Poisson Reduction for Left Invariant Control System

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1 Introduction

Optimal control theory, recognized initially as an engineering problem, reveals a distinct relationship to differential geometry and mechanics [1, 2, 3]. Calculus of variations and Maximum Principle are the fundamental tools in solving the optimal control problems. There is a natural connection between mechanics and optimal control through the Maximum Principle, which yields from optimal control a Hamiltonian system. The solutions of optimal control problems rely on the integration of the Hamiltonian differential equations [4]. In this report, Poisson reduction of the optimal controls for left invariant control systems are investigated [5, 6]. This report closely follows the guideline of [5].

2 Background

In this section, brief description of the definitions and notations are given. Readers can find all the details in [7, 8, 9].

2.1 Left invariant vector fields

Definition 1 (Lie Group) *A* Lie group G *is a differentiable manifold which is also endowed with a group structure such that the map* $G \times G \to G$ *defined by* $(g, h) \mapsto gh^{-1}$ *is* C^∞.

Definition 2 (Lie Algebra) *A* Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ *is a real vector space* $\mathfrak{g}$ *together with a bilinear operator* $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that for all* $x, y, z \in \mathfrak{g}$,

$$[x, y] = -[y, x], \qquad \text{(anti-commutativity)}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \qquad \text{(Jacobi identity)}$$

Let G be a finite dimensional Lie group with identity e, and let X_e be a tangent vector to G at e. We may construct a vector field defined on all of G in the following way. For $g \in G$, define *left translation by* g to be a map $L_g : G \to G$ such that $L_g(h) = gh$, where $h \in G$. Since G is a Lie group, L_g is a diffeomorphism of G for each g. Taking the differential of L_g at e results in a map $T_eL_g : T_eG \mapsto T_gG$ such that $X_g = T_eL_g(X_e)$. The vector field formed by assigning $X_g \in T_gG$ for each $g \in G$ is called a *left invariant* vector field. It is easy to verify that all the left invariant vector fields of G form an algebra under Lie bracket operation on vector fields. We call it the *Lie algebra of the Lie group* G and denote it as $\mathfrak{g}$. It is actually a subalgebra of the Lie algebra of all the smooth vector fields on G. A left invariant control system is defined by letting X_e be a (controlled) curve in $\mathfrak{g}$. The system described in this report has a state which can be represented as an element $g \in G$. The differential equation which describes the time evolution of g can be written as:

$$\dot{g} = T_eL_gX_e(u) \tag{1}$$

where each control $u(\cdot)$ determines a curve $X_e(u(\cdot)) \subset \mathfrak{g}$. Here we limit ourselves to vector valued control functions $u(\cdot)$, and

$$X_e(u) = X_0 + \sum_{i=1}^{m} u_iX_i, \tag{2}$$

where u_i are controls and the X_i span an $(m+1)$-dimensional subalgebra of $\mathfrak{g}$, $m+1 \leq n = \dim(\mathfrak{g})$.

2.2 Poisson Manifolds

Definition 3 *A* Poisson bracket *(or a* Poisson structure*) on a manifold* P *is a bilinear operation* $\{\,,\,\}$ *on* $C^\infty(P)$ *such that* *(a)* $(C^\infty(P), \{\,,\,\})$ *is a Lie algebra; and*

(b) $\{\,,\,\}$ *is a derivation in each factor, that is,*

$$\{FG, H\} = \{F, H\}G + F\{G, H\}$$

for all F, G, *and* $H \in C^\infty(P)$.

A Poisson manifold is denoted by $(P, \{\,,\,\})$ or simply by P if there is no danger of confusion.

Remark 1 Any symplectic manifold (P, Ω) is a Poisson manifold. The Poisson bracket of two functions $F, G \in C^\infty(P)$ is defined by the symplectic form as

$$\{F, G\}(z) = \Omega(z)(X_F(z), X_G(z)),$$

where X_F is the *Hamiltonian vector field* on P satisfying $i_{X_F}\Omega = dF$.

Remark 2 If $\mathfrak{g}$ is a Lie algebra, then its dual $\mathfrak{g}^*$ is a Poisson manifold with Poisson structures given by

$$\{F, G\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle \tag{3}$$

for $\mu \in \mathfrak{g}^*$ and F, $G \in C^\infty(\mathfrak{g}^*)$. The *functional derivative* of F at μ in (3) is the unique element $\delta F/\delta\mu$ of $\mathfrak{g}$ defined by

$$DF(\mu) \cdot \delta\mu = \left\langle \delta\mu, \frac{\delta F}{\delta \mu} \right\rangle, \tag{4}$$

where $\langle , \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Following [10, 11], this bracket on $\mathfrak{g}^*$ is called the *Lie-Poisson bracket.*

Proposition 1 *The map* $H \mapsto X_H$ *is a Lie algebra antihomomorphism, i.e.,*

$$[X_H, X_K] = -X_{\{H,K\}}. \tag{5}$$

Proposition 2 *Let* $H \in C^\infty(P)$ *and* ϕ_t *be the flow of* X_H. *Then*

(a) $\dfrac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H\}$, *or for short* $\dot{F} = \{F, H\}$.

(b) $H \circ \phi_t = H$ *(*conservation of energy*).*

A function $C \in C^\infty(P)$ is called a *Casimir function* if $\{C, F\} = 0$ for all $F \in C^\infty(P)$, that is, C is constant along the flow of all Hamiltonian vector fields or, equivalently, $X_C = 0$, that is, C generates trivial dynamics.

2.3 Quotients of Poisson manifolds

Definition 4 *Let P be a manifold and let G be a Lie group. A (*left*) *action* of a Lie group G on P is a smooth mapping $\Phi : G \times P \to P$ such that:*

(a) $\Phi(e, x) = x$ *for all* $x \in P$*; and*

(b) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ *for all* g, $h \in G$ *and* $x \in P$.

For every $g \in G$ let $\Phi_g : P \to P$ be given by $x \mapsto \Phi(g, x)$. An action is said to be:

1. *free* if it has no fixed points, that is, $\Phi_g(x) = x$ implies $g = e$ or, equivalently, if for each $x \in P$, $g \mapsto \Phi_g(x)$ is one-to-one.
2. *proper* if the mapping $\tilde{\Phi} : G \times P \to P \times P$, defined by $\tilde{\Phi}(g, x) = (x, \Phi(g, x))$, is proper.

If Φ is an action of G on P and $x \in P$, the *orbit* of x is defined by

$$\mathrm{Orb}(x) = \{\Phi_g(x) | g \in G\} \subset P. \tag{6}$$

An action of Φ of G on a manifold P defines an equivalence relation on P by the relation of belonging to the same orbit; explicitly, for x, $y \in P$, we write $x \sim y$ if there exists a $g \in G$ such that $g \cdot x = y$, that is if $y \in \mathrm{Orb}(x)$. We let P/G be the set of these equivalence classes, that is, the set of orbits, also called the *orbit space*. Let $\pi : P \to P/G : x \mapsto \mathrm{Orb(x)}$, and give P/G the quotient topology by defining $U \subset M/G$ to be open if and only if $\pi^{-1}(U)$ is open in P.

Proposition 3 (Proposition 4.2.23, [12]) *If $\Phi : G \times P \to P$ is a proper and free action, then P/G is a smooth manifold and $\pi : P \to P/G$ is a smooth submersion.*

Now we are ready to give the simplest version of a general construction of Poisson manifolds based on symmetry. This construction represents the first steps in a general procedure called *reduction*.

Suppose that G is a Lie group that acts on a Poisson manifold and that each map $\Phi_g : P \to P$ is a Poisson map. Let us also suppose that the action is free and proper, so that the quotient space P/G is a smooth manifold and the projection $\pi : P \to P/G$ is a submersion.

Proposition 4 (Theorem 10.7.1, [9]) *Under these hypotheses, there is a unique Poisson structure on P/G such that π is a Poisson map.*

The unique Poisson structure $\{\, , \}_{P/G}$ on the quotient manifold P/G is induced from the one on P satisfying

$$\{f, k\}_{P/G} \circ \pi = \{f \circ \pi, k \circ \pi\}, \tag{7}$$

where $f,\, k : P/G \to \mathbb{R}$.

Now G-invariant dynamics on P induces dynamics on P/G. To see this, let $H : P \to \mathbb{R}$ be a G-invariant Hamiltonian function on P, i.e.,

$$H(\Phi_g(x)) = H(x), \quad \forall\, g \in G. \tag{8}$$

It defines a corresponding function h on P/G such that for any equivalence class $[x] \in P/G$, $h([x]) = H(x)$, that is, $h \circ \pi = H$. Since π is a Poisson map, it transforms the Hamiltonian vector field X_H on P to X_h on P/G; that is, $T\pi \circ X_H = X_h \circ \pi$, and

$$X_h f = \{f, h\}_{P/G}, \quad \forall\, f \in C^\infty(P/G). \tag{9}$$

It follows that the Hamiltonian vector field X_h leaves invariant the symplectic leaves of P/G. Thus any Casimir function on P/G is an integral of motion for X_h. The integral curves of X_H project under π to integral curves of X_h.

A key example of the Poisson reduction is when $P = T^*G$ and G acts on itself by left translations. Then $P/G \cong \mathfrak{g}^*$ and the reduced Poisson bracket is none other than the Lie-Poisson bracket.

3 Maximum principle

For control systems, optimal control problems have been typically cast in a different setting from the classical variational problems. The basic difference lies in the way in which the trajectories are formulated; in the optimal control setting the trajectories are parametrized by the controlled vector field, while in the traditional variational setting trajectories are simply constrained. The other basic difference is that the necessary conditions for extremals in the optimal control setting are typically expressed using a Hamiltonian formulation using the Pontryagin maximum principle, rather than the Lagrangian settings. Readers can find the detail of the topic in various classic optimal control books, e.g., [13, 4].

The following problem and discussion are taken from [5] directly. Consider an optimal control problem of the form

$$\min_{u(\cdot)} \int_0^T L(u)\, dt \tag{10}$$

subject to the condition that $u(\cdot)$ steers (1)-(2) from g_0 at $t = 0$ to g_1 at $t = T$. Clearly, the Lagrangian L is G-invariant.

To state the necessary conditions dictated by the Pontryagin Maximum Principle, we introduce a parametrized Hamiltonian function on T^*G

$$H(g, p, u) = -p_0 L(u) + \langle p, T_e L_g X_e(u) \rangle, \tag{11}$$

where $p_0 \geq 0$ and $p \in T^*G$. We denote by $t \mapsto u^*(t)$ a curve which satisfies the following relationship along a trajectory $t \mapsto (g(t), p(t))$ in T^*G:

$$H^*((g(t), p(t)) = H(g(t), p(t), u^*(t)) = \max_{u(\cdot)} H(g(t), p(t), u(t))). \tag{12}$$

The Hamiltonian function H^* defines a Hamiltonian vector field X_{H^*} on T^*G, with respect to the canonical symplectic structure on T^*G. The Pontryagin Maximum Principle gives necessary conditions for extremals as follows: an extremal trajectory $t \mapsto g(t)$ of problem (10) is the projection onto G of a trajectory of the flow of the vector field X_{H^*}, which satisfies the boundary condition $g(0) = g_0$ and $g(T) = g_1$, and for which $t \mapsto (p(t), p_0)$ is not identically zero on $[0, T]$. The extremal is called *normal* when $p_0 \neq 0$ (in which case we set $p_0 = 1$). When $p_0 = 0$ we call the extremal *abnormal*, corresponding to the case where the extremal is determined by constraints alone. The abnormal case occurs often but are ruled out under suitable hypotheses. We are concerned solely with normal extremals in this report.

Since there is no constraint on u_i, optimal controls for (10) subject to (1) and (2) satisfy

$$\frac{\partial H}{\partial u_i} = -\frac{\partial L}{\partial u_i} + \frac{\partial}{\partial u_i} \langle p, T_e L_g X_e(u) \rangle = 0, \tag{13}$$

for all $i = 1, \ldots, m$. From (2),

$$\begin{aligned}\langle p, T_eL_gX_e(u)\rangle &= \langle p, T_eL_g(X_0 + \sum_{i=1}^{m} u_iX_i)\rangle \\ &= \langle T_eL_g^*p, X_0 + \sum_{i=1}^{m} u_iX_i\rangle \\ &= \langle \mu, X_0\rangle + u_i\sum_{i=1}^{m}\langle \mu, X_i\rangle,\end{aligned} \tag{14}$$

where $\mu = T_eL_g^*p \in \mathfrak{g}^*$. From (13) and (14)

$$-\frac{\partial L}{\partial u_i} + \langle \mu, X_i\rangle = 0. \tag{15}$$

for all $i = 1, \ldots, m$. Observe that, from (13) and (14) the Hamiltonian H^* is G-invariant. More explicitly, suppose $L(u) = \frac{1}{2}\sum_{i=1}^{m}\lambda_iu_i^2$, the constants $\lambda_i > 0$. Then the optimal u_i are given by

$$u_i = \frac{\langle \mu, X_i\rangle}{\lambda_i} \tag{16}$$

and the Hamiltonian on T^*G is

$$H^* = \langle \mu, X_0\rangle + \frac{1}{2}\sum_{i=1}^{m}\frac{\langle \mu, X_i\rangle^2}{\lambda_i}. \tag{17}$$

Clearly the Hamiltonian H^* is G-invariant. The Hamiltonian vector field X_{H^*} on T^*G corresponding to the Hamiltonian H^* can be reduced to a Hamiltonian vector field X_h on $\mathfrak{g}^*$. The latter is Hamiltonian in a non-canonical (Lie-Poisson) sense. Thus questions about explicit solvability of X_{H^*} are turned into corresponding questions about X_h.

4 Lie-Poisson reduction

The detail of the discussion in this section can be found in [9].

A function $F_L : T^*G \to \mathbb{R}$ is called *left invariant* if, for all $g \in G$,

$$F_L \circ T^*L_g = F_L, \tag{18}$$

where T^*L_g denotes the cotangent lift of L_g, so T^*L_g is the pointwise adjoint of TL_g. Given $F : \mathfrak{g}^* \to \mathbb{R}$ and $\alpha_g \in T^*G$, set

$$F_L(\alpha_g) = F(T_e^*L_g \cdot \alpha_g) \tag{19}$$

which is the *left invariant extension* of F from $\mathfrak{g}^*$ to T^*G. One similarly defines the *right invariant extension* by

$$F_R(\alpha_g) = F(T_e^* R_g \cdot \alpha_g). \tag{20}$$

The main content of the Lie-Poisson reduction theorem is the pair of formulae

$$\{F, H\}_- = \{F_L, H_L\}|\,\mathfrak{g}^* \tag{21}$$

and

$$\{F, H\}_+ = \{F_R, H_R\}|\,\mathfrak{g}^*, \tag{22}$$

where $\{,\}_\pm$ is the Lie-Poisson bracket on $\mathfrak{g}^*$ and $\{,\}$ is the canonical bracket on T^*G. For simplicity, we will only discuss the left invariant case hereafter.

Theorem 1 (Lie-Poisson reduction theorem) *Identifying the set of functions on $\mathfrak{g}^*$ with the set of left invariant functions on T^*G endows $\mathfrak{g}^*$ with Poisson structures give by*

$$\{F, H\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right] \right\rangle. \tag{23}$$

The space $\mathfrak{g}^*$ with this Poisson structure is denoted $\mathfrak{g}^*_-$.

Another version of Lie-Poisson reduction theorem is as the following.

Theorem 2 (Lie-Poisson reduction of dynamics) *Let G be a Lie group and $H : T^*G \to \mathbb{R}$. Assume H is left invariant. Then the function $H^- := H|\,\mathfrak{g}^*$ on $\mathfrak{g}^*$ satisfies*

$$H(\alpha_g) = H^-(\lambda(\alpha_g)) \quad \textit{for all} \quad \alpha_g \in T_g^* G, \tag{24}$$

where $\lambda : T^*G \to \mathfrak{g}^*_-$ is given by $\lambda(\alpha_g) = T^* L_g \cdot \alpha_g$. The flow F_t of H on T^*G and the flow F_t^- of H^- on $\mathfrak{g}^*_-$ are related by

$$\lambda(F_t(\alpha_g)) = F_t^-(\lambda(\alpha_g)). \tag{25}$$

In other words, a left invariant Hamiltonian on T^*G induces Lie-Poisson dynamics on $\mathfrak{g}^*_-$. The result is a direct consequence of the Lie-Poisson reduction theorem and the fact that a Poisson map relates Hamiltonian systems and their integral curves to Hamiltonian systems.

Let $\{X_1, \ldots, X_n\}$ be a basis for the Lie algebra $\mathfrak{g}$, $\{X^1, \ldots, X^n\}$ be the corresponding dual basis for $\mathfrak{g}^*$, thus $\langle X_i, X^j \rangle = \delta_{ij}$. The *structure constants* C_{ab}^d are defined by

$$[X_a, X_b] = \sum_{d=1}^{n} C_{ab}^d X_d \tag{26}$$

where a, b run from 1 to n. Any $\mu \in \mathfrak{g}_-^*$ can be expressed as $\mu = \sum_{i=1}^n \mu_i X^i$, and the $(\pm)$ Lie-Poisson brackets become

$$\begin{aligned}
\{F, G\}_{\pm}(\mu) &= \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right] \right\rangle \\
&= \pm \left\langle \sum_{i=1}^{n} \mu_i X^i, \left[\sum_{a=1}^{n} \frac{\partial F}{\partial \mu_a} X_b, \sum_{b=1}^{n} \frac{\partial G}{\partial u_b} X_b\right] \right\rangle \\
&= \pm \left\langle \sum_{i=1}^{n} \mu_i X^i, \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{d=1}^{n} C_{ab}^d X_d \frac{\partial F}{\partial \mu_a} \frac{\partial G}{\partial u_b} \right\rangle \\
&= \pm \sum_{a,b,d=1}^{n} C_{ab}^d \mu_d \frac{\partial F}{\partial \mu_a} \frac{\partial G}{\partial \mu_b}
\end{aligned} \tag{27}$$

Theorem 3 *Let G be a Lie group and let $H : T^*G \to \mathbb{R}$ be a left invariant Hamiltonian. Let $h : \mathfrak{g}^* \to \mathbb{R}$ be the restriction of H to T_e^*G. For a curve $p(t) \in T_{g(t)}^*G$, let $\mu(t) = (T_e^* L_{g(t)}) \cdot p(t) = \lambda(p(t))$ be the induced curve in $\mathfrak{g}^*$. Assuming that $g(t)$ satisfies the differential equation*

$$\dot{g} = T_e L_g \frac{\delta h}{\delta \mu}, \tag{28}$$

where $\mu = p(0)$, the following are equivalent:

(a) $p(t)$ is an integral curve of X_H; i.e., Hamilton's equations on T^*G hold;

(b) for any $F \in C^\infty(T^*G)$, $\dot{F} = \{F, H\}$, where $\{,\}$ is the canonical bracket on T^*G;

(c) for any $f \in C^\infty(\mathfrak{g}^*)$, we have

$$\dot{f} = \{f, h\}_-, \tag{29}$$

where $\{,\}_-$ is the minus Lie-Poisson bracket;

(d) $\mu(t)$ satisfies the Lie-Poisson equations

$$\frac{d\mu}{dt} = ad_{\delta h/\delta \mu}^* \mu \tag{30}$$

where $ad_\xi : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad_\xi \eta = [\xi, \eta]$ and ad_ξ^* is its dual, i.e.,

$$\dot{\mu}_i = \{\mu_i, h\}_- = - \sum_{b,d=1}^{n} C_{ib}^d \mu_d \frac{\partial h}{\partial \mu_b}. \tag{31}$$

By the Proposition 1, it is immediate that h is constant along trajectories of (31). Additionally, the Casimir functions on $\mathfrak{g}^*$ are also constant along trajectories of (31). For $\mathfrak{g} = SO(3)$ the Lie algebra of skew symmetric matrices, any Casimir function is of the form $\Phi(\mu_1^2+\mu_2^2+\mu_3^2)$. In general, there may not be any nontrivial (non-constant) Casimir functions.

Returning to the optimal control problem of this paper, since the Hamiltonian H in (17) is already expressed as function on $\mathfrak{g}^*$, we note that the reduced Hamiltonian is

$$h = \langle \mu, X_0 \rangle + \frac{1}{2}\sum_{i=1}^{m} \frac{\langle \mu, X_i \rangle^2}{\lambda_i}. \tag{32}$$

We have in effect shown the following reduction of the Maximum Principle.

Theorem 4 *Consider the optimal control problem of the form*

$$\min_{u(\cdot)} \int_0^T L(u)\, dt \tag{33}$$

subject to

$$\dot{g} = T_e L_g (X_0 + \sum_{i=1}^{m} u_i X_i), \tag{34}$$

$g(0) = g_0$ *and* $g(T) = g_1$. *Then every regular extremal is given by*

$$u_i = \frac{\langle \mu, X_i \rangle}{\lambda_i}, \tag{35}$$

where μ *is an integral curve of the vector field* X_h *on* $\mathfrak{g}^*_-$ *corresponding to the Hamiltonian*

$$h = \langle \mu, X_0 \rangle + \frac{1}{2}\sum_{i=1}^{m} \frac{\langle \mu, X_i \rangle^2}{\lambda_i} \tag{36}$$

and the Poisson bracket $\{,\}_-$ *on* $\mathfrak{g}^*$ *is given by (3). In coordinates on* $\mathfrak{g}^*$ *the integral curves satisfy the ordinary differential equations (31).*

5 Example

The following example and solution are taken from [5] directly. We apply the reduction procedure to the steering of a unicycle as shown in Figure 1. If u_1 denotes the steering velocity and u_2 the driving velocity, the functional form of the state equations for this system is

$$\begin{aligned} \dot{x} &= \cos\phi\, u_2 \\ \dot{y} &= \sin\phi\, u_2 \\ \dot{\phi} &= u_1 \end{aligned} \tag{37}$$

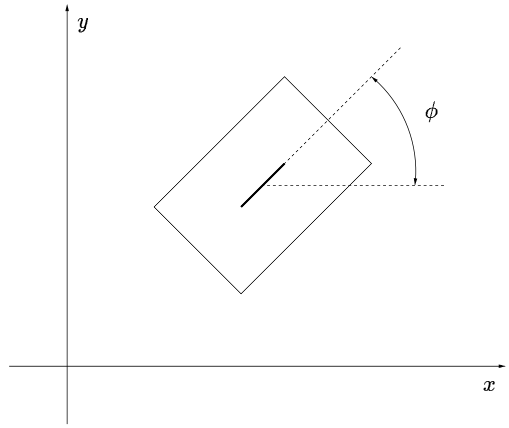

Figure 1: Steerable unicycle. The unicycle has two independent inputs: the *steering* input controls the angle of the wheel, ϕ; the *driving* input controls the velocity of the cart in the direction of the wheel. The configuration of the cart is its Cartesian location and the wheel angle.

Set

$$g = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}, \tag{38}$$

thus $g \in SE(2)$, the rigid motion group of the plane. Then the unicycle equation takes the form

$$\dot{g} = g \cdot \begin{bmatrix} 0 & -u_1 & u_2 \\ u_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{39}$$

This is a left invariant system in $SE(2)$. Further, the Lie algebra of $SE(2)$ is $se(2)$, which is defined by

$$se(2) = \left\{ \xi = \begin{bmatrix} \omega & v \\ 0 & 0 \end{bmatrix}, \omega^T + \omega = 0, v \in \mathbb{R}^3 \right\}, \tag{40}$$

We have that $se(2)$ is spanned by $\{X_1, X_2, X_3\}$, where

$$X_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{41}$$

Since $[X_1, X_2] = X_3$, $[X_1, X_3] = -X_2$, and $[X_2, X_3] = 0$, the structure constants are

$$\begin{aligned} C_{12}^1 &= 0, \quad C_{12}^2 = 0, \quad C_{12}^3 = 1; \\ C_{13}^1 &= 0, \quad C_{13}^2 = -1, \quad C_{13}^3 = 0; \\ C_{23}^1 &= 0, \quad C_{23}^2 = 0, \quad C_{23}^3 = 0. \end{aligned} \tag{42}$$

Now consider the optimal control problem

$$\min_{u(\cdot)} \int_0^T \frac{1}{2}\left(u_1^2 + u_2^2\right)\, dt \tag{43}$$

subject to the boundary conditions. The cost function aims for minimizing the control energy. By the Theorem 3, regular extremals are given by integral curves of a reduced Hamiltonian h on $se(2)^*_-$. The Hamiltonian is in fact, in the coordinates corresponding to the dual basis $\{X^1, X^2, X^3\}$, given by

$$h = \frac{\langle \mu, X_1\rangle^2}{2} + \frac{\langle \mu, X_2\rangle^2}{2} = \frac{\mu_1^2 + \mu_2^2}{2}. \tag{44}$$

The Poisson bracket of two functions ϕ and ψ can be calculated through (27):

$$\begin{aligned} \{\phi, \psi\}_-(\mu) &= -\sum_{a,b,d=1}^{3} C_{ab}^d \mu_d \frac{\partial \phi}{\partial \mu_a}\frac{\partial \psi}{\partial \mu_b} \\ &= \nabla\phi^T\, \Gamma(\mu)\nabla\psi, \end{aligned} \tag{45}$$

where

$$[\Gamma(\mu)]_{ab} = -\sum_{d=1}^{3} C_{ab}^d \mu_d. \tag{46}$$

Therefore, we have

$$\Gamma(\mu) = \begin{bmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & 0 \\ -\mu_2 & 0 & 0 \end{bmatrix}. \tag{47}$$

From (31), the reduced Hamilton's equations are

$$\begin{aligned} \dot\mu_1 &= -\mu_2\mu_3 \\ \dot\mu_2 &= \mu_1\mu_3 \\ \dot\mu_3 &= -\mu_1\mu_2. \end{aligned} \tag{48}$$

The Casimir functions are of the form $\Phi = \Phi(\mu_2^2 + \mu_3^2)$, (equivalently $\nabla\Phi$ is in the kernel of $\Gamma(\mu)$. The level sets of Casimir functions (i.e., symplectic leaves in $\mathfrak{g}^*_-$ are cylinders $\{\mu : \mu_2^2 + \mu_3^2 = c\}$. Integral curves of (48) are intersections of level sets of h, also cylinders $\{\mu : \mu_1^2 + \mu_2^2 = 2h\}$, with the symplectic leaves. Note that equation (48) can be solved by

$$\ddot\mu_2 = -(2h + c)\mu_2 + 2\mu_2^3. \tag{49}$$

This is the equation of an harmonic oscillator with quartic potential term. The general solution to (49) is given by

$$\mu_2(t) = \beta\, \mathrm{Sn}(\lambda(t - t_0), k), \tag{50}$$

where $\mathrm{Sn}(u,k)$ is Jacobi's elliptic sine function, $\lambda^2 < 2h + c < 2\lambda^2$, t_0 is arbitrary, $k^2 = \frac{2h+c}{\lambda^2} - 1$, $\beta^2 = 2h + c - \lambda^2$. Then μ_1 and μ_3 are determined from $\mu_1 = \sqrt{2h - \mu_2^2}$ and $\mu_3 = \sqrt{c - \mu_2^2}$. The optimal controls are given by $u_1 = \mu_1$ and $u_2 = \mu_2$.

We show that the Lie-Poisson reduction simplifies the derivation of the optimal control to left invariant control systems. This simplification allows us to efficiently compute the control inputs.

6 Conclusion

We have worked out explicitly the Poisson reduction of certain G-invariant optimal control problems on Lie groups. The approach presented here yields an algorithm for constructing regular extremals.

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