( t \). Find \( \mathbf{u}(t) \).
   (c) Find the limiting position of the normal \( \mathbf{u}(t) \) as \( t \) approaches zero. [Hint: use l’Hôpital’s rule.]
   (d) Find a normal vector for the tangent plane at \( P \) to the cylinder \( x^2 + y^2 = 1 \).
   (e) Comment on the relation between the answers to parts (c) and (d). Do you find it surprising?

2. [12 points] The level curves of a certain function \( f(x, y) \) are the lines parallel to the line \( y = -x \), and the value of the function increases as one moves from lower left to upper right.
   (a) Where on the ellipse \( x^2 + 2y^2 = 9 \) does \( f \) attain its maximum value?
   (b) Where does it attain the minimum value?

3. [15 points] Find the area of the part of the paraboloid \( z = 9 - x^2 - y^2 \) that lies above the plane \( z = 5 \).

4. [15 points] (a) Describe the (solid) region of integration \( E \) for the integral
   \[
   \int_{-5}^{5} \int_{\sqrt{25-x^2}}^{\sqrt{25-y^2}} \int_{\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} \sqrt{x^2+y^2+z^2} \, dz \, dx \, dy
   \]
   (b) Evaluate the integral by using spherical coordinates.

5. [15 points] In each of parts (a) and (b), evaluate as many as possible of the following five expressions. If the expression is not defined, say so. Be sure to distinguish between “zero” and “not defined”. (1) The divergence of the curl; (2) the curl of the gradient; (3) the gradient of the curl; (4) the divergence of the gradient; (5) the gradient of the divergence.
   There should be ten answers in all. Please present them in the order: 1a, 2a, 3a, 4a, 5a, 1b, 2b, 3b, 4b, 5b.
   (a) \( F(x, y, z) = x^4 \mathbf{i} + y^4 \mathbf{k} \).
   (b) \( f(x, y, z) = x^4 - yz \).

PLEASE TURN OVER THE PAGE FOR THE REMAINING PROBLEMS
6. [12 points] Let $E$ be the solid region between the sphere $x^2 + y^2 + z^2 = 1$ and the larger sphere $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 100$. An expanding gas is flowing into $E$ through the small sphere, and the flux of the velocity field $\mathbf{F}$ through that sphere (with respect to the normal pointing into $E$ from the hole in the middle) is equal to 20. Furthermore, the velocity field $\mathbf{F}$ has divergence everywhere equal to 3 on $E$. Find the flux of $\mathbf{F}$ through the larger sphere with respect to the outward normal.

7. [15 points]
(a) Find a vector field $\mathbf{F}$ whose curl is $(1 + x) \mathbf{k}$. [Hint: guess a solution in the simplest possible form, then check, and correct if necessary.]
(b) Let $S$ be the surface which bounds the solid region which is inside the cylinder $x^2 + y^2 = 1$, above the $xy$-plane, and below the plane $z = 10 + x$. We may think of $S$ as a “can with a slanted top”. Use Stokes’ theorem (and NOT the divergence theorem) to show that the flux of $(1 + x) \mathbf{k}$ through the elliptical top of the can is equal to the flux of the same vector field through the circular bottom (with top and bottom both oriented by the “upward” normal). [Hint: don’t forget to take the cylindrical “side” of the can into account.]
(c) Find the value of this flux.

8. [18 points] Let $D$ be the L-shaped region in the plane whose boundary $C$ consists of the straight line segments connecting the following points in the given order:

$(-1, -1), (2, -1), (2, 1), (1, 1), (1, 3), (-1, 3), (-1, -1)$. 

Find the line integral around $C$ of each of the following vector fields.
(a) $\nabla(x^2 + 1 - \sin(xe^y))$

(b) $y \mathbf{i} - 3x \mathbf{j}$

(c) $\frac{y \mathbf{i} - x \mathbf{j}}{x^2 + y^2}$

[Hint: replace $C$ by a simpler curve.]
1. \( r_1(t) = \langle \cos t, \sin t, t^2 \rangle, \quad r_2(t) = \langle \cos t, -\sin t, 0 \rangle \)

(a) At \( t = 0 \), \( r_1(t) = r_2(t) = \langle 1, 0, 0 \rangle = P \).

(b) The tangent vectors are
\[
\begin{align*}
  r'_1(t) &= \langle -\sin t, \cos t, 2t \rangle = \langle 0, 1, 0 \rangle \quad \text{at } t = 0 \\
  r'_2(t) &= \langle -\sin t, -\cos t, 2t \rangle = \langle 0, -1, 0 \rangle \quad \text{at } t = 0.
\end{align*}
\]

(b) The plane contains lines parallel to the vectors
\[
\begin{align*}
  \mathbf{r}_1(t) - \mathbf{r}_2(t) &= \langle 0, 2 \sin t, t^2 \rangle = 2 \sin t \mathbf{j} + t^2 \mathbf{k} \\
  \mathbf{r}_2(t) - \mathbf{P} &= \langle \cos t - 1, -\sin t, 0 \rangle = (\cos t - 1) \mathbf{i} - \sin t \mathbf{j}.
\end{align*}
\]

So a normal vector is given by the cross product
\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  0 & 2 \sin t + 2t \mathbf{k} \\
  \cos t - 1 & -\sin t & 0
\end{vmatrix} = 2 \sin t \mathbf{i} + \cos t \mathbf{k}.
\]

And a normal with first component 1 is
\[
\langle \mathbf{i} + \frac{\cos t - 1}{\sin t} \mathbf{j} + \frac{2(1 - \cos t)}{t^2} \mathbf{k} \rangle.
\]

(c) As \( t \to 0 \), we use l'Hôpital's rule to find
\[
\lim_{t \to 0} \frac{\cos t - 1}{\sin t} = \lim_{t \to 0} \frac{-\sin t}{\cos t} = \frac{0}{1} = 0,
\]
\[
\lim_{t \to 0} \frac{2(1 - \cos t)}{t^2} = \lim_{t \to 0} \frac{2 \sin t}{2t} = \lim_{t \to 0} \frac{2 \cos t}{2} = 1,
\]
so the limiting normal is \( \langle 1, 0, 1 \rangle \).
(d) A normal to the cylinder is given by
\[ \text{grad} (x^2 + y^2) = 2x \hat{i} + 2y \hat{j} \] at \( P = (1, 0, 0) \).
This is \( 2\hat{i} \).

(e) The vector in (d) does not point in the same direction as the vector in (e). This is somewhat surprising. It shows that "secant planes" to a surface do not always approach the tangent plane.

So critical points along the ellipse occur when
\[
\begin{align*}
\lambda \cdot 2x &= 1 \\
\lambda \cdot 4y &= 1
\end{align*}
\Rightarrow \lambda = \frac{1}{2} = \frac{1}{4y} \Rightarrow 4y = 2x \Rightarrow x = 2y \quad [\text{See the figure.}]
\]
(continued on next page)
Now we apply the constraint: \( x^2 + 2y^2 = 9 \)

\[
(2y)^2 + 2y^2 = 9 
\]

Since \( f \) increases as one moves from lower left to upper right,

(a) The maximum value is attained at \( \left(2\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}} \right) \)

\[
4y^2 + 2y^2 = 9 \\
6y^2 = 9 \\
y^2 = \frac{3}{2} \\
y = \pm \sqrt{\frac{3}{2}} \\
x = \pm 2\sqrt{\frac{3}{2}}. 
\]

(b) The minimum value is attained at \( (-2\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}) \).

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3. The integrand for surface area is

\[
\sqrt{1 + \left( \frac{2x}{\partial x} \right)^2 + \left( \frac{2y}{\partial y} \right)^2} = \sqrt{1 + (2x)^2 + (2y)^2} \\
= \sqrt{1 + 4x^2 + 4y^2} 
\]

So the surface area is

\[
A = \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA , \text{ where } D \text{ is the disc} \\
\text{given by } 9 - x^2 - y^2 \geq 5, \text{ or } x^2 + y^2 \leq 4, \text{ i.e. the disc of radius } 2.
\]
\[ A = \int_{0}^{2\pi} \int_{0}^{2} \sqrt{1+4r^2} \, r \, dr \, d\theta \]

\[ = 2\pi \int_{0}^{2} \sqrt{1+4r^2} \, r \, dr \]

Put \( s = r^2 \), \( ds = 2r \, dr \)

\[ = \pi \int_{0}^{4} \sqrt{1+4s} \, ds \]

\[ = \pi \cdot (1+4s)^{3/2} \bigg|_{0}^{4} \cdot \frac{1}{2} \]

\[ = \frac{\pi}{6} (17^{3/2} - 1) \]

\[ 4 \quad \text{(a) we have} \quad y \in [-5, 5], \quad x \in [0, \sqrt{25-y^2}], \]

which define a half circle in the (x,y) plane.

The last inequality means

\[ z \in \left[ -\sqrt{25-x^2-y^2}, \sqrt{25+x^2+y^2} \right] \]

means that we have a hemispherical region ("half-ball") defined by \( x^2+y^2+z^2 \leq 25, \, x \geq 0 \).

Continued
4(b). In spherical coordinates, we have

\[ \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \ \phi \in [0, \pi], \ \rho \in [0, 5], \]

and the integral becomes

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\phi} \int_{0}^{5} \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi
\]

\[
= \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{5} \rho^2 \ d\rho \ \sin \phi \ d\phi \ d\theta
\]

\[
= \frac{5^4}{4} \cdot \pi \cdot 2 = \frac{5^4}{2} \pi = \frac{625}{2} \pi.
\]

\[\Box\]

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Now \[ \iiint_{E} \text{div} \mathbf{F} \, dV = \iiint_{E} 3 \, dV = 3 \cdot \text{Vol} \, E \]
\[ = 3 \cdot \left( \frac{4}{3} \pi \cdot 10^2 - \frac{4}{3} \pi \cdot 1^3 \right) \]
\[ = 4 \pi (999) = 3996 \pi \]

On the other hand, \( \partial E = S_2 + S_1 \) with the orientation in the figure. We are given that
\[ \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = 20, \text{ so we have } \]
\[ \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = 20 + 3996 \pi \]
(a) Let \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \). Then \( \text{curl } \mathbf{F} = (Q_y - P_z) \mathbf{k} + (P_z - Q_x) \mathbf{i} + (Q_x - P_y) \mathbf{j} \).

If we wish to get \((1+x) \mathbf{k}\) as the result, the simplest choice is \( Q = x + \frac{1}{2} x^2, \ P = R = 0 \), i.e.
\[
\mathbf{F} = (x + \frac{1}{2} x^2) \mathbf{j}.
\]

(b) Since \( S \) is the boundary of a solid region, \( \partial S = \emptyset \), so Stokes' theorem yields
\[
0 = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iiint (1+x) \mathbf{k} \cdot d\mathbf{S}.
\]

\[
= \iiint (1+x) \mathbf{k} \cdot d\mathbf{S} = \iint_{S_{\text{top}}} (1+x) \mathbf{k} \cdot d\mathbf{S} + \iint_{S_{\text{side}}} (1+x) \mathbf{k} \cdot d\mathbf{S} = \iint_{S_{\text{bottom}}} (1+x) \mathbf{k} \cdot d\mathbf{S}.
\]
when all pieces have the \underline{outward normal}.

Now \( \iint_{S_{\text{top}}} = 0 \) because \((1+x) \mathbf{k}\) is tangent to the sides cylindrical surface.

Also, \( S_{\text{bottom}} \) has the \underline{downward normal}, so we get
\[
\iint_{S_{\text{top}}} (1+x) \mathbf{k} \cdot d\mathbf{S} + \iint_{S_{\text{side}}} (1+x) \mathbf{k} \cdot d\mathbf{S} = \iint_{S_{\text{bottom}}} (1+x) \mathbf{k} \cdot d\mathbf{S} = 0,
\]
and we are done.

(c) \[ \iint_{S_{\text{bottom}}} (1+x) \mathbf{k} \cdot d\mathbf{S} = \iint (1+x) dA = \int 1 dA + \int x dA = \pi + 0 = \pi \]
\[
\text{unit disc unit disc unit disc (The } 0 \text{ is by symmetry.)}
\]
(a) The integral is zero, since the integral of any gradient around a closed curve is zero.

(b) \[ \int \int (Q_x - P_y) \, dA = \int \int (P_x + Q_y) \cdot \, dx \cdot dy \]

Here, \( P = y \) and \( Q = -3x \),

So \( Q_x - P_y = -3 - 1 = -4 \), and

The integral is \( -4 \cdot \text{(Area } D) = -4 \cdot 40 = \boxed{-160} \)

(c) Here, \( P = \frac{y}{x^2 + y^2} \), so \( P_y = \frac{(x^2 + y^2) \cdot 1 - y (-2y)}{(x^2 + y^2)^2} \)

Similarly, \( Q_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} \)

So \( Q_x - P_y = 0 \). By Green's theorem, we may replace the curve \( C \) by another curve surrounding the singular point \((0,0)\) of our vector field. We choose a circle \( C_1 \) of radius \( \varepsilon \), parametrized by \( x = \varepsilon \cos \theta, \, y = \varepsilon \sin \theta \). Then the integral becomes

\[ \int \int \frac{y^2 - x^2}{x^4 + y^4} \cdot (-\varepsilon \sin \theta \, d\theta + \varepsilon \cos \theta \, d\theta) = \int \frac{-\varepsilon^2 (\sin^2 \theta + \cos^2 \theta)}{\varepsilon^2} \, d\theta = -2\pi \]