Kepler Problem and SO(4) Momentum Map

Xiang Tang

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Abstract

A complete solution of the Kepler problem is given; the SO(4) action on (TS^3, Ω_4) is discussed; and the incomplete Kepler hamiltonian vector field is regularized by the LS map, embedding it into $(T^+S^3, \tilde{\omega}_4)$.

[2] did excellent work on these topics. I read through them and write this report.

O Introduction

Kepler problem is a famous and classical problem in both mathematics and physics. Since Kepler, there are various solutions. In Section 1, we are going to solve it using symplectic geometry.

The hamiltonian vector field in the solution to the Kepler problem is not complete, and we will regularize it by embedding it into a complete flow. The LS map provides a nice way to regularize the negative energy Keplerian orbits all at once. This is discussed in section 3.

In the regularization, the LS map sends negative energy set $(\Sigma_- = \{(q,p) \in T\mathbb{R}^3 | H(q,p) < 0\}, \omega_3|\Sigma_-)$ to $(T^+S^3, \tilde{\omega}_4)$, which is SO(4) symmetric. To see the connections between these two vector fields, we devote section 2 to the discussion of some properties of the SO(4) momentum map.

1 The Kepler Problem

Kepler Problem: Consider two particles in $\mathbb{R}^3$. One is fixed at the origin, and the other one moves under the influence of gravitational field of the fixed particle. The problem, named after Kepler who first gave the solution, is to describe the motion of the second particle.

In this section, we will give a complete solution using symplectic geometry. This is done in three steps:

1.1 We define the Kepler Hamiltonian system $(H, T\mathbb{R}^3_0, \omega_3)$ and discuss some properties of the Kepler Hamiltonian vector field X_H.

1.2 On $(\Sigma_-, \tilde{\omega}_3 = \omega_3|\Sigma_-)$ (Σ_- is the open subset of $(T\mathbb{R}^3_0, \omega_3)$ where the energy H is negative), we define a representation of $so(4)$ in the space of Hamiltonian vector fields which has a momentum map $\mathcal{G}$. And by studying $\mathcal{G}$, we get the characterization of the orbits of X_H.

1.3 Kepler's equation.

There is a similar discussion in [1]$II.3$. The idea of the proof of our theorems just comes from it.

1.1 Kepler Hamiltonian system $(H, T\mathbb{R}_0^3, \omega_3)$

On the phase space $T\mathbb{R}_0^3 = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ with coordinates (q,p) and $\omega_3 = \sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i$ the symplectic form, consider the Kepler Hamiltonian

$$H : T\mathbb{R}_0^3 \to \mathbb{R} : (q,p) \longmapsto \frac{1}{2} < p, p > - \frac{\mu}{|q|}.$$

(Here $<,>$ is the Euclidean inner product on $\mathbb{R}^3$ and $|q|$ is the length of vector q. As we consider the case where the force is attractive, μ is supposed to be > 0.)

This is what we call Kepler Hamiltonian system. The integral curves of the Hamiltonian vector field X_H on $T\mathbb{R}_0^3$ satisfy the equations

$$\begin{cases} \dot{q} & = p \\ \dot{p} & = -\mu \frac{q}{|q|^3}. \end{cases}$$

In physics, we know that energy h, angular momentum J and Runge-Lenz vector e, so called eccentricity vector, are conserved quantities in a two-body system. Thus we get some integrals of the Kepler vector field X_H.

Define:

$$\begin{aligned} h &\overset{def}{=} \tfrac{1}{2} < p, p > - \tfrac{\mu}{|q|}; \\ J &\overset{def}{=} (J_1, J_2, J_3) = q \times p; \\ e &\overset{def}{=} (e_1, e_2, e_3) = -\tfrac{q}{|q|} + \tfrac{1}{\mu} p \times (q \times p). \end{aligned}$$

Proposition 1. *h,J,e are integrals of X_H.*

Proof:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}} h &= < p, \tfrac{\mathrm{d}}{\mathrm{dt}} p > + \tfrac{\mu}{|q|^3} < q, \tfrac{\mathrm{d}}{\mathrm{dt}} q > \\ &= - < p, \tfrac{\mu}{|q|^3} q > + \tfrac{\mu}{|q|^3} < q, p > = 0 \\ \tfrac{\mathrm{d}}{\mathrm{dt}} J &= \tfrac{\mathrm{d}}{\mathrm{dt}} q \times p + q \times \tfrac{\mathrm{d}}{\mathrm{dt}} p \\ &= p \times p - \tfrac{\mu}{|q|^3} q \times q = 0 \\ \tfrac{\mathrm{d}}{\mathrm{dt}} e &= - \tfrac{\mathrm{d}}{\mathrm{dt}} \tfrac{q}{|q|} + \tfrac{1}{\mu} \tfrac{\mathrm{d}}{\mathrm{dt}} p \times J \\ &= \tfrac{1}{|q|^3} < \tfrac{\mathrm{d}}{\mathrm{dt}} q, q > q - \tfrac{1}{|q|} \tfrac{\mathrm{d}}{\mathrm{dt}} q + \tfrac{1}{\mu} \tfrac{\mathrm{d}}{\mathrm{dt}} p \times J \\ &= \tfrac{1}{|q|^3} (< p, q > q - < q, q > p - q \times J) \\ &= \tfrac{1}{|q|^3} (q \times (q \times p) - q \times (q \times p)) = 0. \quad \square \end{aligned}$$

Proposition 2. *If the energy h is negative, then the image of each integral curve of the Kepler vector field under the bundle projection $\tau : T\mathbb{R}_0^3 \to \mathbb{R}^3 : (q,p) \longmapsto q$ is bounded.*

Proof: We consider different situations according to the value of J.

Case 1. J=0. Since e is an integral of X_H and J=0, the direction $e = -\frac{q}{|q|}$ of the motion is constant. Therefore the motion takes place on the line $q(t) = r(t)e$. From the conservation of energy, $h + \frac{\mu}{r} = \frac{1}{2} < p, p > \geq 0$. So $|q| = |r| \leq -\frac{\mu}{h}$.

Case 2. J$\neq$0. Since $J^2 = |q \times p|^2 = |q|^2|p|^2 - < q, p >^2$,

$$\begin{aligned} h = \tfrac{1}{2} < p, p > - \tfrac{\mu}{|q|} &= \tfrac{1}{2}\tfrac{(J^2 + <q,p>^2)}{|q|^2} - \tfrac{\mu}{|q|} \geq \tfrac{1}{2}\tfrac{J^2}{|q|^2} - \tfrac{\mu}{|q|} \\ &= \tfrac{1}{2}J^2(\tfrac{1}{|q|^2} - \tfrac{2\mu}{J^2}\tfrac{1}{|q|} + \tfrac{\mu^2}{2J^2}) - \tfrac{\mu}{2J^2} \\ &= \tfrac{1}{2}J^2(\tfrac{1}{|q|} - \tfrac{\mu}{J^2})^2 - \tfrac{\mu^2}{2J^2} \end{aligned}$$

When $h < 0$,

$$\frac{1}{2}J^2(\frac{1}{|q|} - \frac{\mu}{J^2})^2 \leq \frac{\mu^2}{2J^2} + h,$$

$$\frac{1}{(\frac{\mu}{J^2} + \sqrt{\frac{\mu^2}{J^4} + \frac{2h}{J^2}})} \leq |q| \leq \frac{1}{(\frac{\mu}{J^2} - \sqrt{\frac{\mu}{J^2} + \frac{2h}{J^2}})} \qquad \square$$

Proposition 3. *The flow X_H is not complete.*

Proof: Consider a bounded motion with J=0 and $h < 0$ which starts at $(r(0), \dot{r}(0)) = (\frac{\mu}{-h}, 0)$. The time it takes to reach the origin is

$$T = \int_0^{\frac{\mu}{h}} \frac{\mathrm{dr}}{\sqrt{\frac{2\mu}{r} + 2h}} \qquad (\frac{1}{2}\dot{r}^2 = h + \frac{\mu}{r}) = \frac{\pi}{2}\frac{\mu}{(-2h)^{\frac{3}{2}}},$$

which is finite. $\square$

1.2 The $so(4)$ Momentum Map

Definition: Σ_- is the open subset of $(T\mathbb{R}_0^3, \omega_3)$ where the energy h is negative.

On Σ_-, we will show that the components of angular momentum J and modified eccentricity vector $\tilde{e} = -ve(where\ v = \frac{\mu}{\sqrt{-2h}})$ form a Lie algebra under Poisson bracket which is isomorphic to $so(4)$. This defines a representation of $so(4)$ in the space of Hamiltonian vector fields on $(\Sigma_-, \tilde{\omega}_3 = \omega_3|\Sigma_-)$ which has a momentum map $\mathcal{G}$. In fact $\mathcal{G}$ is a surjective submersion from Σ_- to C=$\{(J, \tilde{d}) \in \mathbb{R}^6 | < J + \tilde{e}, J + \tilde{d} > = < J - \tilde{e}, J - \tilde{e} >> 0\}$ each of whose fibers is a unique bounded orbit of X_H. From this, we get the orbits of the particle in the Kepler's problem.

Theorem 1. *On $(\Sigma_-, \tilde{\omega_3})$ the components of J and $\tilde{e}$ satisfy the Poisson bracket relations*

$$\{J_i, J_j\} = \sum_k \varepsilon_{ijk}J_k, \ \{J_i, \tilde{e}_j\} = \sum_k \varepsilon_{ijk}\tilde{e}_k \quad and \ \{\tilde{e}_i, \tilde{e}_j\} = \sum_k \varepsilon_{ijk}J_k$$

Thus the relations define a Lie algebra which is isomorphic to so(4).

This is (46) in [1]($P56$).

Proof: Notice that $\{q_l, p_m\} = \delta_{lm},\ \{q_i, q_j\} = 0\ \ and\ \ \{p_i, p_j\} = 0$. The relations in the Theorem 1 are pure calculation from there, so we omit them.

For i=1,2,3, we define $\xi_i \overset{def}{=} \frac{1}{2}(J_i + \tilde{e}_i)$ and $\eta_i \overset{def}{=} \frac{1}{2}(J_i - \tilde{e}_i)$. In terms of $\xi_i\ and\ \eta_i$, the bracket relations become

$$\{\xi_i, \xi_j\} = \sum_k \varepsilon_{ijk}\xi_k, \quad \{\eta_i, \eta_j\} = \sum_k \varepsilon_{ijk}\eta_k, \quad and\ \ \{\xi_i, \eta_j\} = 0.$$

These relations define the Lie algebra $so(3)\times so(3)$ which is just $so(4)$. □

Thus the maps $J_i \mapsto ad_{J_i} = -X_{J_i}$ and $\tilde{e}_i \mapsto ad_{\tilde{e}_i} = -X_{\tilde{e}_i}$ define a representation of $so(4)$ in the space of Hamiltonian vector fields on $(\Sigma_-, \tilde{\omega}_3)$. In other words, we have a Hamiltonian action of Lie algebra $so(4)$ on $(\Sigma_-, \tilde{\omega}_3)$. Associated to this action is the map

$$\mathcal{G} : \Sigma_- \subseteq T\mathbb{R}^3_0 \to \mathbb{R}^6 : (q,p) \mapsto (J, \tilde{e}) = (q \times p, v(\frac{q}{|q|} - \frac{1}{\mu} p \times (q \times p))). \quad (v = \mu/\sqrt{-2H})$$

If we choose $\{\varepsilon_i\}_{1\leq i \leq 6} = \{J_i, \tilde{e}_i\}$ as the basis for $so(4)$ with Lie bracket $\{,\}$, one let that $\mathcal{G}^{\varepsilon_i}$ be the i^{th} component of map $\mathcal{G}$, then the bracket relations may be written as $\{\mathcal{G}^{\varepsilon_i}, \mathcal{G}^{\varepsilon_j}\} = \mathcal{G}^{\{\varepsilon_i, \varepsilon_j\}}$. Therefore, the map $\mathcal{G}$ is the equivariant momentum map for the $so(4)$ action on $(\Sigma_-, \tilde{\omega}_3)$.

Next we investigate the geometric properties of $\mathcal{G}$, so that we can find a characterization of the orbits of the Kepler hamiltonian vector field.

By calculation, we get

$$\begin{cases} < J, \tilde{e} > & = 0 \\ < J, J > & + < \tilde{e}, \tilde{e} >= v^2 > 0. \end{cases}$$

These relations define a smooth 4-dimensional manifold C_v which is diffeomorphic to $S^2_v \times S^2_v$ since the relations are equivalent to

$$\begin{cases} < J + \tilde{e}, J + \tilde{e} > & = v^2 > 0 \\ < J - \tilde{e}, J - \tilde{e} > & = v^2 > 0. \end{cases}$$

If we write $v = \frac{\mu}{\sqrt{-2h}}$ for some $h < 0$ and consider the map

$$\mathcal{G}_h = \mathcal{G}|_{H^{-1}(h)} : H^{-1}(h) \subseteq \Sigma_- \to C_v \subseteq \mathbb{R}^6,$$

we have the following theorem.

Theorem 2. *$\mathcal{G}$ is a surjective submersion whose fiber $\mathcal{G}_h^{-1}(c)$ is*

1. *an oriented ellipse, when $c \notin C_v \cap \{J = 0\}$;*
2. *a line which is the union of two half open line segments $\{(\frac{\sigma}{\mu}\tilde{e}, \pm\frac{1}{v}(\sqrt{2h + \frac{2\mu}{\sigma}})\tilde{e}) \in T_0\mathbb{R}^3 | \sigma \in (0, \frac{\mu}{-h}]\}$ that join smoothly at $(\frac{\mu}{-vh}\tilde{e}, 0)$, when $c \in C_v \cap \{J = 0\}$.*

This is the claim in $P58$ of [2]

Theorem 3. *$\mathcal{G}:\Sigma_- \subseteq T\mathbb{R}^3_0 \to C \subset \mathbb{R}^6$ is a surjective submersion each of whose fibers are unique bounded orbits of Kepler vector field X_H, where C is the submanifold of $\mathbb{R}^3 \times (\mathbb{R}^3 - (0,0))$ defined by $< J, \tilde{e} >= 0$.*

This is the claim in $P61$ of [2].

Before proving these two theorems, we will integrate the two results. In fact, the orbit of the Kepler vector field X_H is just the orbit of the motion of the particle in the Kepler's problem. Now Theorem 3 tells us the manifold C is the space of the Kepler vector field $X_H's$ orbits. And when we fix $H = h$, C_v is just the orbits space corresponding to these Kepler vector field $X_H(H = h)$. From theorem 2, we know that the orbit in C_v is either an oriented ellipse or a line. So combining the two theorems, we know that the orbit of the particle is either an oriented ellipse or a union of two line segment provided that the total energy of the system is negative and fixed. In the nature, because the angular momentum $J \neq 0$, the line situation doesn't happen. And usually the eccentricities of the planets orbits are very small, the orbits look very close to a circle. Hence, originally Kepler formulated his first law as follows: the planets move around the sun in circles, but the sun is not at the center. The similar conclusion can be found in [1]

Now we come to the proof of these two theorems. I rewrite the proof in [2] $P58-62$. For the proof of theorem 2, we divide it into three steps.

1) We are going to prove that $\mathcal{G}$ is a surjective submersion.
Let $(q,p) \in H^{-1}(h)$,and $V(q,p) = span\{X_{J_i}(q,p), X_{\tilde{e}_i}(q,p)\}_{i=1,2,3}$. Since J and $\tilde{e}$ are integrals of X_H, it follows that $V(q,p) \subseteq ker(\mathrm{dH(q,p)})$, which is $T_{(q,p)}H^{-1}(h)$. Therefore

$$P \stackrel{def}{=} \mathrm{d}\mathcal{G}_{\mathrm{(q,p)}}|\mathrm{V_{(q,p)}} = \begin{pmatrix} \mathrm{dJ_i(q,p)} \\ \mathrm{d\tilde{e}_i(q,p)} \end{pmatrix} \mathrm{V_{(q,p)}} = \begin{pmatrix} \{J_i, J_j\}_{(q,p)}, & \{J_i, \tilde{e}_j\}_{(q,p)} \\ \{\tilde{e}_i, J_j\}_{(q,p)}, & \{\tilde{e}_i, \tilde{e}_j\}_{(q,p)} \end{pmatrix}$$

Because P is conjugate to the matrix

$$\begin{pmatrix} \{\xi_i, \xi_j\}(q,p) & 0 \\ 0 & \{\eta_i, \eta_j\}(q,p) \end{pmatrix} = \begin{pmatrix} \sum_k \varepsilon_{ijk}(J_k + \tilde{e}_d)(q,p) & 0 \\ 0 & \sum_k \varepsilon_{ijk}(J_k - \tilde{e}_k)(q,p) \end{pmatrix}$$

, on C_v, rank(P)=4. (On C_v, $\|J_K + \tilde{e}_k\|^2 = \|J_k - \tilde{e}_k\|^2 = v^2 > 0$). Therefore, $\mathcal{G}_h$ is a submersion.

Now we will show that $\mathcal{G}_h$ is surjective. Let $(J, \tilde{e}) \in C_v$. Then $e_0 = |e| = \frac{1}{v}|\tilde{e}| \in [0,1]$. ($v^2 =< J, J >$ $+ < \tilde{e}, \tilde{e} > \geq < \tilde{e}, \tilde{e} >$). Choose

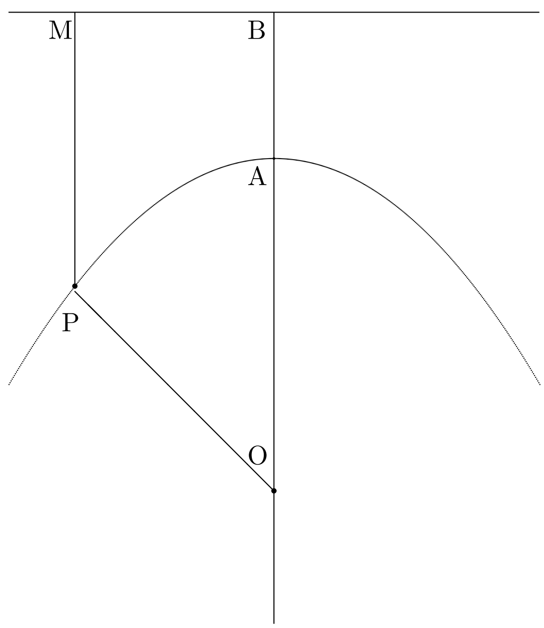
$$(q,p) = \begin{cases} (-\frac{v}{\mu}\frac{(1-e_0)}{e_0}\tilde{e}, -\frac{\mu}{v^3}\frac{1}{e_0(1-e_0)}J \times \tilde{e}) & when\ e_0 \in (0,1)\ and\ J \neq 0; \\ (-\frac{\mu^2}{v}p \times J, p) & when\ e_0 = 0\ and\ J \neq 0.\ \ Here\ < p, J >= 0.\ |p| = \frac{\mu}{v}; \\ (-\frac{v}{\mu}\tilde{e}, \frac{v}{\mu^2}\tilde{e}) & when\ \ J = 0. \end{cases}$$

A straightforward calculation shows that $(q,p) \in H^{-1}(h)$ and $\mathcal{G}_h(q,p) = (J, \tilde{e})$.

2) Consider the situation that c$\notin C_v \cap \{J = 0\}$. We prove that when c=$(J, \tilde{e}) \in C - (C_v \cap \{J = 0\})$, the fiber $\mathcal{G}_h^{-1}(c)$ lies over an oriented ellipse . Let $h = -\frac{\mu^2}{2v^2}$. We will show $h < 0, J \neq 0$ and $e = -\frac{1}{v}\tilde{e}$

determine a unique oriented ellipse which is traced out by projection $t \mapsto q(t)$ of an integral curve $t \mapsto (q(t), p(t))$ of X_H. Because $J \neq 0$ and $< q(t), J >=< p(t), J >= 0$, the curves $t \mapsto q(t)$ and $t \mapsto p(t)$ lies in a plane $\pi \subseteq \mathbb{R}^3$ which is perpendicular to J. Since $< J, e >$=0, the eccentricity vector e also lies in π. Therefore we may write $< q, e >= |q| e_0 \cos f$(f is the true anomaly)=$-|q| + \frac{1}{\mu} J_0^2 (= |J|^2)$(the definition of e). Therefore $|q| e_0 \cos f = -|q| + \frac{1}{\mu} J_0^2$.

Suppose that $e_0 = 0$. Then $|q| = \frac{1}{\mu} J_0^2$ which defines a circle $\mathcal{S}$ in π whose center is the origin. Since $< q, p >= 0$, tangent vector $p(t)$ to $\mathcal{S}$ at $q(t)$ is perpendicular to $q(t)$. Because $\{q, p, q \times p\}$ is an oriented basis of $\mathbb{R}^3$, $\{q, p\}$ is a positively oriented basis for π. Hence the circle traced out by $t \mapsto q(t)$ is positively oriented.



Ellipse in the plane π. The ratio of the distance OP to the distance PM to the line MB is constant e.

Suppose that $e_0 \neq 0$. Then equation $|q| e_0 \cos f = -|q| + \frac{1}{\mu} J_0^2$ may be written as $e_0 (\frac{J_0^2}{\mu e_0} - |q| \cos f) = |q|$. Calculate

$$\begin{aligned} e_0^2 &= |e|^2 = 1 - \frac{2}{\mu |q|} |q \times p|^2 + \frac{1}{\mu^2} |p \times (q \times p)|^2 \\ &= 1 - \frac{2}{\mu |q|} J_0^2 + \frac{1}{\mu^2} (|p|^2 J_0^2 - < p, J >^2) \quad (J = q \times p) \\ &= 1 + \frac{2}{\mu^2} J_0^2 h \quad ((p, J) = 0. \;\; h = \frac{1}{2} < p, p > - \frac{\mu}{|q|}) \\ &< 1 \;(h < 0) \end{aligned}$$

So $|q| = \frac{J_0^2}{\mu}\frac{1}{1+e_0\cos f}$ is an ellipse in π with eccentricity e and major semi axis lying along e of length $a = \frac{J_0^2}{\mu}\frac{1}{1-e^2} = \frac{\mu}{-2h}$.

From the fact q, p is a positively oriented basis of the plane π, we obtain

$$
\begin{aligned}
J = |q \times p| &= \textit{area of the positively oriented parallegram spanned by } q, p. \\
&= det \begin{pmatrix} < q, e_0^{-1} e > & < q, (J_0 e_0)^{-1}(J \times e) > \\ < p, e_0^{-1} e > & < p, (J_0 e_0)^{-1}(J \times e) > \end{pmatrix}, \\
& \{e_0^{-1} e,\ (J_0 e_0)^{-1} J \times e\} \textit{ is a positively oriented orthonormal basis of } \pi. \\
&= |q|^2 \frac{\mathrm{d}f}{\mathrm{dt}}
\end{aligned}
$$

Thus $\frac{\mathrm{df}}{\mathrm{dt}} > 0$, which means this ellipse is oriented in the direction of increasing true anomaly f.

3) Consider the situation that c=$(J, \tilde{e}) \in C_v \cap \{J = 0\}$, we will prove $\mathcal{G}_h^{-1}(c)$ is a union of two half segments.. Since $J = 0$, the modified eccentricity vector $\tilde{e} = v\frac{q}{|q|}$. Since $\tilde{e}$ is constant along integral curves $t \mapsto (q(t), p(t))$ of X_H and $h < 0$, the image of $t \mapsto q(t)$ lies along $\tilde{e}$ and is half open line segment $\{\frac{\sigma}{v}\tilde{e} \in \pi | \sigma \in (0, \frac{\mu}{-h}]\}$. From $J = 0$, it follows that $p = \lambda\tilde{e}$ for some $\lambda \in \mathbb{R}$. In order that $(\frac{\sigma}{v}\tilde{e}, p) \in H^{-1}(h)$, where $h = -\frac{\mu^2}{2v^2}$, we must have $\lambda^2 =< p, p >= 2h + \frac{2\mu}{\sigma}$. Therefore $\mathcal{G}_h^{-1}(c)$ is the union of the two half open line segments

$$\{(\frac{\sigma}{v}\tilde{e}, \pm\frac{1}{v}(\sqrt{2h + \frac{2\mu}{\sigma}})\tilde{e}) \in T\mathbb{R}_0^3 | \sigma \in (0, \frac{\mu}{-h}]\}$$

which join smoothly at $(\frac{\mu}{-hv}\tilde{e}, 0)$.

Combine the three parts, we complete the proof of theorem 2 $\square$

Now we come to the proof of theorem 3:

$\forall c = (J, \tilde{e}) \in C_v \subset C,\ (J, \tilde{e}) \in C_v;\ (v^2 = \|J + \tilde{e}\|^2 = \|J - \tilde{e}\|^2),\ \textit{let } h = -\frac{\mu^2}{2v^2}$. From the result of theorem 2, $\mathcal{G}_h^{-1}(c)$ is nonempty. Hence $\mathcal{G}^{-1}(c)$ is nonempty. So $\mathcal{G}$ is surjective. Because $\mathcal{G}_h^{-1}(c)$ is a unique oriented bounded orbit of Kepler vector field, $\mathcal{G}^{-1}(c)$ is as well.

Now we prove $\mathcal{G}$ is submersion. Since C is 5-dimensional (C=$\{(J, \tilde{e})| < J, \tilde{e} >= 0\}$), we need to compute the rank of $\mathrm{d}\mathcal{G}(\mathrm{q}, \mathrm{p})$, $\forall$ (q, p). Actually because we have the following facts. it suffices to show that for every $(q, p) \in H^{-1}(h)$ the vector $\mathrm{d}\mathcal{G}(\mathrm{q,p})(\mathrm{gradH(q,p)})$ is normal to C_v at $\mathcal{G}(q, p)$.

1 $\mathrm{d}\mathcal{G}(\mathrm{q,p})\mathrm{T_{(q,p)}H^{-1}(h)} = \mathrm{T}_{\mathcal{G}(\mathrm{q,p})}\mathrm{C_v}$;

2 A normal space to H^{-1}(h), in Σ_- at (q,p) is spanned by gradH(q,p);

3 As a submanifolds of C, the manifold C_v is defined by $F(J, \tilde{e}) =< J, J > + < \tilde{e}, \tilde{e} > -v^2 = 0$ where $v = \frac{\mu}{\sqrt{-2H}}$

Since the normal space to C_v at $\mathcal{G}(q, p) = (J, \tilde{e}) \in C$ is spanned by $\mathrm{gradF(J, \tilde{e})} = 2(\mathrm{J}, \tilde{\mathrm{e}})$, it is sufficient to check that $< d\mathcal{G}(q, p)\mathrm{gradH(q,p)}, \mathrm{gradF}(\mathcal{G}(\mathrm{q,p})) >\neq 0$. The following calculations show

$d\mathcal{G}(q,p)gradH(q,p) \perp C_v$.

$$\begin{aligned}
&0 \neq < \mathrm{gradH(q,p)}, \mathrm{gradH(q,p)} >= \mathrm{dH(q,p)gradH(q,p)}\\
&= d(-\tfrac{\mu^2}{2}(< J, J > + < \tilde{e}, \tilde{e} >^{-1}))(q,p)\mathrm{gradH(q,p)}\\
&(H = -\tfrac{\mu^2}{2v^2} \quad and\ F(J,\tilde{e}) =< J, J > - < \tilde{e}, \tilde{e} > -v^2)\\
&= \tfrac{\mu^2}{2}(< J, J > + < \tilde{e}, \tilde{e} >)^{-2}(< J, dJ(qp)\mathrm{gradH(q,p)} > + < \tilde{\mathrm{e}}, \mathrm{dJ(q,p)gradH(q,p)} >)\\
&= \tfrac{2H(q,p)}{\mu^2} < d\mathcal{G}(q,p)\mathrm{gradH(q,p)}, \mathrm{gradF}(\mathcal{G}(\mathrm{q,p})) > \quad \square.
\end{aligned}$$

1.3 Kepler's Equation

In 1.2, we used the constants of the motion to describe the orbits of the Kepler vector field X_H of negative energy. In this subsection, we will determine the particle at a given time, and complete the solution. The similar topic can be found on $P62-63$ of [2].

In order to give a time parametrization of the bounded Keplerian orbit, we define a new time scale, the eccentric anomaly s , by

$$\frac{ds}{dt} = \frac{\sqrt{-2h}}{|q|}.$$

First, we find a differential equation for $|q(t)|$

$$\begin{aligned}
&h && = \tfrac{1}{2}|p|^2 - \tfrac{\mu}{|q|};\\
&|q|^2|p|^2 && = 2\mu|q| + 2h|q|^2;\\
&while\ \ |q|^2|p|^2 && = |q \times p|^2 + |< q, p >|^2 = J^2 + < q, p >^2 = J^2 + |q|^2(\tfrac{\mathrm{d}|\mathrm{q}|}{\mathrm{dt}})^2.
\end{aligned}$$

Hence

$$2\mu|q| + 2h|q|^2 = J^2 + |q|^2(\frac{d|q|}{dt})^2.$$

Changing to the time variables s and dividing by -2h, gives

$$(\frac{d|q|}{ds})^2 + a^2(1-e_0^2) = 2a|q| - |q|^2 (a = \frac{\mu}{-2h} = \frac{J^2}{\mu}\frac{1}{1-e_0^2}).$$

Let $e_0 a\rho = a - |q|$. Then we have

$$\begin{cases} (\frac{d\rho}{ds})^2 + \rho^2 & = 1 \\ \rho(0) = \frac{1}{e_0 a} & (a - |q(0)|) = 1. \end{cases}$$

Therefore $|q(s)| = a - ae_0 \cos s$.

To find the relation between the time scale s and physical time scale t, using the facts that $|q(s)| = a - ae\cos s$ and $\frac{ds}{dt} = \frac{\sqrt{-2h}}{|q|}$,

$$\sqrt{-2h}(t-\tau) = \sqrt{(-2h)}\int_\tau^t dt = \int_0^s (a - ae\cos s)ds = as - ae\sin s.$$

Dividing the above equation by a and using $a = \frac{\mu}{-2h} = \frac{v^2}{\mu}$ gives Kepler's equation

$$s - e\sin s = \frac{\mu^2}{v^2}(t-\tau) = nl$$

where l is the mean anomaly and $n = \frac{\mu^2}{v^3}$ is the mean motion. Noting that

$$< q, p >=< q, \frac{dq}{dt} >= |q|\frac{d|q|}{ds}\frac{ds}{dt} = \sqrt{-2h}ae\sin s = ve\sin s$$

when $t = \tau$, from the Kepler's equation, it follows that $s = 0$. Let $\bar{\tau}$ be the physical time corresponding to $s = 2\pi$. Then $\tau - \bar{\tau}$ is the period of elliptical motion, which is

$$\frac{2\pi}{n} = 2\pi\frac{a^{3/2}}{n^{1/2}}.$$

This is Kepler's third law of motion.

2 SO(4) Momentum Map

In last section, we got a complete solution of the Kepler's problem. But in 1.1, we know that the flow X_H is not complete. We will regularize it into a complete flow by constructing an map from $(\Sigma_-, \omega_3|\Sigma_-)$ to $(T^+S^3, \tilde{\Omega}_4)$. So in this section, we mainly discuss the SO(4) momentum map on (TS^3, Ω_4). This is a corresponding part to $II.2$ in [2]$(P44-53)$.

2.1 Monemtum Map ρ

$<,>$ is the Euclidean inner product on $\mathbb{R}^4$. This induces a Riemannian metric on $\mathbb{R}^4$. Using it, we pull back the canonical symplectic 2-form Ω on $T^*\mathbb{R}^4$ onto $T\mathbb{R}^4$, $\omega_4 = -\mathrm{d} < y, \mathrm{dx} >$. On $(T\mathbb{R}^4, \omega_4)$, we consider the Hamiltonian function

$$H : T\mathbb{R}^4 \to \mathbb{R} : (x, y) \mapsto \frac{1}{2} < y, y >$$

H describes the motion of a free particle in $\mathbb{R}^4$. When the particle is subjected to the force $\lambda(x, \dot{x}) = - < \dot{x}, \dot{x} > x$, it is constrained to S^3. So we have defined the hamiltonian function $H \overset{def}{=} \frac{1}{2} < y, y >$, on $(TS^3, \Omega_4 = \omega_4|_{TS^3})$.($\Omega_4$ is a well defined symplectic 2-form on TS^3).

Then the integral curves of the symplectic gradient X_H on TS^3 satisfy

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= - < y, y > x. \end{cases}$$

The flow of X_H on TS^3 is

$$\phi_t^H = \begin{pmatrix} cos(\sqrt{2H}) & \sin(t\sqrt{2H})/\sqrt{2H} \\ -\sqrt{2H}\sin(t\sqrt{2H}) & cos(t\sqrt{2H}) \end{pmatrix} \begin{pmatrix} x \\ y. \end{pmatrix}$$

Now we construct the momentum map associated to the $SO(4)$ action on (TS^3, Ω_4) . First, let $T^+S^3 = \{(x, y) \in TS^3 | y \neq 0\}$.

Consider the linear action of $SO(4)$ on $T\mathbb{R}^4$ defined by

$$\Phi : SO(4) \times T\mathbb{R}^4 \to T\mathbb{R}^4 : (A, (x, y)) \mapsto (Ax, Ay).$$

This is a hamiltonian action which we can define momentum map by

$$\mathcal{J} : T\mathbb{R}^4 \to so(4)^* : \mathcal{J}(x, y)(a) \stackrel{def}{=} < ax, y > \qquad \forall a \in so(4).$$

It's easy to prove that $\mathcal{J}$ is equivariant.

Then restricting Φ to an action $\hat{\Phi}$ on TS^3, we have

$$\hat{\Phi} : SO(4) \times TS^3 \to TS^3 : (A, (x, y)) \mapsto (Ax, Ay).$$

$\hat{\Phi}$ is a hamiltonian action on (TS^3, Ω_4) with momentum map

$$\tilde{\mathcal{G}} = \mathcal{J}|_{TS^3} : TS^3 \subseteq T\mathbb{R}^4 \to so(4)^*.$$

In order to study the geometry of the momentum map $\tilde{\mathcal{G}}$, we transform it into an easier to understand map.

We have known the anti symmetric matrices $\{e_{ij}\}_{1\le i<j\le 4}$ form a basis for Lie algebra (so(4),[,]), and the vector $\{e^*_{ij}\}_{1\le i<j\le 4}$, where $e^*_{ij} = e^t_{ij}$, form the standard dual basis for $so(4)^*$. The Lie bracket $\{,\}_{so(4)^*}$ on $so(4)^*$ is defined by

$$\{e^*_{ij}, e^*_{lk}\}_{so(4)^*} = -\sum_{mn} C^{mn}_{ij,lk} e^*_{mn}, \ where \qquad [e_{ij}, e_{lk}] = \sum_{mn} C^{mn}_{ij,lk} e_{mn}.$$

Let $\vartheta : \wedge^2\mathbb{R}^4 \to so(4)$ be the linear map defined by

$$\vartheta(u \wedge v)w =< w, v > u- < w, u > v \qquad \forall\ u, v, w \ \in \mathbb{R}^4.$$

Using the basis $\{e_i \wedge e_j\}_{1\le i<j\le 4}$ of $\wedge^2\mathbb{R}^4$, we see that $\vartheta(e_i \wedge e_j) = e_{ij}$. Consequently $\vartheta^t : so(4)^* \to (\wedge^2\mathbb{R}^4)^* = \wedge^2(\mathbb{R}^4)^*$ sends e^*_{ij} to $e^*_i \wedge e^*_j$.

Since

$$\begin{aligned} \vartheta^t(e^*_{ij})(x, y) &= (e^*_i \wedge e^*_j)(x, y) = e^*_i(x)e^*_j(y) - e^*_i(y)e^*_j(x) = x_i y_j - x_j y_i \\ &= S_{ij}(x, y) =< x, e_{ij}(y) > \qquad \forall x, y \in \mathbb{R}^4, \end{aligned}$$

$(\wedge^2\mathbb{R}^4)^*$ is the space $\mathcal{S}$ of homogeneous quadratic functions on $T\mathbb{R}^4$ which is spanned by $\{S_{ij}\}_{1\le i<j\le 4}$. As a subspace of $C^\infty(T\mathbb{R}^4)$, $\mathcal{S}$ has Poisson bracket $\{,\}_{\mathcal{S}}$ which is induced from the standard Poisson bracket $\{,\}$ on the space of the smooth functions on $(T\mathbb{R}^4, \omega_4)$

Proposition 4. *Lie algebras $(\mathcal{S}, \{,\}_{\mathcal{S}})$ and $(so(4)^*, \{,\}_{so(4)^*})$ are isomorphic.*

This is claim in $P47$ of [2].

Proof: From the definition,

$$dS_{ij} =< e_{ij}(y), \mathrm{dx} > - < \mathrm{e_{ij}(x), dy} > .$$

Since $\omega_4(\mathrm{dx}) = -\frac{\partial}{\partial y}$, $\omega_4(\mathrm{dy}) = \frac{\partial}{\partial \mathrm{x}}$, we find that

$$X_{S_{ij}}(x, y) = \omega_4(\mathrm{dS_{ij}}) = - < \mathrm{e_{ij}(x)}, \frac{\partial}{\partial \mathrm{x}} > - < \mathrm{e_{ij}(y)}, \frac{\partial}{\partial \mathrm{y}} > .$$

Therefore,

$$\begin{aligned} \{S_{ij}, S_{lk}\}_{\mathcal{S}}(x, y) &= (X_{S_{lk}} \lrcorner \mathrm{dS_{ij}})(x, y) = - < e_{lk}(x), e_{ij}(y) > + < e_{lk}(y), e_{ij}(x) > \\ &= - < x, (e_{ij}e_{lk} - e_{lk}e_{ij})y >= - < x, [e_{ij}, e_{lk}]y > \\ &= -\vartheta^t([e_{ij}, e_{lk}]^*)(x, y) = \vartheta^t(\{e_{ij}^*, e_{lk}^*\}_{so(4)^*})(x, y) \end{aligned}$$

which tells us ϑ^t is a Lie algebra isomorphism. □

On $\wedge^2\mathbb{R}^4$, we can define an inner product

$$\mathcal{K} : \wedge^2\mathbb{R}^4 \times \wedge^2\mathbb{R}^4 \to \mathbb{R} : (u \wedge v, x \wedge y) \mapsto det \begin{pmatrix} < u, x > & < u, y > \\ < v, x > & < v, y > . \end{pmatrix}$$

Since obviously $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$ is an orthonormal basis of $(\wedge^2\mathbb{R}^4, \mathcal{K})$, we may identify $\wedge^2\mathbb{R}^4$ with $(\wedge^2\mathbb{R}^4)^*$.

Thus combining what we have got, instead of studying the momentum map $\tilde{\mathcal{G}}$, we study the map

$$\rho : T^+S^3 \subset T\mathbb{R}^4 \to \wedge^2\mathbb{R}^4 : (x, y) \mapsto x \wedge y = \sum_{1 \le i < j \le 4} S_{ij}(x, y) e_i \wedge e_j,$$

where $S_{ij} = det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$ satisfying

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0.$$

Proposition 5. *Let C be the set of all non zero 2-vectors on $\mathbb{R}^4$ whose coordinates satisfy the above equation, $im(\rho) = C$.*

This is the claim on $P48$ of [2].

Proof: Suppose that $\theta \in C$. Then θ is decomposable, that is, there are vectors $u, v \in \mathbb{R}^4$, such that $\theta = u \wedge v$. (To see this, we let $\theta = S_{ij}e_i \wedge e_j$. Since $\theta \neq 0, \exists\ S_{ij} \neq 0$. We suppose $S_{12} \neq 0$, let $u = (1, 0, -\frac{S_{23}}{S_{12}}, -\frac{S_{24}}{S_{12}})$ and $v = (0, S_{12}, S_{13}, S_{14})$. Then $\theta = u \wedge v$.) Let $\{x, y\}$ be an orthonormal basis of the 2-plane spanned by $\{u, v\}$. Then $u \wedge v = \lambda x \wedge y$ for some non zero λ. Therefore, $\rho(x, \lambda y) = \theta$. □

On the level of hamiltonian $H^{-1}(h) = \{(x, y) \in T^+S^3 \subseteq T\mathbb{R}^4 | \frac{1}{2} < y, y >= h\}$, consider the map $\rho_h : H^{-1}(h) \subseteq T^+S^3 \to C \subseteq \wedge^2\mathbb{R}^4 : (x, y) \mapsto x \wedge y$, which is the restriction of ρ to $H^{-1}(h)$. From the identity $\sum_{1 \le i < j \le 4}(x_iy_j - x_jy_i)^2 =< x, x >< y, y > - < x, y >^2$, we see that the image of ρ_h is contained in the submanifolds C_h of C defined by $\sum_{1 \le i < j \le 4} S_{ij}^2 = 2h$. In fact, C_h is diffeomorphic to $S^2_{\sqrt{h/2}} \times S^2_{\sqrt{h/2}}$.

2.2 The properties of the momentum ρ

Having the map $\rho : T^+S^3 \to \wedge^2\mathbb{R}^4$, in this subsection, we discuss its geometric properties.

Lemma 1. *For every $h > 0$, the map $\rho_h : H^{-1}(h) \to C_h$ is a surjective submersion each of whose fibers is a single orbit of the symplectic gradient of H.*

This is the claim on $P49$ of [2].

Proof: To show that ρ is surjective. Suppose that $S = S_{ij} \in C_h$. Since C_h is contained in $C = \rho(T^+S^3)$, there is a $(x,y) \in T^+S^3$ such that $\rho(x,y) = S$. But 2h= $\sum_{1\le i<j\le 4} S_{ij}^2$, since S$\in C_h$. From $\sum_{1\le i<j\le 4}(x_iy_j - x_jy_i)^2 =< x,x >< y,y > - < x,y >^2$, the definition of S_{ij}, and the fact that $(x,y) \in T^+S^3$, we find that $\frac{1}{2} < y,y >= h$. Hence $(x,y) \in H^{-1}(h)$.

To show that ρ_h is a submersion, we must verify that the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x,y) \in H^{-1}(h)$, because C_h is 4-dimensional. Toward this goal, let $V(x,y) = span\{X_{S_{ij}}(x,y)\}_{1\le i<j\le 4}$, where $X_{S_{ij}}$ is the Hamiltonian vector field on $(T\mathbb{R}^4, \omega_4)$ corresponding to the Hamiltonian function

$$S_{ij} : T\mathbb{R}^4 \to \mathbb{R} : (x,y) \mapsto x_iy_j - x_jy_i$$

Since $S_{ij}|T^+S^3$ is an integral of the geodesic vector field X_H on T^+S^3, we see that $V(x,y) \subseteq ker\mathrm{d}H(x,y) = T_{(x,y)}H^{-1}(h)$ for every $(x,y) \in T^+S^3$. Now $(T_{(x,y)}\rho_h)|V_{x,y} = \mathrm{d}H(x,y) = \{S_{ij}, S_{lk}\}_{\mathcal{S}} = \tilde{\rho}$. Set

$$\begin{aligned}
\xi_1 &= \tfrac{1}{2}(S_{12} + S_{34}), \quad \eta_1 = \tfrac{1}{2}(S_{12} - S_{34});\\
\xi_2 &= \tfrac{1}{2}(S_{13} + S_{24}), \quad \eta_2 = \tfrac{1}{2}(S_{13} - S_{24});\\
\xi_3 &= \tfrac{1}{2}(S_{14} + S_{23}), \quad \eta_3 = \tfrac{1}{2}(S_{14} - S_{23});
\end{aligned}$$

$\tilde{\rho}$ is conjugate to

$$\begin{pmatrix} \{\xi_i, \xi_j\}_{\mathcal{S}} & 0 \\ 0 & \{\eta_i, \eta_j\}_{\mathcal{S}} \end{pmatrix} = \begin{pmatrix} \sum_k \varepsilon_{ijk}\xi_k & 0 \\ 0 & \sum_k \varepsilon_{ijk}\eta_k. \end{pmatrix}$$

We can get $\sum \xi_i^2 = \sum \eta_i^2 = h/2$, so each of the 3×3 skew symmetric matrices $\sum_k \varepsilon_{ijk}\xi_k$ *and* $\sum_k \varepsilon_{ijk}\eta_k$ is non zero and hence has rank 2. Therefore , the rank of $T_{(x,y)}\rho_h$ is 4, for every $(x,y) \in H^{-1}(h)$. Thus ρ_h is a submersion.

Given $S = (S_{ij}) \in C_h$, the fiber $w = \rho_h^{-1}(S)$ is a union of orbits of the geodesic vector field X_H of energy h because $S_{ij}|T^+S^3$ are integral of X_H. By the definition of ρ_h, w is the set of all ordered pairs $\{x,y\}$ of orthogonal vectors in $\mathbb{R}^4$ such that $< x,x >= 1, < y,y >= 2h$ and the 2-plane Π spanned by $\{x,y\}$ has Plucker coordinates (S_{ij}). Since any two such basis of Π are related by a counterclockwise rotation in Π, we find that

$$W = \{(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \in H^{-1}(h) | \theta \in [0, 2\pi]\}$$

Therefore W is the unique related orbit of X_H traced out by an integral curve of X_H. □

The goal of the following discussion is to construct a symplectic form on C_h, such that we can give a description of ρ. First , we define a Poisson bracket $\{,\}$ on $C^\infty(\mathcal{S})$ of smooth functions on Lie algebra $(\mathcal{S},\{,\}_{\mathcal{S}})$. For f,g$\in C^\infty(\mathcal{S})$, let

$$\{f,g\} = \sum_{1\le i<j\le 4, 1\le l<k\le 4} \frac{\partial f}{\partial S_{ij}}\frac{\partial g}{\partial S_{lk}}\{S_{ij},S_{lk}\}_{\mathcal{S}}.$$

On $C^\infty(\mathcal{S})$, define a multiplication $(f*g)(s) = f(s)g(s)\ \forall s\in\mathcal{S}$. Then $(C^\infty(\mathcal{S}),^*)$ is a commutative ring with unit. Using the above $\{,\}$, we can check that $\mathcal{A} = (C^\infty(\mathcal{S}),\{,\}_{\mathcal{S}})$ is a Poisson algebra.

Consider $C_1 = \sum_{1\le i<j\le 4} S_{ij}^2 - 2h$ and $C_2 = S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23}$. They are Casimirs for $\mathcal{A}$. Let $\mathcal{B}$ be the ideal in $(C^\infty(\mathcal{S}),*)$ which is generated by C_1 and C_2. Then $\mathcal{B}$ is a Poisson ideal in $\mathcal{A}$. Therefore we can define a Poisson bracket $\{,\}_{C_h}$ on $C^\infty(\mathcal{S}/\mathcal{B})$ by

$$\{f+\mathcal{B}, g+\mathcal{B}\}_{C_h} = \{f,g\}$$

From the analytic results, we can identify the space $C^\infty(\mathcal{S}/\mathcal{B})$ with the space $C^\infty(C_h)$ of smooth functions on C_h. Consequently, we may define the quotient Poisson algebra

$$\mathcal{D} = \mathcal{A}/\mathcal{B} = (C^\infty(C_h),\{,\}_{C_h},*).$$

Because $\{S_{ij}+\mathcal{B}, S_{lk}+\mathcal{B}\}_{C_h} = \{S_{ij},S_{lk}\}_{\mathcal{S}}$, the matrix of Poisson brackets $\{,\}_{C_h}$ has rank 4. Therefore $\{,\}_{C_h}$ is non degenerate and defines a symplectic form ω_h on C_h. Moreover

$$\rho_h^*\omega_h = \omega_4|H^{-1}(h).$$

Theorem 4. *The map $\rho : T^+S^3 \subseteq T\mathbb{R}^4 \to C \subseteq \wedge^2\mathbb{R}^4 : (x,y)\mapsto x\wedge y$ is a surjective submersion, each of whose fibers is a unique oriented orbit of the geodesic vector field X_H on (T^+S^3,ω_4).*

This is the claim on $P52$ of [2].

Proof: We have known that ρ is surjective. So we need to show each of the fiber is a unique oriented orbit of X_H. Suppose that $S = (S_{ij}) \in C$. Because S is non zero, $\sum_{1\le i<j\le 4} S_{ij}^2 = 2h$ for some $h > 0$. Therefore $S \in C_h$. Since the fiber $\rho_h^{-1}(S)$ of ρ_h is a unique oriented orbit of X_H of energy h, so is the fiber $\rho^{-1}(s)$ of ρ because $\rho = \rho_h$ on $H^{-1}(h)$.

To show that ρ is submersion, the method is similar to the proof of Theorem 2. We can compute

$$< T_{(x,y)}\rho(\mathrm{gradH(x,y)}), \mathrm{gradF}(\rho(\mathrm{x,y})) > \neq 0.$$

Here we omit the calculation.(The details can be found in [2]) $\square$

We have an interesting consequence.

Corollary 1. *The space of orbits of the geodesic vector field with positive energy is the manifold C. The orbit map is $\rho : T^+S^3 \to C$*

At the end of this section, we might find that the result of this section and the first section are similar. In 1.2, we have $so(4)$ action, and in this section, we have $SO(4)$ action. From theorem 2,3 and 4, we find that their orbits are so similar! Is there any relations between them? This is discussed in the last section.

3 Regularization of the Kepler vector field

In this section, we will remove the incompleteness on the flow of the Kepler vector field by embedding it into a completed flow. This process is called regularization. We follow the method in [2](II 3.4). On the subset of phase space where the Kepler Hamiltonian is negative, one can perform regularization in such a way that the embedding not only is symplectic but also that it linearizes and integrates the so(4) action. That there is such a large hidden symmetry in the Kepler problem is quite remarkable because this symmetry doesn't arise from a lift of a symmetry on the configuration space.

We will regularize all negative energy Keplerian orbits at once using the Ligon-Schaaf map the LS. We show that the LS is the only symplectic map from the negative energy set ($\Sigma_- = \{(q,p) \in T_0\mathbb{R}^3 | H(q,p) < 0\}, \omega_3|\Sigma_-$) to (T^+S^3, ω_4), which has the following properties.

1. It intertwines the Kepler and Delaunay vector fields(we will define the latter below.), X_H and $X_{\mathcal{H}}$,respectively;

2. It intertwines their $so(4)$ momentum map $\mathcal{G},\ and\ \tilde{\mathcal{G}}$;

3. It maps Σ_- onto $T^+(S - np)$, where np=(0,0,0,1).

And Ligon-Schaaf map is a diffeomorphism because it maps one period of a parametrized integral curve of the Kepler vector field of negative energy onto one period of a parametrized integral curve of the Delaunay vector field on $T^+(s^3 - np)$. We'll omit most of the calculation here and give an outline of the whole approach. For the details, readers are encouraged to read [2] 3.4($P64 - 75$).

First, we define the Delaunay vector field on T^+S^3.

$T^+S^3 = \{(x,y) \in TS^3 | y \neq 0\}$ be the tangent bundle of S^3, less its zero section. As T^+S^3 is a open subset of TS^3, $\Omega_4|T^+S^3$ is a sympletic form $\tilde{\omega}_4$ on T^+S^3. On $(T^+S^3, \tilde{\omega_4})$ consider the Delaunay Hamiltonian:

$$\tilde{\mathcal{H}} : T^+S^3 \subseteq T\mathbb{R}^4 \to \mathbb{R} : (x,y) \mapsto -\frac{1}{2}\frac{\mu^2}{< y,y >}$$

The integral curves of the Delaunay vector field $X_{\mathcal{H}}$ satisfy

$$\begin{cases} \frac{dx}{dt} &= \frac{\mu^2}{<y,y>} y \\ \frac{dy}{dt} &= -\frac{\mu^2}{<y,y>} x. \end{cases}$$

The flow of the Delaunay vector field is

$$\phi_t^{\tilde{\mathcal{H}}}(x,y) = \begin{pmatrix} \cos\frac{\mu^2}{v^3}t & \frac{1}{v}\sin\frac{\mu^2}{v^3}t \\ -v\sin\frac{\mu^2}{v^3} & \cos\frac{\mu^2}{v^3}t. \end{pmatrix}$$

Note that on the energy surface $\tilde{\mathcal{H}}^{-1}(-\frac{1}{2}\frac{\mu^2}{v^3})$, all the integral curves of the Delaunay vector field $X_{\tilde{\mathcal{H}}}$ are periodic of period $2\pi\frac{v^3}{\mu^2}$. Then we do the regularization by the following steps.

Step 1. the smooth map $\Phi : \Sigma_- \subseteq T\mathbb{R}_0^3 \to T^+S^3 : (q,p) \mapsto (x,y)$ intertwines the $so(4)$ momentum map $\mathcal{G}$ and $\tilde{\mathcal{G}}$, that is $\Phi^*\tilde{\mathcal{G}} = \mathcal{G}$, if and only if

$$(x,y) = \Phi(q,p) = (A\sin\phi + B\cos\phi, -vA\cos\phi + vB\sin\phi),$$

where

$$\begin{aligned} A &= (\tilde{A}_4) = (\tfrac{q}{|q|} - \tfrac{1}{\mu} < q,p > p, \tfrac{1}{v} < q,p >), \\ B &= (\tilde{B}, B_4) = (\tfrac{1}{v}|q|p, \tfrac{1}{\mu} < p,p > |q| - 1), \end{aligned}$$

$v = \frac{\mu}{\sqrt{-2h}}$, and $\phi = \phi(q,p)$ is an arbitrary smooth real valued function on Σ_-.

The proof is just straightforward to check that $\Phi^*\tilde{\mathcal{G}} = \mathcal{G}$ is equivalent to

$$\tilde{x} \times \tilde{y} = q \times p$$

$$x_4\tilde{y} - y_4\tilde{x} = v(\frac{q}{|q|} - \frac{1}{\mu}p \times (q \times p)) = Mq + Np,$$

where

$$M = v(\frac{1}{|q|} - \frac{1}{\mu} < p,p >) \; and \; N = \frac{v}{\mu} < q,p > .$$

We omit the details.

Step 2 Denote Φ_ϕ be the map Φ defined in step 1 corresponding to smooth function ϕ.

Φ_ϕ intertwines the Kepler and Delaunay vector fields that is $T\Phi_\phi \circ X_H = X_{\tilde{\mathcal{H}}} \circ \Phi_\phi$ if and only if

$$\phi = \phi(q,p) = \frac{1}{v}(q,p) - F(q,p),$$

Where F is a smooth integral of X_H. The proof is still straightforward computation, so we omit it.

We define the LS, the Ligon-Schaaf map, to be

$$\Phi_\phi, \qquad \phi_{LS} = \frac{1}{v} < q,p > .$$

Corollary 2. *For every (q,p)* $\in \Sigma_-$, $\Phi_\phi(q,p) = \Psi_{\frac{v^3}{\mu^2}}(LS(q,p))$, Ψ_t *be the flow of Delaunay vector field* $X_{\tilde{\mathcal{H}}}$.

Step 3 $\Phi_\phi = T^+S^3_{np}$ and $\Phi_\phi\tilde{\mathcal{G}} = \mathcal{G}, T\Phi_\phi \circ X_H = X_{\tilde{\mathcal{H}}} \circ \Phi_\phi$ *and* $\Phi_\phi\tilde{\omega}_4 = \tilde{\omega}_3$ if and only if $\Phi_\phi = LS$. And the LS is a diffeomorphism. The proof is also omitted.

We now put the preceding arguments into perspective. Geometrically, we think T^+S^3 as a regularized model for the negative energy subset Σ_- of the phase space of the Kepler problem. On T^+S^3 the SO(4) symmetry $\hat{\Phi}$ is globally defined and its orbits

$$T_{\frac{\mu}{\sqrt{-2h}}}S^3 = \{(x, y) \in T^+S^3| < y, y >= -\frac{\mu^2}{2h}\}$$

are the regularization of the energy surface $H^{-1}(h) \subseteq \Sigma_-$ of the Kepler problem. Let $\mathcal{S} = \{(x, y)|T^+S^3|x_4 = 1\}$.From the above, we can see that $LS^{-1}(\mathcal{S})$ is the collision set $\{(q, p) \in T\mathbb{R}^3|q = 0\}$ of the Kepler problem. Since $T_{\frac{\mu}{\sqrt{-2h}}}S^3 \cap \mathcal{S}$ is non empty for every $h < 0$, we conclude that the $SO(4)$ symmetry is not globally defined for the Kepler problem. Such a large hidden symmetry of the Kepler's problem is really remarkable and interesting.

This completes our report.

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