Marsden-Weinstein Reductions for Kähler, Hyperkähler and Quaternionic Kähler Manifolds

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1 Introduction

If a Lie group G acts on a symplectic manifold (M, ω) and preserves the symplectic form ω, then in some cases there may exist a moment map ([C]) Φ from M to the dual of the Lie algebra. When the action is (locally) free, the preimage of a point in the dual of the Lie algebra modulo the isotropy group of this point will still be a symplectic manifold (orbifold). This process is called the Marsden-Weinstein reduction (MW reduction) and the reduced manifold is called Marsden-Weinstein quotient (MW quotient) ([MW]). It has been generalized to hyperkähler and quaternionic Kähler maniflods by Hitchin et al. ([HKLR]) and Galicki and Lawson ([GL]), respectively. In this term paper, we will show how Marsden-Weinstein reduction works in Kähler, hyperkähler and quaternionic Kähler cases and then give some examples to see how MW reduction gives a new approach to get manifolds or orbifolds in each case.

2 What Is a Moment Map and What is the MW Reduction?

Let a Lie group G act on a symplectic manifold (M, ω) preserving the symplectic form. Then a map Φ from M to $\mathfrak{g}^*$ satisfying the following conditions is called a moment map.

1. $d < \Phi(x), v >= i(v_M)\omega|_x$, where v_M is the vector field generated by the $\mathfrak{g}$ action of v on M, i.e. $v_M(p) = \frac{d}{dt}|_{t=0} \exp tv \cdot p$.

2. Equivariance, i.e. $\Phi a = \mathrm{Ad}^*_a \Phi$, for all $a \in G$.

G is then said to act on M hamiltionianly. For every vector field X on M and v in $\mathfrak{g}$, we have $< T\Phi(X), v >= X(< \Phi, v >) = d < \Phi(x), v > (X) = \omega(v_M, X)$; therefore, Φ is unique up to a "local constant", i.e. something in $H^0(M)$.

Let G_μ be the isotropy group at point $\mu \in \mathfrak{g}^*$. Since G_μ acts equivariantly, it preserves the μ-level set of Φ, i.e. there is also a G_μ action on $\Phi^{-1}(\mu)$. Although the original symplectic form ω on M when restricting on $\Phi^{-1}(\mu)$ is not nondegenerate any more, after dividing by the group G_μ, we will get a well-defined symplectic form on $\Phi^{-1}(\mu)/G_\mu$. This whole process is called Marsden-Weinstein reduction (or Marsden-Weinstein-Meyer reduction). If G acts freely on the μ-level set, then this quotient space is actually a manifold (thus a symplectic manifold); if the action is only locally free (for example when we only know that μ is a regular value of the moment map), this quotient space may be an orbifold. This process allows one to construct and recover a lot of symplectic manifolds.

Example 2.1 *(Complex projective spaces).* Let $U(1) = S^1$ act on $\mathbb{C}^n$ by scalar multiplication on each component as $\theta \cdot (z_1, \dots, z_n) = (e^{-i\theta} z_1, \dots, e^{-i\theta} z_n)$. Then, one may verify that the moment map is given by

$$\Phi(z_1, z_2, \dots, z_n) = (1/2) \sum_{j=1}^{n} |z_j|^2.$$

Since S^1 is commutative, the isotropy group at every point is the whole S^1. Then the MW quotient at the point 1/2 is

$$\Phi^{-1}(1/2)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}.$$

It's well known that $\mathbb{CP}^{n-1}$ has a hermitian metric

$$(i/2)\partial\bar{\partial}\log(|z|^2)$$

inherited from $\mathbb{C}^n$ and it is indeed a Kähler metric (see the next section), so its imaginary part is a symplectic form. If the MW reduction is nice enough, this symplectic form should be the "reduced" symplectic form got from the MW reduction. And it is! This is so because the "reduced" symplectic form is also inherited from the standard symplectic form on $\mathbb{C}^n$, which is compatible with its own complex structure.

Example 2.2 *(Coadjoint orbits of a compact Lie group).* Let G be a compact Lie group. Then T^*G carries a natural symplectic structure $\omega = -d\alpha$, where α is the tautological 1-form (see [C], p. 8), such that, $\forall \beta \in T^*G$, $\forall X \in T_\beta(T^*G)$, π, the projection from T^*G to G,

$$\alpha|_\beta(X) = \beta(\pi_* X).$$

Since v_{T^*G} is infinitesimal (lifted) left translation for all v in the Lie algebra, $L_{v_{T^*G}}\alpha = 0$, and then

$$i(v_{T^*G})d\alpha + d(i(v_{T^*G})\alpha) = 0.$$

Take the value at β, and notice that $\pi_* v_{T^*G}$ is the differential of a left translation, hence a right invariant vector field, then we will have,

$$d < \text{right translate } \beta \text{ to } e, v >= i(V_{T^*G})\omega,$$

i.e. if we take the trivialization of $T^*G \cong G \times \mathfrak{g}^*$ by the right translation, then the moment map now is just the projection onto the second component. So $\Phi^{-1}(\mu) \cong G$ and the MW quotient at point μ is

$$\Phi^{-1}(\mu)/G_\mu = \text{the coadjoint orbit of } \mu.$$

Example 2.3 *(Toric manifolds and symplectic cut).* A toric manifold M is a compact $2n$-dimensional symplectic manifold with an effective hamiltonian n-dimensionals torus $\mathbb{T}^n$-action. The image of the moment map $\Phi : M \longrightarrow \mathfrak{t}^* = \mathbb{R}^n$ is then a convex polytope in $\mathbb{R}^n$, called Delzent polytope (see [C], Section 28.), and inversely, M can be constructed as

$$M = \Phi(M) \times \mathbb{T}/\sim,$$

where $\sim$ means that $(p,t) \sim (q,s) \Longleftrightarrow p = q$ and $ts^{-1} \in \exp(($ the face containing $p)^\perp)$. Therefore, there is a one-to-one correspondence between Delzant polytopes and toric manifolds.

A symplectic manifold (M, ω) with an S^1-action and a moment map Φ: $M \longrightarrow \mathbb{R}$ can be operated on by the following process:

(i) Let S^1 act on $M \times \mathbb{C}$ by $e^{i\theta}(m,z) = (e^{i\theta} \cdot m, e^{-i\theta} z)$;

(ii) Let the moment map be $\Phi_1 : (m,z) \mapsto \Phi(m) + (1/2)|z|^2 - \lambda$;

(iii) Do the reduction: $\Phi_1^{-1}(0)/S^1$.

This process is called symplectic cutting.

We can see why it is called symplectic cutting in the case of toric manifolds. Let $M = (\mathbb{R}_+)^n \times \mathbb{T}/\sim$, ($\sim$ is as defined just now and $(\mathbb{R}_+)^n$ is the first quadrant of

$\mathbb{R}^n$), S^1 = a subtorus of $\mathbb{T}$: $(e^{i\theta}, 1, ..., 1)$. Then after the symplectic cutting, we'll end up with a quotient manifold $([0, \lambda_1] \times (\mathbb{R}_+)^{n-1}) \times \mathbb{T}/\sim$, as if we cut $\mathbb{R}^n_+$ by the hyperplane corresponding to this S^1. Geometrically, different S^1's correspond to different directions of the hyperplane, different λ_1's correspond to different places of the cutting.

Since a polytope can always be obtained by n cuts on $\mathbb{R}^n_+$, a toric manifold is in fact a result of a series of symplectic cuttings, hence a MW quotient.

In fact, $\mathbb{R}^n_+ \times \mathbb{T}/\sim = \mathbb{C}^n$, and we can write all the reductions in the n cuts as one if we ask

(i) $\mathbb{T}$ to act on $\mathbb{C}^n \times \mathbb{C}^n$ by "double" ordinary action, i.e.

$$(\exp i\theta_1, \dots, \exp i\theta_n) \cdot (w_1, \dots, w_n, z_1, \dots, z_n)$$
$$= (\exp(-i\theta_1)w_1, ..., \exp(-i\theta_n)w_n, \exp(-i\theta_1)z_1, ..., \exp(-i\theta_n)z_n);$$

(ii) the moment map Φ:

$$(w, z_1, ..., z_n) \mapsto (\Phi_1(w) + (1/2)|z_1|^2 - \lambda_1, ..., \Phi_n(w) + (1/2)|z_n|^2 - \lambda_n),$$

where Φ_j is the moment map of the j-th subtorus in the process of cutting with the action on $\mathbb{C}^n$ induced from the ordinary $\mathbb{T}$-action.

So all toric manifolds can be viewed as the MW quotient from the simple space $\mathbb{C}^{2n}$.

3 Kähler Reduction

Definition 3.1. A Kähler manifold (M, g, I, ω) is a symplectic complex Riemannian manifold such that these three structures are compatible, i.e.

$$g(I\cdot, \cdot) = \omega(\cdot, \cdot).$$

The hermitian form $g + i\omega$ is then called the Kähler metric of M.

Remark 3.2. From the point of view of holonomy bundles (see [S]), Kähler $\Longleftrightarrow$ the holonomy bundle is contained in the $U(n)$ frame bundle. When the manifold M is compact, an isometry group automatically preserves the Kähler metric. (see [S], Section 8.)

Theorem 3.3 ([HKLR]). *If a compact Lie group G acts on a Kähler manifold M isometrically, hamiltonianly and freely, then its MW quotient is still a Kähler manifold.*

Actually, this theorem is not hard to prove. First, look at the symplectic manifold $Q = \Phi^{-1}(0)/G$. The G-orbit and the level set $\Phi^{-1}(0)$ are symplectic orthogonal, i.e. if we ask $W = \{v_M : v \in \mathfrak{g}\} \subset V = T_x M$, then

$$T_{[x]}Q = W^{\perp\omega}/W = (I \cdot W)^{\perp}/W = (I \cdot W)^{\perp} \cap W^{\perp}.$$

(Here $\perp \omega$ means the symplectic orthogonal, $\perp$ means the ordinary orthogonal under the Riemannian structure). So $T_{[x]}Q$ is invariant under the I-action. Hence I can be reduced to it. Its integrability will be inherited from the original manifold since the group action is isometric. So, together with the reduced symplectic form we have known and the reduced Remannian metric by the isometric property, the only thing left is to check the compatibility.

Secondly, if the theorem is true in the 0-case, it will be true in a general case because instead of considering the G-action and the moment map Φ, we can consider the G_μ-action and the moment map $\Phi - \mu$, and reduce the μ-case to the 0-case.

Example 3.4. Recalling from Example 2.1, the S^1-action preserves the metric, so the result manifold $\mathbb{CP}^{n-1}$ is a Kähler manifold. Unsurprisingly, the reduced Kähler metric is the same one inherited from $\mathbb{C}^n$.

Example 3.5. Since $\mathbb{T}$ acts on $\mathbb{C}^n \times \mathbb{C}^n$ isometrically, and toric manifolds are MW quotients of $\mathbb{C}^n \times \mathbb{C}^n$, toric manifolds are Kähler manifolds.

Remark 3.6. The coadjoint orbits of a compact semisimple Lie group G are Kähler manifolds; however, one can't expect them to be deduced from T^*G by the usual Kähler reduction. R.S. Filippini explained this in his paper [F].

4 Hyperkähler Reduction

Definition 4.1. A hyperkähler manifold is a Riemannian manifold (M, g) carrying three complex structures I_1, I_2, I_3, and three symplectic structures $\omega_1, \omega_2, \omega_3$ such that they are compatible respectively as in the Kähler case and I_1, I_2, I_3 behave algebraically like quaternions, i.e.

$$I_1^2 = I_2^2 = I_3^2 = -1, \qquad I_1 I_2 = I_3, \qquad \text{etc..}$$

Remark 4.2. From the point of view of holonomy bundles (see [S]), hyperkähler $\Longleftrightarrow$ the holonomy bundle is contained in the $U(n, \mathbb{H})$ frame bundle. (Some people like to call it as $Sp(n)$ as well.)

Given an isometric G action on a hyperkähler manifold M with structures $(g, I_1, I_2, I_3, \omega_1, \omega_2, \omega_3)$, we may define their moment maps Φ_1, Φ_2, Φ_3 corresponding to each symplectic form, which can be written as a single map

$$\Phi : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

The generalized theorem is as following:

Theorem 4.3 ([HKLR]). *Let a compact Lie group G acts on a hyperkähler manifold M isometrically, hamiltonianly (with respect to all the 3 symplectic structures) and freely. Then the MW quotient is again a hyperkähler manifold.*

The proof is similar to the case of Kähler manifolds. (The authors only gave a proof for the 0-level set, but again it's not hard to generalize it to the general case.)

Remark 4.4. The coadjoint orbits of the complexification G^c of a compact semisimple Lie group G are hyperkähler ([K]), but I don't know how one can get them by hyperkähler reduction.

5 Quaternionic Reduction

Definition 5.1. 4n-dimensional Manifold M is said to be almost quaternionic if there is a 3-dimensional subbundle $\mathfrak{G}$ of $\mathrm{Hom}(TM, TM)$ such that at each point it is isomorphic to $\mathrm{Im}\mathbb{H}$ (i.e. the image part of the quaternionic numbers).

So at each point x of M, we will have a basis J_1, J_2, J_3 in the fibre $\mathfrak{G}$ such that:

$$J_1J_2 = J_3, \qquad J_2J_3 = J_1, \qquad J_3J_1 = J_2, \qquad J_jJ_j = -1, \qquad j = 1,2,3. \quad (1)$$

Comparing with the hyperkähler case, at each point, we won't have a canonical choice of J_1, J_2, J_3. What we have is only a unit sphere of $\mathrm{Im}\mathbb{H}$.

A Riemannian metric g is called adapted to the quaternion structure of M if each $J \in \mathfrak{G}$ is orthogonal, i.e.,

$$< Jv, Jw > = < v, w > \quad (2)$$

for all unit $J \in \mathfrak{G}$ and all $v, w \in T_xM$. By a local trivialization of $\mathfrak{G}$, we can always choose some orthonormal basis,

$$v_1, J_1v_1, J_2v_1, J_3v_1, v_2, ..., J_3v_n, \qquad 4n = \dim M, \ J_j \text{ as in (1)} . \quad (3)$$

Thus we always have some "local metric" which is adapted. Then, by partition of unity, (globally) adapted metric always exists.

Given a metric, we can identify T^*M with TM; hence, $\mathfrak{G} \subset \mathrm{Hom}(TM, TM)$ = $T^*M \otimes TM$ =$T^*M \otimes T^*M$. In fact, this embedding can be given by

$$J \longmapsto \omega : \omega(v, w) =< Jv, w > . \tag{4}$$

If the metric is adapted in addition, we can easily discover that ω in fact is in $\wedge^2 T^*M$. So we can view $\mathfrak{G}$ as a subbundle of $\wedge^2 T^*M$. Let ω_j be the corresponding basis of J_j in (1), then by a direct calculation the associated exterior 4-form

$$\Omega = \sum_{j=1}^{3} \omega_j \wedge \omega_j$$

is invariant no matter how ω_j is chosen.

Definition 5.2. The Riemannian manifold M together with $\mathfrak{G}$ is quaternionic Kähler if $\nabla\Omega = 0$, where ∇ denotes the Levi-Civita connection.

This is equivalent saying that the holonomy bundle is contained in the $U(n, \mathbb{H})$ $U(1, \mathbb{H})$ frame bundle. ([S]) So this terminology can be confusing, because M may not be a Kähler manifold in the ordinary sense. Locally, it is then possible to make a smooth choice of J_1, J_2, J_3, but they cannot be assumed to be complex structures. However, what is nice for quaternionic Kähler manifold is that it must be Einstein, i.e. constant Ricci curvature. (See [S], Section 9.) This implies that the scalar curvature of M must be constant. When the scalar curvature of M is 0, one can show that $\mathfrak{G}$ is flat and the metric is hyperkähler ([GL]), hence reducing to the case of the last section. When the scalar curvature of M is not 0, we do not necessarily have a canonical choice of a closed 2-form to define the moment map as before. All we have now is a closed 4-form Ω ($d\Omega = 0$ because $\nabla\Omega = 0$ by a direct calculation. See [J] p. 134 for details). However, we can also define a $\mathfrak{G}$-valued 2-form:

$$\sum_{j=1}^{3} \omega_j \otimes \omega_j \qquad \text{for } \omega_j \text{ in (4).}$$

It is also well defined as in the case of Ω. For a Lie group H acting on M preserving everything it should, then corresponding to this $\mathfrak{G}$-valued 2-form we might hope that the moment map Φ goes from M to $\mathfrak{h}^* \otimes \mathfrak{G}$ hence a section of the bundle $\mathrm{Hom}(\mathfrak{h}, \mathfrak{G})$. And we might also hope that this Φ satisfies:

$$\nabla < \Phi(x), v >= \sum_{j=1}^{3} i(v)\omega_j \otimes \omega_j =: \Theta(v) \text{ (for the future use).} \tag{5}$$

In fact, K. Galicki and H. B. Lawson proved that we will always have a moment map provided that the scalar curvature is nonzero. The following result belongs to them. [GL] First consider the Lie group

$$\mathrm{Aut}(M) := \{a : a^*\Omega = \Omega \text{ and } a \text{ is an isometry } \}$$

and its Lie algebra

$$\mathfrak{aut}(M) := \{v : L_v\Omega = 0 \text{ and } v \text{ is a Killing vector field on } M \}.$$

Lemma 5.3. *Assume that the scalar curvature of M is not zero. Then to each* $v \in \mathfrak{aut}(M)$ *there corresponds a unique section* $f_v \in \Omega^0(\mathfrak{G})$ *such that*

$$\nabla f_v = \Theta_v. \tag{6}$$

In fact, since $\mathfrak{G}$ *is a 3-dimensional oriented vector bundle, we can identify SkewEnd* $(\mathfrak{G})$ *with* $\mathfrak{G}$ *via the cross product. From this point of view,* f_v *is given explicitly by the formula*

$$f_v = (1/\lambda)(L_v - \nabla_v), \tag{7}$$

where λ *is some nonzero constant.*

Definition 5.4. If H is a compact Lie subgroup of Aut(M), then a moment map Φ associated to it is a section of the bundle $\mathfrak{h}^* \otimes \mathfrak{G}$ such that,

$$< \Phi(x), v >= f_v(x), \forall x \in M, \forall v \in \mathfrak{h}.$$

Remark 5.5. By the definition itself and Lemma 5.3 we can see that the moment map uniquely exists and satisfies (5).

Remark 5.6. By the uniqueness in Lemma 5.3, Φ is H-equivariant. This means that for $a \in H$ we must have $\Phi(ax) = (a_*\Phi)(x)$, $\forall x \in M$. Viewing $\mathfrak{G}$ as the subbundle in $\wedge^2 T^*M$, $a_*\Phi$ is naturally defined by $Ad_a^* \otimes (a^{-1})^*$, i.e.

$$< (a_*\Phi)(x), v >= (a^{-1})^* < \Phi(ax), Ad_{a^{-1}} v >, \forall v \in \mathfrak{h}.$$

Then equivariance means that, $\forall v \in \mathfrak{h}$, $a \in H$,

$$\begin{aligned} f_v(a^{-1}x) =& < \Phi(a^{-1}x), v >=< (a_*^{-1}\Phi)(x), v > \\ &= a^* < \Phi(a^{-1}x), Ad_a v >= a^* f_{Ad_a v}(a^{-1}x). \end{aligned}$$

However

$$\nabla(a^* f_{Ad_a v}) = a^*\nabla(f_{Ad_a v}) = a^*\Theta_{Ad_a v} = \Theta Ad_a v = \nabla(f_v).$$

This means $a^* f_{Ad_a v} = f_v$, hence the equivariance holds.

Remark 5.7. The action of H on $\mathfrak{h} \otimes \mathfrak{G}$ is linear on each fibre, so it preserves the zero section

$$\mathfrak{L}_H = \{x \in M : \Phi(x) = 0\} = \{x \in M : f_v(x) = 0, \forall v\},$$

So H acts on it and it is the only natural level set of Φ that can have an H-action.

The generalized theorem in this case is as the following:

Theorem 5.8 (Quaternionic Reduction). *Let M be a quaternionic Kähler manifold with nonzero scalar curvature. Let $H \subset aut(M)$ be a compact subgroup with moment map Φ. Let $\mathfrak{L}_H^M$ be the H-invariant subset of the zero section $\mathfrak{L}_H$ where Φ intersects the zero section transversally and where H acts freely. Then $\mathfrak{L}_H^M/H$ equipped with the induced quotient metric is again a quaternionic Kähler manifold.*

Applying the theorem above to the case $H = S^1$, we can end up with some quaternionic Kähler orbifolds. This is described by the following Corollary:

Corollary 5.9. *Let M be as above and suppose $H \cong S^1 \subset Aut(X)$ is a closed 1-parameter subgroup generated by a vector field $v \in \mathfrak{aut}(M)$. If v_x is not 0 at all points $x \in \mathfrak{L}_H$, then $\mathfrak{L}_H/H$ is a compact quaternionic Kähler orbifold.*

Example 5.10 *(Quaternionic Kähler manifold $\mathbb{HP}^n$ and some quaternionic reduction of it).* Viewing a matrix in $U(n+1,\mathbb{H})$ as an orthogonal frame in $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$, $U(n+1,\mathbb{H})/U(n,\mathbb{H})$ tells us how to choose a unit quaternionic vector in $\mathbb{H}^{n+1}$. So $U(n+1,\mathbb{H})/U(n,\mathbb{H}) \times U(1,\mathbb{H})$ gives us the quaternionic projective space $\mathbb{HP}^n$ because a unit quaternionic vector moduloing the $U(1,\mathbb{H})$-action gives exactly a quaternionic line in $\mathbb{H}^{n+1}$. Similarly, $U(n+1,\mathbb{H})/U(n,\mathbb{H}) \times U(1,\mathbb{C})$ will give us $\mathbb{CP}^{2n+1}$ because a unit quaternionic vector can also be viewed as a unit complex vector on its underlying complex space. Hence we have the Hopf fibration:

$$\mathbb{CP}^{2n+1} = \frac{U(n+1,\mathbb{H})}{U(n,\mathbb{H})\times U(1,\mathbb{C})} \xrightarrow{\ \pi\ } \frac{U(n+1,\mathbb{H})}{U(n,\mathbb{H})\times U(1,\mathbb{H})} = \mathbb{HP}^n$$

which assigns to a complex line in $\mathbb{H}^{n+1}$ its quaternionic span. Each fibre is a $\mathbb{CP}^1$. But π is not holomorphic (in fact $\mathbb{HP}^n$ does not even admit a global almost complex structure). Each point $z \in \pi^{-1}(x)$ determines an almost complex structure I_z on the real tangent space $T_x\mathbb{HP}^n$ by pulling back the complex structure on $\mathbb{CP}^{2n+1}$. To be more specific, to apply I_z to a vector v, apply the complex structure in $T_x\mathbb{CP}^{2n+1}$ to any lift of v, and then project back to $\mathbb{HP}^n$. The family of almost complex structures determined in this manner may be identified with the 2-sphere of unit imaginary quaternions. This shows that $\mathbb{HP}^n$ is almost quaternionic. It is actually quaternionic Kähler. Let $(u_0, \ldots, u_n)$ be the linear coordinates on the quaternionic vector space $\mathbb{H}^{n+1}$, where scalar multiplication is defined from the right. Think them as the "homogeneous coordinates" for the quaternionic projective space $\mathbb{HP}^n = (\mathbb{H}^{n+1} - \{0\})/\mathbb{H}^*$.

For notational convenience, we write the homogeneous coordinates as (u_0, u) where $u = (u_1, \ldots, u_n)$. For each pair of integers $p, q \in \mathbb{Z}^+$ with $(p, q) = 1$ and $0 < q/p \leq 1$, we shall consider the action on $\mathbb{HP}^n$ defined in homogeneous coordinates by

$$\phi_t(u_0, u) = (e^{2\pi iqt}u_0, e^{2\pi ipt}u),$$

where $t \in [0, 1)$ if $(p+q)$ is odd and where $t \in [0, 1/2)$ if $(p+q)$ is even. This action is in Aut($\mathbb{HP}^n$), and gives a vector field:

$$V(u_0, u) = (iqu_0, ipu). \tag{8}$$

Consequently, we can consider the quaternionic moment map for this action. The moment map is a section on $\mathbb{R} \otimes \mathfrak{G}$ and $\mathfrak{G}$ at a point is made up by all the complex structures got by pulling back the complex structure on $\mathbb{CP}^{2n+1}$ along the fibre.

For a particular vector field V as in (8), $f_V(x) = 0$ means that in the direction of $V(x)$, all the complex structures pulled back don't change, which can only happen if $V(x) \perp x$. Hence,

$$\mathfrak{L}_{S^1} = \{(u_0, u) \in \mathbb{HP}^n : q\bar{u}_0 iu_0 + p\bar{u}iu = 0\}.$$

One can verify that the circle action is locally free. (See [GL] Theorem 4.4.) So the corollary tells us the quaternionic reduction gives a compact quaternionic orbifold $\mathfrak{O}_{q,p}(n-1)$

$$\mathfrak{O}_{q,p}(n-1) := \mathfrak{L}_{S^1}/S^1.$$

In the case where $p = q = 1$, we see that the set $\mathfrak{L}_{S^1}$ is invariant under left $U(n+1, \mathbb{C})$ action with respect to i. Explicit calculation about the stabilizer shows that

$$\mathfrak{O}_{q,p}(n-1) = \frac{U(n+1, \mathbb{C})}{U(n-1, \mathbb{C}) \times U(2)}.$$

with its symmetric quaternionic Kähler metric. [GL] In the case $q \neq p$, the reduction will end up with quaternionic Kähler orbifolds which are not locally symmetric. In fact, H. B. Lawson and K. Galicki discovered that for $q/p < 1$, when

$q/p \to 1$ the metric on $\mathfrak{O}_{q,p}(n-1)$ converges locally to the metric on $\mathfrak{O}_{1,1}$ and when $q/p \to 0$, these metrics converge locally to a hyperkähler metric on $T\mathbb{CP}^{n-1}$. [GL].

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