A Convexity Theorem For Isoparametric Submanifolds

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1 Introduction.

The main objective of this paper is to discuss a convexity theorem for a certain class of Riemannian manifolds, so-called isoparametric submanifolds, and how this relates to other convexity theorems.
In the introduction we will present the convexity theorems. In Section 2 we will describe the geometry of isoparametric submanifolds and in Section 3 we will relate this to the geometries of the other convexity theorems. Finally in Section 4 we will give the proof of the convexity theorem for isoparametric submanifolds.

We first state a convexity theorem by Kostant ([K], 1973) for symmetric spaces, and then show how this relates to the convexity theorem for a Hamiltonian torus action (Atiyah [A] and Guillemin-Sternberg [GS], 1982) on one side and to a convexity theorem for isoparametric submanifolds (Terng, [T1], 1986) on the other.

Theorem 1 (Kostant)[T1 p.487] *Let G/K be a symmetric space, $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition of the the Lie algebra, $\mathcal{T}$ a maximal abelian subspace of $\mathcal{P}$, W the associated Weyl group of G/K acting of $\mathcal{T}$, and $u : \mathcal{P} \to \mathcal{T}$ the orthogonal projection. Let M be an orbit of the isotropy representation of G/K through $z \in \mathcal{P}$. Then $u(M) = cvx(W \cdot z)$, the convex hull of $W \cdot z$.*
Remark: the isotropy representation of G/K is the action of K on $T_{[K]}(G/K)$

induced by the action of G on G/K given by $(g, hK) \to ghK$.

Now let G be a compact connected Lie group with a biinvariant metric, σ an involution of G, and K the fixedpoint set of σ. Then G/K is a symmetric space with the following metric: the derivative at the identity of $\pi : G \to G/K$ identifies $\mathcal{K}^\perp$ and $T_{[K]}G/K$. The restriction of Ad to K acts on $\mathcal{K}^\perp$, and is equivalent to the isotropy representation. Since this is an orthogonal representation of K on $T_{[K]}G/K$, we can extend the inner product on $T_{[K]}G/K$ to a Riemannian metric on G/K, which makes G/K into a symmetric space.
$G \times G$ is also a connected compact Lie group with a biinvariant metric, and choosing the involution σ on $G \times G$ to be $\sigma(g_1, g_2) = (g_2, g_1)$ we see that $(G \times G)/\Delta G$ is a symmetric space, and it is canonically diffeomorphic to G. The isotropy representation of ΔG on $(G \times G)/\Delta G$ is just $Ad : G \times \mathcal{G} \to \mathcal{G}$, which we can identify with the coadjoint action of G on $\mathcal{G}^*$ (by identifying $\mathcal{G}$ and $\mathcal{G}^*$).
So, if T is a maximal torus of G and M is a coadjoint orbit of G, Kostant's theorem tells us that the image of the orthogonal projection of M on $\mathcal{T}^*$ is convex [T2 p.9].
Coadjoint orbits, endowed with the Kostant-Kirillov two form, are symplectic manifolds. For any $\xi \in \mathcal{G}$ the height function $x \mapsto \langle x, \xi \rangle$ on M is Hamiltonian for the vectorfield ξ_M, and the moment map for the T-action on M is the orthogonal projection of M on $\mathcal{T}^*$, so - as seen above - it has a convex polyhedron as image [T2 p.10].
The situation of coadjoint orbits of compact Lie groups is generalized by the following theorem:

Theorem 2 (Atiyah and Guillemin-Sternberg)[T1 p.487] *Let N be a compact connected symplectic manifold with a symplectic action of a torus T, and let $J : N \to \mathcal{T}^*$ be the moment map. Then $J(N)$ is a convex polyhedron.*

Kostant's theorem has been generalized in other directions too. The principal orbits of the isotropy representations of symmetric spaces are always isoparametric submanifolds , i.e. submanifolds (in this case of Euclidean space) with flat normal bundle and constant principal curvatures along parallel normal fields. (The non-principal orbits are not isoparametric in general, see [PT] p.170 and 6.5.6.). If M is an isoparametric submanifold we can canonically associate to M a Weyl group, acting on $p + \nu_p M$ for every

$p \in M$.
The next theorem states that the Riemannian geometric condition of being isoparametric is enough to ensure a convexity theorem. Not all isoparametric submanifolds arise as orbits of isotropy representations of symmetric spaces. There are infinitely many families of counterexamples in codimension 2, but it is a theorem that all isoparametric submanifolds of codimension greater or equal than 3 which are irreducible (i.e. can not be written as a product of two nontrivial isoparametric submanifolds) are principal orbits of such representations [O]. So this theorem gives a stronger statement than Kostant's theorem [T1 p.488].

Theorem 3 (Terng)[PT p.167] *Let M be a full, compact, isoparametric submanifold of $\mathbb{R}^{n+k}$ and $p \in M$. The Weyl group W of M acts on $\nu_p := p + \nu_p M$. Let P be the orthogonal projection of $\mathbb{R}^{n+k}$ onto ν_p and u its restriction to M. Then $u(M) = cxv(W \cdot p)$, the convex hull of $W \cdot p$.*
Remark: an isoparametric submanifold of $\mathbb{R}^{n+k}$ is full if it is not contained in any affine hyperplane, and if it is compact it is necessarily contained in some $(n+k-1)$-sphere of $\mathbb{R}^{n+k}$, so one can assume that it is contained in some sphere centered at the origin [PT 6.3.11].

2 The geometry of isoparametric submanifolds.

The purpose of this section is to define isoparametric submanifolds and present some of their geometric properties, expecially those that will be needed in the proof of theorem 3.

Definition [T1 p.488] A submanifold M of $\mathbb{R}^{n+k}$ is called *isoparametric* if
1) the normal bundle of M is flat
2) the principal curvatures along parallel normal fields are constant.

From property 1) it follows that for the normal curvature tensor we have $R^{\perp} = 0$, so by the Ricci equation any two shape operators at any point x commute. Since the shape operators at x are symmetric and they commute, they are simultaneusly diagonalizable, and we obtain a decomposition $T_xM = \bigoplus\{E_i(x)|i \in I\}$ (where I is some finite index set) into maximal common eigenspaces of the shape operators at x. The shape operator A_v at x depends linearly on $v \in \nu_x M$, so for each $E_i(x)$ there is a vector $n_i(x) \in \nu_x M$ such that $A_v|_{E_i(x)} = \langle n_i(x), v\rangle Id_{E_i(x)}$. Property 2) implies that we have a decomposition $TM = \bigoplus\{E_i|i \in I\}$ into smooth subbundles of TM, and that the n_i's (which are called *curvature normals*) are parallel.

Now, for every $i \in I$, we define the *focal hyperplanes* $l_i(x) = \{x + v|v \in \nu_x M, \langle v, n_i(x)\rangle = 1\}$. Reflections about any focal hyperplane permute the other focal hyperplanes, so the the reflections generate a finite group acting on $\nu_x := x + \nu_x M$. The focal hyperplanes meet in exactly one point [PT 5.3.7], and the closure of a connected component of $\nu_x \backslash \cup\{l_i(x)\}$ is called *Weyl chamber*. Since the curvature normals are parallel, the focal hyperplanes at different points are obtained from one another by parallel translation, and we obtain a well-defined group acting on ν_x for every $x \in M$, called the *Weyl group* associated to M [T1 p.489, PT p.113 and 119].

Now we state some properties of isoparametric submanifolds that we will need in the proof of Theorem 3. We assume that M is an n-dimensional full isoparametric submanifold of $\mathbb{R}^{n+k}$.

Fact 1 [T1 p.490] Let v be a parallel normal field on M such that $y = p + v(p)$ lies in an i-face σ of a Weyl chamber of ν_p. Then $M_v := \{q+v(q)|q \in M\}$ is a submanifold of $\mathbb{R}^{n+k}$, the natural map $\pi_v : M \to M_v$ is a submersion, and $\{M_v|p+v(p) \in \Delta_p\}$ is a singular foliation of $\mathbb{R}^{n+k}$, where Δ_p is the Weyl chamber of ν_p containing p.
If y is W-regular, i.e. if y does not lie on any focal hyperplane of ν_p, then π_v is a diffeomorphism and M_v is isoparametric. In general we have:
i) If w is another parallel normal field such that $z = p + w(p) \in \sigma$, then $\pi_v^{-1}(y) = \pi_w^{-1}(z)$, which we will denote by $S_{p,\sigma}$.
ii) $S_{p,\sigma}$ is a full isoparametric submanifold of codimension $k-i$ in a suitable subspace $V(p,\sigma)$ of $\mathbb{R}^{n+k}$. Furthermore $p+\tilde{\nu}_p S_{p,\sigma} \subset \nu_p$, where $\tilde{\nu}_p S_{p,\sigma}$ denotes the normal space of $S_{p,\sigma}$ in $V(p,\sigma)$ at p.
iii) $S_{p,\sigma}$ is connected [Z, lemma 1.4]
iv) If $\sigma \subset \gamma$ are two faces in ν_p then $S_{p,\gamma} \subset S_{p,\sigma}$.

Fact 2 [T1 p.490] Let $a \in \mathbb{R}^{n+k}$ be nonzero, $f : M \to \mathbb{R}$ the height

function given by $f(x) = \langle a, x\rangle$, and $C(f)$ the set of critical points of f. Then
i) $x \in C(f) \Leftrightarrow a \in \nu_x$
ii) f has a unique local maximum (minimum) value.
Furthermore, f has a maximum at $x \Leftrightarrow a \in \Delta_x$,
and if $a \in \sigma \subset \Delta_x$ then $f^{-1}(f(x)) = S_{x,\sigma}$. [PT 8.6.5, 8.3.6 and remarks on p. 167]
iii) if $a \in \nu_x$ is W-regular, then $C(f) = W \cdot x$
iv) if $a \in \sigma \subset \nu_x$ is W-singular, then

$$C(f) = \cup_{y\in W\cdot x}\{S_{y,\gamma}| \cap \{l_i(x)|\gamma \subset l_i(x)\} = \cap\{l_i(x)|\sigma \subset l_i(x)\}\}.$$

3 Momentum maps and isoparametric submanifolds.

Now we will try to determine similarities between the geometric situations of theorems 2 and 3. We will use the same notation as in the statements of the theorems in the introduction.
The basic idea for the proof of Theorem 3 goes back to the proof of Theorem 2 by Atiyah and Guillemin-Sternberg, despite the fact that there is no torus acting on the isoparametric submanifold M.

We first consider the case of a coadjoint orbit $G\cdot x$ of a compact connected Lie group G with a maximal torus T. As seen in Section 1, the moment map of the T-action is the orthogonal projection on $\mathcal{T}^*$, and $G \cdot x$ meets $\mathcal{T}^*$ orthogonally. Indeed, if $G \cdot x$ is a principal orbit (hence an isoparametric submanifold of $\mathcal{G}^*$) then $\nu_x := x + \nu_x(G \cdot x)$ is just $\mathcal{T}^*$ [T1 p.499]. So the components of the moment map are the height functions $\langle\cdot, a\rangle : G \cdot x \to \mathbb{R}$, where $a \in \nu_x$ is nonzero. We will see that, in the general settings of Theorems 2 and 3, the components $J(\cdot)v$ (where $v \in \mathcal{T}$) of the moment map J - which are Hamiltonian functions for the torus action - have geometric properties similar to the components of the orthogonal projection u.
If G/K is a symmetric space, $K \cdot x$ a principal orbit of the isotropy representation and $\mathcal{T}$ the maximal abelian subalgebra of $\mathcal{P}$ through x, then - generalizing the above remark for coadjoint orbits - $\mathcal{T} = x + \nu_x(G \cdot x)$ [T1 p.490]. This suggests that, while looking for a convexity theorem for isoparametric submanifolds, one should consider the orthogonal projection on the

normal space to some point of the manifold.

Now we will consider the maxima, the fibers and the critical points of the functions $J(\cdot)v$ and$\langle \cdot, a\rangle$. (In the following we will assume that $v \in \mathcal{T}$ and $a \in \nu_p$ are both nonzero).
Each function $J(\cdot)v$ has a unique local maximum [GS Theorem 5]. The same holds for the funcion $\langle \cdot, a\rangle$ by Fact 2ii), and this fact is used in Step v) of the proof of Theorem 3.

The fibers of $J(\cdot)v$ are always connected (or empty) [A, p.5]. Similarly, the fiber of $\langle \cdot, a\rangle$ over x is connected if we make the assumption that the function $\langle \cdot, a\rangle$ has a maximum at x (Fact 2ii) and Fact 1iii)).

The set of critical points of the moment map J (resp. of the orthogonal projection u) is the union of the set of critical points of its components $J(\cdot)v$ (resp of $\langle \cdot, a\rangle$). The critical point set $C(J(\cdot)v)$ of $J(\cdot)v$ consists of the fixed points of the closure S of $exp(\mathbb{R}v) \subset T$, as one can compute directly from the definition of moment map. Each component of $C(J(\cdot)v)$ is a symplectic manifold and it is preserved by the torus action, as can be shown using the fact that any two points of a torus orbit have the same stabilizer. The torus action on a component of $C(J(\cdot)v)$ is not effective, since $S \subset T$ acts trivially, so there is an induced action by a quotient torus of dimension less than the one of T. By induction over the codimension of S in T one can prove that the image of $C(J(\cdot)v)$ under the moment map J is a union of convex polyhedra [GS Theorem 3.8].
The critical points of $\langle \cdot, a\rangle$ are a union of the connected manifolds $S_{y,\sigma}$ (where $y \in W \cdot p$ and σ is a face of ν_p, see Fact 2iv) and Fact 1iii)). Since a is nonzero, each $S_{y,\sigma}$ is an isoparametric submanifold of codimension less than $k = codim(M)$ by Fact 1ii). Applying induction on the codimension, Step i) in the proof of theorem 3 shows that $u(S_{y,\sigma})$ is also a union of convex polyhedra.
As we pointed out in this section, if M is a principal orbit of a coadjoint action of a compact Lie group G with maximal torus T, the dimension of T is the codimension of the full isoparametric submanifold M.

4 The proof of Theorem 3.

In this section we prove a lemma and give the proof of Theorem 3 as in [T1 p.490]. We assume the notation of theorem 3.

Lemma *Let $C(u)$ be the set of critical points of u; then*

$$C(u) = \cup_{q \in W \cdot p}\{S_{q,\sigma} | \sigma \textit{ is a 1-face of some Weyl chamber of } \nu_p\}.$$

Proof We may assume that $\nu_p = \mathbb{R}^k$. Let $t_1, \cdots, t_k$ be the standard basis for $\mathbb{R}^k$, so that u has components $u_i(x) = \langle t_i, x\rangle$. Then
$x \in M$ is a critical point of $u \Leftrightarrow$
There is a nonzero vector $a \in \mathbb{R}^k$ such that $a_1 du_1(x) + \cdots + a_k du_k(x) = 0 \Leftrightarrow$
x is a critical point of some height function $\langle \cdot, a\rangle$ where $a \in \mathbb{R}^k$ is nonzero.
The last equivalence uses Fact 2i).
Let $a \in \nu_p$ be nonzero and σ the face in ν_p containing a. The critical points of $\langle a, \cdot\rangle$ are $\cup_{y \in W \cdot x}\{S_{y,\gamma} | \cap \{l_i(x) | \gamma \subset l_i(x)\} = \cap\{l_i(x) | \sigma \subset l_i(x)\}\}$ by Fact 2iv), so $C(u)$ is contained in

$$\cup_{q \in W \cdot p}\{S_{q,\sigma} | \sigma \text{ is an i-face of } \nu_p, i > 0\} = \cup_{q \in W \cdot p}\{S_{q,\sigma} | \sigma \text{ is a 1-face of } \nu_p\}.$$

For the inclusion we used that the only 0-face is $\{0\}$, and for the equality we used Fact 1iv).
Conversely, if $y \in \cup_{q \in W \cdot p}\{S_{q,\sigma} | \sigma \text{ is a 1-face of } \nu_p\}$ then by Fact 2iv) y is a critical point for $\langle \cdot, a\rangle$ where a is some vector in σ, so $y \in C(u)$.

Proof of theorem 3
We proceed by induction over the codimension k of M.
In the case $k = 1$ M must be $r \cdot S^n$ for some positive r, so $u(M)$ is the line segment from p to $-p$. For the outward pointing normal unit vector ξ we have $A_\xi(p) = -\frac{1}{r} Id_{T_pM}$ (where A denotes the shape operator), so the only curvature normal is $\frac{\xi(p)}{r}$, so the only focal hyperplane is $p - r\xi(p) = \{0\}$, so $W \cdot p = \{p, -p\}$.

Now we assume that the theorem holds for codimensions less than k. Let D denote $cvx(W \cdot p)$ and C be the set of critical points of u. We divide the proof in five steps.

i) $u(C)$ is a finite union of $(k-1)$-polyhedra and $\partial D\subset u(C)\subset D$. Since $u(C)$ is compact, it follows that $D\setminus u(C)$ is open in ν_p.
For any 1-face σ of ν_p and $q\in W\cdot p$, $S_{q,\sigma}$ is an isoparametric submanifold of codimension $k-1$ by Fact 1ii), so by induction hypothesis and by Fact 1ii) $u(S_{q,\sigma}\ )$ is a $k-1$-polyhedron in ν_p. The first assertion above follows because the lemma tells us that

$$u(C)=\cup_{q\in W\cdot p}\{u(S_{q,\sigma})|\sigma \text{ is a 1-face of some Weyl chamber of } \nu_p\}.$$

To show that $\partial D\subset u(C)$, because of the lemma it is enough to show that

$$\partial D\subset\cup_{q\in W\cdot p}\{u(S_{q,\sigma})|\sigma \text{ is a 1-face of the Weyl chamber containing } q\}.$$

∂D is a union of $k-1$-polyhedra. Let μ be one of them and a an outward pointing vector in ν_p normal to μ. Then $\langle a,\cdot\rangle$ has an absolute maximum exactly on $u^{-1}(\mu)$, which according to Fact 2ii) is $S_{p,\gamma}$ where γ is the smallest face of the Weyl chamber of p containing a. Now the assertion follows from Fact 1iv).

ii) $\partial(u(M))\subset u(C)$.
Let $x\in\partial(u(M))\subset u(M), x\notin u(C)$. Then $x=u(q)$ for some regular point $q\in M$ of u. So, since u is a submersion at q, $x=u(q)$ lies in the interior of M, a contradiction.

iii) $u(M)\subset D$.
By i) $u(C)\subset D$, and by ii) $\partial(u(M))\subset u(C)$, so $\partial(u(M))\subset D$, and the assertion follows because $u(M)$ is compact and D convex.

iv) Let O be a connected component of $D\setminus u(C)$. Then $O\subset u(M)$ or $O\cap u(M)=\emptyset$.
$D\setminus u(C)$ is open in ν_p by i), so O is open, so $O\subset u(M)\Leftrightarrow O\subset u(M)^\circ$ and $O\cap u(M)=\emptyset\Leftrightarrow O\cap u(M)^\circ=\emptyset$ (here $u(M)^\circ$ denotes the open hull of $u(M)$ in ν_p). Suppose that both statements above are not true. Then $O\cap u(M)^\circ$ is a nonempty proper subset of O. Since $O\setminus u(M)^\circ$ is nonempty, there is a sequence (y_n) in $O\cap u(M)^\circ$ converging to a point $y\in O\setminus u(M)^\circ$, and $y\in u(M)$ because $u(M)$ is compact. But $y\notin u(C)$ because otherwise $y\in O\cap u(C)=\emptyset$, and by ii) $\partial u(M)\subset u(C)$, so we have $y\in(u(M)\setminus\partial u(M))=u(M)^\circ$, a contradiction.

v) $D \setminus u(C) \subset u(M)$.

Suppose not. Then by iv) for a connected component O of $D \setminus u(C)$ we have $O \cap u(M) = \emptyset$. By i) the boundary of O is a union of $(k-1)$-polyhedra. Let μ be one of these and t the outward pointing unit vector normal to μ. The function $\langle t, \cdot \rangle$ on M has a local minimum on μ becuase t is outward pointing and O is open. By Fact 2ii) it has a global minimum there, say c. So $u(M)$ is contained in $\{y \in \nu_p | \langle y, t \rangle \geq c\}$, and since $\partial D \subset u(M)$ by i) the same holds for ∂D and therefore for D. Since t is outward pointing there is a face of $O \subset D$ contained in $\{y \in \nu_p | \langle y, t \rangle \leq c\}$, so $M \subset \{y \in \nu_p | \langle y, t \rangle = c\}$, which is contradition to the fact that M is full in $\mathbb{R}^{n+k}$.

Final remarks We notice that, in the setting of Theorem 2, the fibers of the moment map J are connected [A]. This is not true in general for the orthogonal projection $u : M \to \nu_p$ of an isoparametric submanifold M, an easy example being $S^1 \subset \mathbb{R}^2$.

Given a Hamiltonian action of a compact Lie group on a symplectic manifold, one can apply symplectic reduction to obtain new interesting symplectic manifolds. This is an important tool in symplectic geometry. There does not seem to be a way to apply a similar construction to an isoparametric submanifold M, since - even though the projection $u : M \to \nu_p$ plays the role of the moment map - there is no Lie group acting on M in general.

Given the setting of Theorem 2 and $v \in \mathcal{T}$, knowing $J(\cdot)v$ one can reconstruct the (symplectic) action of $\{exp(tv) | t \in \mathbb{R}\} \subset T$ on N by integrating the vectorfield v_N given by $v_N(p) = \frac{d}{dt}|_{t=0}((exp(tv) \cdot p)$. In the setting of Theorem 3, in the attempt to construct some kind of action on M, one could try to consider the flow of the Riemannian gradient of the height function $\langle \cdot, a \rangle$ (where $a \in \nu_p$), but - beside not giving a generalisation of the case of coadjoint orbits - this flow does not usually preserve the Riemannian structure of M, as can be seen considering $M = S^2$.

In conclusion, the analogy between isoparametric submanifolds and symplectic manifolds seems to be mostly limited to the convexity theorem.

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