QUANTUM MOMENT(UM) MAP(PING)S
A HOPF ALGEBRA APPROACH

GIZEM KARAALI

1. Introduction

A basic question related to group actions on manifolds preserving some geometric structure is the existence of moment maps [1]. The notion is well developed in the case of symplectic or hamiltonian actions. A natural generalization to Poisson actions was made in [L1], and we will review this shortly in one of the sections of this note. As a natural next step, one would like to quantize moment maps. Up to now, there have been two different methods to approach this problem [2]. One of these is via the method of deformation quantization, and this is the topic of Bursztyn's note, see [B]. The topic of the present note is the second method, basically developed in [L2].

The main ingredients in this method are those of quantum algebra: Hopf algebras, and more particularly quantum groups. Therefore we will start with a basic introduction to these concepts. Fundamental definitions and basic facts about quantum groups will be described in Section 2. Section 3 of this paper will mainly be a review of Poisson geometry, and more particularly Poisson-Lie groups, in order to give finally a (sketchy) recap of [L1] where we see how the notion of moment maps is generalized to the setting of Poisson actions. Finally in Section 4, we will review [L2], where Lu develops her quantum moment map. Although Lu develops a lot more for a quantum theory including reduction and semidirect products in [L2], we will only be concerned with the moment map and restrict ourselves mainly to Section 3 of her paper.

[1] Among the many versions (as hinted in the title of this note), we choose to use the term moment map. This choice is made only to minimize typing efforts and has no philosophical basis.

[2] Perhaps one should take this distinction with a grain of salt; when one gets into the details of the method discussed in this paper, one sees that the concepts of deformation and quantization are quite basic to the constructions involved. However the quantum group notions which arise here do not appear in the papers discussed in [B].

2. Hopf Algebras and Quantum Groups

2.1. **Definitions:** Here we give the basic definitions regarding Hopf algebras and quantum groups. One may check [CP],[CW] or [M] for a more transparent version of these definitions involving commutative diagrams.

Definition 1. A k-*algebra* is a k-vector space A together with two k-linear maps, multiplication $m : A \otimes A \to A$ and unit $u : k \to A$, such that m is associative (i.e. satisfies $m(m \otimes id) = m(id \otimes m)$) and the unit map satisfies the unit axiom:

$$s_1 = m(u \otimes id) = m(id \otimes u) = s_2$$

where s_1 is the scalar multiplication from $k \otimes A$ into A, and s_2 is the scalar multiplication from $A \otimes k$ into A.

Definition 2. A k-*coalgebra* is a k-vector space C together with two k-linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to k$, such that Δ is coassociative (i.e. satisfies $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$) and the counit map satisfies the counit axiom:

$$(1 \otimes \ ) = (\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = (\ \otimes 1)$$

where $(1 \otimes \ )$ is the map $C \to k \otimes C$ which sends $c \in C$ to $1 \otimes c$, and $(\ \otimes 1)$ is the map $C \to C \otimes k$ which sends $c \in C$ to $c \otimes 1$.

Definition 3. An algebra A is called *commutative* if $m_A \circ \sigma = m_A$. A coalgebra C is called *cocommutative* if $\sigma \circ \Delta_C = \Delta_C$. (In both cases, σ is the twist map $x \otimes y \mapsto y \otimes x$.)

We note that if C is a coalgebra and A is an algebra, then $Hom_k(C, A)$ becomes an algebra under the *convolution product* $\star$ defined as

$$(f \star g)(c) = m \circ (f \otimes g)(\Delta c)$$

for all $f, g \in Hom_k(C, A), c \in C$.The unit element in $Hom_k(C, A)$ is $u\epsilon$. We continue with the definitions:

Definition 4. A k-space B is a *bialgebra* if (B, m, u) is a k-algebra, (B, Δ, ϵ) is a k-coalgebra, and either of the following (equivalent) conditions holds:

(1) Δ and ϵ are algebra morphisms. (A map $f : A \to B$ is a *algebra morphism* if $f \circ m_A = m_B \circ (f \otimes f)$ and $u_B = f \circ u_A$.)

(2) m and u are coalgebra morphisms. (A map $f : C \to D$ is a *coalgebra morphism* if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and $\epsilon_C = \epsilon_D \circ f$.)

Definition 5. Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. then H is a *Hopf algebra* if there exists an element $S \in Hom_k(H, H)$ which is an inverse to id_H under convolution $\star$. S is called an *antipode* for H.

A standard example of a Hopf algebra is the group algebra $k[G]$ of a group G. The coproduct is given by $\Delta(g) = g \otimes g$ and the counit is given by $\epsilon(g) = 1$ on the elements g of the group G. (More generally for an arbitrary coalgebra C, elements $g \in C$ satisfying $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ are called *group-like.*) The antipode is defined on the group elements as $S(g) = g^{-1}$.

Dually, we can look at the algebra $A = Fun(G)$ of functions on a group G [3]. Here the multiplication is the natural commutative one: $(f_1 f_2)(g) = f_1(g) f_2(g)$, and the unit is the map u mapping elements of the field k to the associated constant maps. We can identify $Fun(G \times G)$ with $A \otimes A$, so the group multiplication considered as a mapping $G \times G \to G$ induces the copruduct $\Delta : A \to A \otimes A$ given by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$. The counit is the map $\epsilon : A \to k$ mapping $f \in A$ to $f(e)$, where e is the identity element of G. Finally the antipode is the map S defined by $S(f)(g) = f(g^{-1})$. Thus we have a commutative Hopf algebra structure on A. (It will also be cocommutative if the group G is commutative.)

Another family of examples is given by universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$. In this case, the coalgebra structure is given as follows: $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ for any $x \in \mathfrak{g}$. (More generally for an arbitrary coalgebra C, elements $x \in C$ satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ are called the *primitive elements* of C.) The antipode is defined on the Lie algebra elements as $S(x) = -x$.

Next, we introduce a useful notation due to Sweedler: Let C be any coalgebra with comultiplication $\Delta : C \to C \otimes C$. The *sigma notation* for Δ is given as follows: For any $c \in C$, we write:

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$$

The subscripts here are just symbolic, the elements $c_{(1)}, c_{(2)}$ do not stand for particular elements of C. Recall that Δ maps C to $C \otimes C$, and so for any $c \in C$ we will have $\Delta(c) = (c_{1,1} \otimes c_{1,2}) + \cdots + (c_{N,1} \otimes c_{N,2})$ for some elements $c_{i,j} \in C$ and for some integer N that depends on c. The Sweedler notation is just a method of separating the $c_{i,1}$ from the $c_{i,2}$. (In a sense, $c_{(1)}$ stands for the generic $c_{i,1}$, and $c_{(2)}$ stands for the generic $c_{i,2}$.) With this notation, our counit axiom, in the definition of a coalgebra (Definition 2), becomes:

$$c = \sum \epsilon(c_{(1)}) c_{(2)} = \sum \epsilon(c_{(2)}) c_{(1)},$$

[3]This example will be the most important example to keep in mind for this review paper. Our major motivation for studying Hopf algebras originates from this example.

for any $c \in C$. The convolution product we defined in $Hom_k(C, A)$ becomes:

$$(f \star g)(c) = \sum f(c_{(1)}) g(c_{(2)}).$$

Now we introduce two notions of duality:

Definition 6. (1) For any k-space V, $V^* = Hom_k(V, k)$ is called the *linear dual* of V. (This defines a nondegenerate bilinear form $\langle\ ,\ \rangle : V^* \otimes V \to k$ via $\langle f, v\rangle = f(v)$).

(2) The *finite dual* of an algebra A is defined to be $A^o = \{f \in A^*; f(I) = 0$ for some ideal I of A such that $dim(A/I) < \infty\}$.

We can define these duals for any space with one or more of the above structures. It can be shown that the linear dual of a coalgebra will have a natural algebra structure, but one needs to introduce the finite dual to conclude the analogous result for an algebra: If A is not finite dimensional, $A^* \otimes A^*$ is a proper subspace of $(A \otimes A)^*$, and so the image of $m^* : A^* \to (A \otimes A)^*$, the dual of the multiplication map, may or may not lie in $A^* \otimes A^*$. In fact, one can show that A^o is the largest subspace V of A^* such that $m^*(V) \subset V \otimes V$.

Now let H be a Hopf algebra and H^* be its (finite) dual Hopf algebra. The pairing $\langle\ ,\ \rangle$ between H and H^* satisfies:

$$\langle x, ab\rangle = \langle \sum x_{(1)} \otimes x_{(2)}, a \otimes b\rangle,$$

$$\langle xy, a\rangle = \langle x \otimes y, \sum a_{(1)} \otimes a_{(2)}\rangle$$

and

$$\langle x, 1_H\rangle = \epsilon_{H^*}(x),$$

$$\langle 1_{H^*}, a\langle = \epsilon_H(a),$$

$$\langle S_{H^*}(x), a\rangle = \langle x, S_H(a)\rangle$$

where $a, b \in H$ and $x, y \in H^*$.

We are finally ready to define quantum groups. Here is the definition from [L2]:

Definition 7. A *quantum group* is a Hopf algebra $(Fun(G), *_h, \Delta, S_h, \epsilon_h)$ consisting of:

(1) a one-parameter family of associative algebra structures $*_h$ on the space of functions $Fun(G)$ of a group G (here G can be C^∞, analytic, or formal);

(2) the map $\Delta : Fun(G) \to Fun(G) \otimes Fun(G)$ defined by: $\Delta(f)(g_1, g_2) = f(g_1 g_2)$ for any $g_1, g_2 \in G$, which is the pullback of the group multiplication map of G [Lu's note here: The comultiplication Δ here is "not quantized" in the sense that it is simply the pullback; we are not given a one-parameter family Δ_h of comultiplications];

(3) (the antipodes) maps S_h from $Fun(G)$ to itself such that S_0 is given by: $S_0(f)(g) = f(g^{-1})$; and

(4) (the counits) maps ϵ_h from $Fun(G)$ to the ground field k such that ϵ_0 is given by: $\epsilon_0(f) = f(e)$ where $e \in G$ is the unit element of G.

We might sum up the above as follows: A quantum group is a collection of Hopf algebra structures on the space of functions of a group G. These structures are parametrized by a (possibly formal) variable h, and are deformations of the natural (commutative) Hopf algebra structure on $Fun(G)$ [4].

Now we have defined our objects of interest. In the next section we will describe the notion of inner action for a Hopf algebra, and the corresponding notion of a quantum group action will thus be formalized. This type of action will turn out to be the one fit for developing the notion of a quantum moment map.

2.2. **Inner actions of Hopf Algebras:** Here our basic reference will be [M]. We start with definitions for general algebras and coalgebras.

Definition 8. For a k-algebra A, a (left) A-*module* is a k-space M with a k-linear map $\gamma : A \otimes M \to M$ such that $\gamma(m \otimes id) = \gamma(id \otimes \gamma)$ and $\gamma(u \otimes id) =$ scalar multiplication.

Translated to the language of actions, we have: For a k-algebra A, we say that A *act*s on the k-space M if M is a left A-module. The action is given by the map γ [5].

The following is the dual notion of a (co-)action of a coalgebra:

Definition 9. For a k-coalgebra C, a (right) C-*comodule* is a k-space M with a k-linear map $\rho : M \to M \otimes C$ such that $(id \otimes \Delta)\rho = (\rho \otimes id)\rho$ and $(id \otimes \epsilon)\rho =$ tensoring with 1. In this case we say that ρ is a *coaction* of C on M.

We will also need:

Definition 10. (1) Let A be an algebra. Let M, N be (left) A-modules, with structure maps γ_M and γ_N respectively. A map $f : M \to N$ is called an A-*module map* if $f \circ \gamma_M = \gamma_N \circ (id \otimes f)$.

---

[4] Historically, the category of quantum groups was defined as a dual to the category of Hopf algebras, with the motivation coming from the duality between the category of groups and commutative Hopf algebras. (One can check [D] for such a description.) Today, the phrase *quantum group* is almost analogous to the phrase *Hopf algebra*. We will stick to Lu's definition here, and look at quantum groups as Hopf algebras obtained from deformations of the function algebra of a group.

[5] Note that this definition of an algebra action is the usual one; we only state it here so that we can see the duality when we define the corresponding notion of a coaction.

(2) Let C be a coalgebra. Let M, N be (right) C-comodules, with structure maps ρ_M and ρ_N respectively. A map $f : M \to N$ is called a *C-comodule map* if $\rho_N \circ f = (f \otimes id) \circ \rho_M$.

So these are maps of algebras and coalgebras which preserve the module and comodule structures on the corresponding spaces.

Let us consider some basic examples:

Example 1. *For any coalgebra C, $M = C$ is a right comodule using $\rho = \Delta$. This gives us a left action of C^* on C: For $f \in C^*, c \in C$, we let $(f \Rightarrow c) = \sum \langle f, c_{(2)} \rangle c_{(1)}$, where $\langle\ ,\ \rangle$ is the nondegenerate pairing between C and its linear dual C^*. Actually one can show that $(\Rightarrow)$ is the transpose of right multiplication in C^*, noting that $\langle g, (f \Rightarrow c) \rangle = \langle gf, c \rangle$.*

Example 2. *There is also a natural right action of C^* on C: For $f \in C^*, c \in C$, we let $(c \Leftarrow f) = \sum \langle f, c_{(1)} \rangle c_{(2)}$. Again we can show that $\langle g, (c \Leftarrow f) \rangle = \langle fg, c \rangle$, so $(\Leftarrow)$ is the transpose of left multiplication in C^*.*

Example 3. *Analogously we can define a left (resp. right) action $(\Leftarrow)$ (resp. $(\Leftarrow)$) of A on A^* for any algebra A which is the transpose of right (resp. left) multiplication on A.*

Now we specialize to Hopf algebra actions:

Definition 11. Let H be a Hopf algebra. An algebra A is a *(left) H-module algebra* if:

(1) A is a (left) H-module via $h \otimes a \mapsto h \cdot a$, (so H, as an algebra, acts on A);

(2) $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ for any $a, b \in A$, (so m_A is an H-module map); and

(3) $h \cdot 1_A = \epsilon(h) 1_A$, (so u_A is an H-module map).

(We say that H *measures* A if only (2) and (3) are satisfied.)

Thus, given a Hopf algebra H and an algebra A, we have a Hopf algebra action of H on A if A is a (left-) H-module (i.e. H acts on A as an algebra) and the algebraic structure of A is compatible with this action. Similarly we can define a Hopf algebra co-action:

Definition 12. An algebra A is a (right) *H-comodule algebra* if

(1) A is a (right) H-comodule, via $\rho : A \to A \otimes H$, (so H, as a coalgebra, acts on A);

(2) $\rho(ab) = \sum (a_{(1)} b_{(1)}) \otimes (a_{(2)} b_{(2)})$ for all $a, b \in A$, (so m_A is a (right) H-comodule map);

(3) $\rho(1_A) = 1_A \otimes 1_A$, (so u_A is an H-comodule map).

Therefore, given a Hopf algebra H and an algebra A, we have a Hopf algebra co-action of H on A if A is a (right-) H-comodule (i.e. H acts on A as a coalgebra) and the algebraic structure of A is compatible with this action.

Let us look at some more examples.

Example 4. *For any Hopf algebra H, we have both an action and a coaction on $M = H$ by right multiplication and Δ respectively.*

Example 5. *The trivial H-module M is defined as follows: For $h \in H$, $m \in M$, $h \cdot m = \epsilon(h)m$.*

Example 6. *Let $H = kG$. Then it is well-known that we can find a correspondence between H-module algebras and group representations of G. A nice characterization of H-comodule algebras is also available; namely on any H-comodule algebra we can find a natural G-graded algebra structure.*

Example 7. *Let H be an arbitrary Hopf algebra. Then the left adjoint action $ad_l : H \otimes H \to H$, of H on itself is defined to be:*

$$(ad_l h)(k) = \sum h_{(1)} k(S(h_{(2)})$$

for all $h, k \in H$. The right adjoint action $ad_r : H \otimes H \to H$, of H on itself is defined to be:

$$(ad_r h)(k) = \sum S(h_{(1)}) k h_{(2)}$$

for all $h, k \in H$.

Finally we make the following:

Definition 13. Let H be a Hopf algebra, and A be an H-module algebra. Consider an action $H \otimes A \to A$ given by: $h \otimes a \mapsto h \cdot a$, such that:

(1) $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ for any $a, b \in A$; and

(2) $h \cdot 1_A = \epsilon(h) 1_A$,

(i.e. H measures A.) Then this action is called *inner* if there exists a convolution invertible map $u \in Hom_k(H, A)$ such that for all $h \in H, a \in A$ we have:

$$h \cdot a = \sum u(h_{(1)}) a u^{-1}(h_{(2)}).$$

(We may always assume $u(1) = 1 = u^{-1}(1)$, for if not, we can simply replace u by $v(h) = u(1)^{-1} u(h)$ for all $h \in H$.)

One can check that the left and right adjoint actions of a Hopf algebra H on itself are inner: Just set $u(h) = h, u^{-1}(h) = S(h)$. Also the trivial action of H on any algebra A will also be inner: $u(h) = u^{-1}(h) = \epsilon(h)$. Note that if we have an inner action of H on some algebra A and g is group-like, then g acts as the inner automorphism of A defined by $x \mapsto u(g)x(u(g))^{-1}$. Conversely, if a group G acts as inner automorphisms of A, then the action of $H = kG$ is inner. On the other hand, if p is a primitive element of H, then p acts as the inner derivation $x \mapsto u(p)x - xu(p) = [u(p), x]$. Conversely if a Lie algebra $\mathfrak{g}$ acts as inner derivations of A, then the action of $H = \mathfrak{U}\mathfrak{g}$ will be inner. Thus inner actions may be seen as a natural generalization of inner automorphisms and inner derivations.

Here we will be mainly interested in inner actions. Under semi-classical limits, inner actions of Hopf algebras will correspond to Poisson actions of related Poisson-Lie groups and the study of moment maps for Poisson actions is thus relevant. Therefore we will develop the theory for Poisson actions in the next section.

3. Poisson-Lie groups, Poisson Actions

3.1. **Basic Notions:** First we will start with the basics and give some fundamental definitions. Here we will mostly be using [CP], the first chapter of which is a recap of the theory of Poisson-Lie groups. Also one may look at [CW] or [R]. For a more detailed development of the Poisson geometry behind all this, one can check [V].

Recall that a Poisson structure on a manifold M is a skew-symmetric R-bilinear bracket $\{\cdot,\cdot\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ satisfying the Jacobi identity (i.e. a Lie bracket on $C^\infty(M)$), and the Leibniz rule:

$$\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + \{f_1, f_3\} f_2.$$

It is easy to see that the Leibniz identity means that for any fixed $f \in C^\infty(M)$, the map $g \mapsto \{f, g\}$ is a derivation of $C^\infty(M)$, and gives us what we call the *Hamiltonian vector field* corresponding to f. Thus we can equivalently say that the Poisson structure on M is a bivector field π (i.e. a skew-symmetric 2-tensor, an element of $TM^{\otimes 2}$) called the *Poisson bivector*. (The Jacobi identity, on the other hand, is equivalent to the vanishing of the Schouten bracket $[\pi, \pi]$, a natural extension of the Lie bracket, which will not be defined here. One can look at [V] for the definition and basic properties of the Schouten bracket.)

A natural definition is:

Definition 14. A map $F : N \to M$ is a *Poisson map* if it preserves the Poisson bracket, i.e.

$$\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N, f_1, f_2 \in C^\infty(M),$$

or equivalently in terms of the Poisson bivector,

$$(dF \otimes dF)\pi_N(x) = \pi_M(F(x)).$$

In other words, a Poisson map between two Poisson manifolds is a map whose pullback on functions is a Lie algebra homomorphism with respect to the Poisson brackets.

Also recall that for two Poisson manifolds $(M, \{\cdot,\cdot\}_M), (N, \{\cdot,\cdot\}_N)$, the product Poisson bracket on $C^\infty(M\times N)$ is given by:

$$\{f_1, f_2\}_{M\times N}(x,y) = \{f_1(\cdot,y), f_2(\cdot,y)\}_M(x) + \{f_1(x,\cdot), f_2(x,\cdot)\}_N(y).$$

Finally here is our basic definition:

Definition 15. A *Poisson-Lie group* is a Lie group G which has a Poisson structure such that the multiplication map $m : G\times G \to G$ is a Poisson map, where $G \times G$ is given the product Poisson structure.

The fact that m is a Poisson map can be expressed as:

$$\{f_1, f_2\}_G(xy) = \{f_1\circ R_y, f_2\circ R_y\}_G(x) + \{f_1\circ L_x, f_2\circ L_x\}_G(y),$$

or in terms of the Poisson bivector π_G:

$$\pi_G(xy) = (dR_y\otimes dR_y)\pi_G(x) + (dL_x\otimes dL_x)\pi_G(y),$$

or by some abuse of notation:

$$\pi_G(xy) = \pi_G(x)\cdot y + x\cdot \pi_G(y). \tag{$*$}$$

The natural definition for a Poisson action is now obvious: Let (G, π_G) be a Poisson-Lie group, and (P, π_P) be a Poisson manifold. Then an action $\sigma : G\times P \to P$ is a *Poisson action* if σ is a Poisson map. In terms of the Poisson bivectors, (note the abuse of notation again!!) this translates to:

$$\pi_P(g\cdot x) = \pi_G(g)\cdot x + g\cdot \pi_P(x).$$

Now let (G, π_G) be a Poisson-Lie group. By $(*)$, it is easy to see that the Poisson bivector vanishes at e, the identity element of G. So we can differentiate π_G and get a linear Poisson structure on $\mathfrak{g}$. This gives a Lie algebra structure on $\mathfrak{g}^*$. In fact this is an example of the correspondence between linear Poisson structures on a Lie algebra $\mathfrak{g}$ and Lie algebra structures on its dual $\mathfrak{g}^*$. Thus every Poisson-Lie group structure on a group G gives rise to a Lie bracket on $\mathfrak{g}^*$, and in fact is determined by this Lie algebra structure. (One can look at any of [CP], [CW], [R], [V] for a basic discussion of the above).

One can prove that, if π is a Poisson structure on G which makes G into a Poisson-Lie group, then $c = d\pi$ must satisfy the cocycle condition:

$$c([X,Y]) = [X, c(Y)] - [Y, c(X)]$$

where $[\cdot,\cdot]$ stands for the Schouten bracket. Conversely, if we start with the usual Lie algebra structure on $\mathfrak{g}$ (i.e. the one coming from the Lie group structure of G; denote it by $[\cdot,\cdot],$) and some Lie algebra structure on $\mathfrak{g}^*$ (denote this by $[\cdot,\cdot]^*$), the compatibility condition for these two to give rise to a Poisson-Lie group structure on G is summarized in the cocycle condition again. In fact, the following are equivalent:

(1) $[\cdot,\cdot]^*$, (more precisely, its dual map from $\mathfrak{g}$ into $\mathfrak{g}\wedge\mathfrak{g}$), satisfies the cocycle condition with respect to $[\cdot,\cdot]$,

(2) $[\cdot,\cdot]$, (more precisely, its dual map from $\mathfrak{g}^*$ into $\mathfrak{g}^*\wedge\mathfrak{g}^*$), satisfies the cocycle condition with respect to $[\cdot,\cdot]^*$,

(3) There is a Lie bracket on the vector space $\mathfrak{g}+\mathfrak{g}^*$ which extends the two brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$, and leaves invariant the natural inner product on $\mathfrak{g}+\mathfrak{g}^*$ given by:

$$(X+\eta, Y+\zeta) = \eta(Y)+\zeta(X) \;\; X,Y\in\mathfrak{g}, \eta,\zeta\in\mathfrak{g}^*,$$

(4) c in the cocycle condition can be integrated to give a multiplicative structure on the underlying group, (so if we are looking at the cocycle condition of (1), we integrate to get a multiplicative structure on G, whereas the corresponding procedure for the cocycle condition of (2) gives a multiplicative structure on the dual group G^*) [6].

Summarizing all the above, we have:

Proposition 1. *Every Poisson-Lie group G is accompanied by its dual G^*, a Poisson-Lie group, which is the (simply connected) Lie group with Lie algebra $\mathfrak{g}^*$, whose Poisson structure is determined by the Lie bracket on $\mathfrak{g}$.*

Observe that the dual of the dual Poisson-Lie group G^* is the universal covering group of G, by the simply connectedness hypothesis.

Next we describe the dressing actions, which will be the generalization of the coadjoint action to Poisson actions; these will be relevant when we want to generalize the notion of G-equivariance to moment maps for Poisson actions. (Also they will provide a basic example of Poisson actions as we will see below).

[6]When we have a compatible pair of Lie algebras $(\mathfrak{g},\mathfrak{g}^*)$ as above, we say that the pair is a Lie bialgebra. To see how this fits in with the general definition of a bialgebra in the second section, one notices first that the Lie coalgebra structure is defined as a skew-symmetric linear map $\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ called the *cocommutator*, which satisfies a certain condition (called the *coJacobi identity*). In other words, we may think of a Lie bialgebra as a space on which two compatible structures are given, one of which is a commutator, which gives a Lie algebra structure to our space, and the other is a cocommutator which gives a Lie algebra structure to its dual. Thus, the analogy with the definitions in the second section is clear: We define a *Lie bialgebra* to be a space which is both a Lie algebra and a Lie coalgebra, with the two structures satisfying the cocycle condition mentioned above.

For $X \in \mathfrak{g}$ let X^{left} (resp. X^{right}) denote the left-invariant (resp. right-invariant) 1-form on G^* which is equal to X at the identity (using the fact that $\mathfrak{g}$ and $\mathfrak{g}^*$ are dual). Given a 1-form α, the Poisson bivector π converts it to a vector field $\pi^{\#}(\alpha)$, (here we denote by $\pi^{\#}$ the bundle map $T^*G \to TG$ associated to π). The fields $\pi^{\#}(X^{left})$ (resp. $\pi^{\#}(X^{right})$) form a Lie algebra isomorphic to $\mathfrak{g}$. These induce two actions of the Lie algebra $\mathfrak{g}$ on G^* which are the exact analogs of the coadjoint action (We actually get: The linear part of the dressing action at the identity is the coadjoint action. Note only that, unlike the coadjoint action, the dressing actions need not be Hamiltonian). We call these actions the *infinitesimal dressing action*s.

We cannot in general integrate the infinitesimal dressing actions to get a global group action. However in the cases where we can, the *dressing action* of G on G^* is defined to be this (global) action. This happens precisely when the dressing action of the dual G^* on G is (globally) defined. The orbits of the dressing action of G^* on G are exactly the symplectic leaves of G.

3.2. **Actions and Moment Maps in the Poisson Case.** Here we will be summarizing [L1].

When G has the zero Poisson structure, its dual Poisson-Lie group is just $\mathfrak{g}^*$, with the abelian Lie group structure, and the left and right dressing actions of G are simply the left and right coadjoint actions of G on $\mathfrak{g}^*$. Recall that, in the symplectic case, the moment map is a map $J : P \to \mathfrak{g}^*$ for a symplectic action $G \times P \to P$, and the G-equivariance of the moment map is with respect to the coadjoint action of the group on the dual of its Lie algebra. Generalizing from the above observations, Lu in [L1] says that the moment map for a general left (resp. right) Poisson action $\sigma : G \times P \to P$ should be a map from P to the dual group G^* with some properties to be specified later, and the G-equivariance of the moment map should be with respect to the left (resp. right) dressing action of G on G^*.

We first wish to study Poisson actions a bit more in detail. We note that working with Poisson actions in general is very difficult, as the conditions for an action to be Poisson are rather weak; or equivalently, Poisson actions do not preserve enough structure. We therefore would like to study classes of actions which satisfy stronger conditions. Hence we make the following:

Definition 16. An action σ of a group G or a Lie algebra $\mathfrak{g}$ on a Poisson manifold (P, π_P) is *tangential* if for any element $X \in \mathfrak{g}$, the vector field generated by the action of X is tangent to each symplectic leaf of P.

In words, this means that an action on a Poisson manifold P is *tangential* if it leaves the symplectic leaves in P invariant. Every action on a symplectic manifold is tangential. The left and right (infinitesimal) dressing actions of $\mathfrak{g}^*$ on G, and the corresponding dressing actions of G^* on G (when they are defined) are tangential as well. On the other hand, the left action of a Poisson-Lie group on itself by left translations is never tangential.

Lu in [L1] gives a Maurer-Cartan-like criterion for a tangential action to be Poisson. (In fact, this is how she proves that the dressing actions, if complete, are Poisson). Thus as long as we are restricted to tangential actions, (which preserve symplectic leaves, so are naturally a good class of actions to restrict to), we have a way to determine whether we have a Poisson action.

We next give Lu's definition for moment maps for a Poisson action.

Definition 17. A C^∞ map $J : P \to G^*$ is called a *moment map* for the Poisson action $\sigma : G \times P \to P$ if for each $X \in \mathfrak{g}$,

$$\sigma_X = {\pi_P}^{\#}(J^*(X^{left})),$$

where σ_X is the vector field on P which generates the action σ_{exptX} on P, and ${\pi_P}^{\#}$ is the bundle map $T^*P \to TP$ associated to π_P. A Poisson action of G on P is said to *have a moment map* if there is a map J as above, which generates the action.

Similarly we can define ${\sigma'}_X = -{\pi_P}^{\#}(J^*(X^{right}))$; where X^{right} is the right-invariant 1-form corresponding to X. This will give us a right action of G on P. In this case one may note that the map $X \mapsto {\sigma'}_X$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on P, while the analogous map for σ_X is a Lie algebra antihomomorphism.

When G has the zero Poisson structure, the above reduces to the usual definition of a moment map (for symplectic actions); recall that in that case $G^* = \mathfrak{g}^*$. Note also that for both the left and the right dressing actions of G^* on G, when they are defined, the identity map of G is a moment map according to this definition.

Lu remarks that the expected version of Noether's theorem still holds for such a moment map:

Theorem 1. *Let $\sigma : G \times P \to P$ be a Poisson action of a Poisson-Lie group G on a Poisson manifold P with a moment map $J : P \to G^*$. If $H \in C^\infty(P)$ is G-invariant, then J is an integral of the Hamiltonian vector field X_H of H.*

Then she proves, modulo a proposition from Bourbaki, the following:

Theorem 2. *Let $\sigma : G \times P \to P$ be a Poisson action of a Poisson-Lie group G on a symplectic simply connected manifold P. Then for every*

$x \in P$ and $w \in G^*$, there is a unique moment map $J : P \to G^*$ with $J(x) = w$.

To make the generalized definition for G-equivariance, we restrict to complete G, i.e. we assume that the dressing vector fields are complete and thus we can integrate the infinitesimal dressing actions to a global action of G:

Definition 18. A moment map $J : P \to G^*$ for a Poisson action $\sigma : G \times P \to P$ is said to be *G-equivariant* if for every $g \in G$ $J \circ \sigma_g = \lambda_g \circ J$ where λ is the (left) dressing action of G on its dual G^*.

Then we get:

Theorem 3. *For a connected Poisson-Lie group G, a moment map $J : P \to G^*$ for a Poisson action $\sigma : G \times P \to P$ is G-equivariant if and only if J is a Poisson map.*

(This fits in well with the case when G has the zero Poisson structure. Recall that in that case there is an affine Poisson structure on $\mathfrak{g}^*$, such that J becomes a Poisson map with respect to this structure, and furthermore it can be shown to be equivariant, too, when we change the G action on $\mathfrak{g}^*$ by a 2-cocycle.)

Lu goes on further to describe reduction in the particular case when there is a tangential action which admits a moment map, but we will stop reviewing [L1] here.

4. Quantum Moment Maps

Finally we are ready to discuss Lu's quantum moment maps [L2]. We first define:

Definition 19. Let P be a Poisson manifold. A one-parameter family of noncommutative algebra structures denoted by $*_h$ on the vector space $Fun(P)$ is called a *quantization* of the Poisson structure $\{\cdot,\cdot\}_P$, if

(1) $*_0$ corresponds to the commutative multiplication; and
(2) the "derivative of $*_h$ at $h = 0$" is the Poisson bracket, i.e.

$$\{f, g\}_P = \lim_{h\to 0} \frac{1}{h}(f *_h g - g *_h f)$$

for $f, g \in Fun(P)$ [7].

Let G be a Lie group and let $(Fun(G), *_h, \Delta, S_h, \epsilon_h)$ be a quantum group. Then the semi-classical limit of $*_h$ is a Poisson structure on G

[7] We repeat here Lu's remark that any one-parameter family of noncommutative algebra structures $\{*_h\}$ on $Fun(P)$ which satisfies (1) will define a Poisson bracket $\{\cdot,\cdot\}_P$ on P given by the formula (2). This Poisson bracket is called the *semi-classical limit of* $*_h$.

such that the usual multiplication map $m : G \times G \to G$ is a Poisson map, i.e. such that G becomes a Poisson-Lie group with this structure. In this case we say that the quantum group $(Fun(G), *_h, \Delta, S_h, \epsilon_h)$ is a *quantization* of the Poisson-Lie group G.

Let H be a quantum group quantizing a Poisson-Lie group G, and let V be a quantization of a Poisson manifold P. If we have a $\sigma : V \to V \otimes H$ that defines a (right) H-comodule structure on V, and if σ is the pull-back (on functions) of some map $\sigma_0 : P \times G \to P$, then one can prove that σ_0 is a (right) Poisson action. In this case we are justified to say that H's (co)action on V is a *quantization* of the Poisson action σ_0. For example, the right H-comodule structure on a quantum group H itself given by the comultiplication of H is a quantization of the right action of the corresponding Poisson-Lie group on itself by right translations.

Recall the definition of a moment map in the Poisson setting (see Definition 17). Here we look at Hopf algebra theory to come up with a similar situation. In other words, we want to find a map, from the dual of a Hopf algebra to some algebra, with which we can define a Hopf algebra action [8]. Here is the well-known fact from Hopf algebra theory that Lu states (one can find this result in [M]):

Proposition 2. *If V is an algebra and if $\Phi : H^* \to V$ is an algebra homomorphism, then the map $D^\Phi : H^* \otimes V \to V$ defined by:*

$$x \otimes v \mapsto x(v) := \sum \Phi(x_{(1)}) v \Phi(S(x_{(2)}))$$

defines a (left) action of H^* on V and makes V into a (left) H^*-module algebra.

We note that the above action D^Φ defined by the map Φ is inner. (Check the definition of inner actions in Section 2.2 (Definition 13). The convolution invertible map u of the definition there will be our Φ, and as S is the inverse to id under convolution, we will have $\Phi^{-1}(x) = \Phi(S(x))$.)

This proposition suggests the following:

Definition 20. The map Φ of the above proposition (i.e. an algebra homomorphism $\Phi : H^* \to V$ defining the corresponding action $D^\Phi : H^* \otimes V \to V$ via

$$x \otimes v \mapsto x(v) := \sum \Phi(x_{(1)}) v \Phi(S(x_{(2)}))$$

[8] Although most of Lu's constructions in [L2] work for general Hopf algebras, we will keep in mind the special case where H is a quantum group quantizing a Poisson-Lie group G; in this case it follows from the Quantum Duality Principle in [STS] that H^* is a quantum group quantizing the dual group G^*. Similarly, we think of the H-module algebras and H^*-module algebras as quantizations of some Poisson manifolds on which our Poisson-Lie group G acts, in a Poisson manner.

of H^* on V) is called the *moment map* for the action D^Φ of H^* on V.

To justify this choice for the moment map, we look at the semi-classical limit of the action D^Φ. For this we assume that our Hopf algebras are associated in the proper way to Poisson objects. More precisely we assume that H^* is a quantization of G^*, the dual group of a Poisson-Lie group G. Also we assume that V is a quantization of a Poisson manifold P, and the map $\Phi : H^* \to V$ is the pullback of a map $\phi : P \to G^*$. Then ϕ is Poisson as Φ is an algebra homomorphism. Explicitly calculating semi-classical limits, we will have:

$$\lim_{h\mapsto 0}\frac{1}{h}(x(v)-\epsilon(x)v) =$$
$$= \lim_{h\mapsto 0}\frac{1}{h}\left[\sum\left(\Phi(x_{(1)})v\Phi(S_h(x_{(2)}))\right) - v\sum\left(\Phi(x_{(1)})\Phi(S_h(x_{(2)}))\right)\right] =$$
$$= \lim_{h\mapsto 0}\frac{1}{h}\sum\left[\left(\Phi(x_{(1)})v - v\Phi(x_{(1)})\right)\Phi(S_h(x_{(2)}))\right] =$$
$$= \sum\lim_{h\mapsto 0}\frac{1}{h}\left[\left(\Phi(x_{(1)})v - v\Phi(x_{(1)})\right)\Phi(S_h(x_{(2)}))\right] =$$
$$= \sum\left(\left\{\phi^*(x_{(1)}), v\right\}\phi^*(S_0(x_{(2)}))\right)$$

The bracket and the multiplication in the last line are respectively the Poisson bracket and the commutative multiplication of *Fun(P)*; in other words, the former is the semi-classical limit of $*_h$ and the latter is $*_0$ of V. The multiplications in the first four lines are the noncommutative multiplications $*_h$ of V. Also note that $(x,v)\mapsto \epsilon(x)v$ is the trivial action of H^* on V.

Define the vector field σ_x on P by

$$v \mapsto \{\phi^*(x_{(1)}), v\}\phi^*(S_0(x_{(2)}), v \in Fun(P)$$

or by

$$\sigma_x = -\phi^*(S_0(x_{(2)}))X_{\phi^*(x_{(1)})}$$

where $X_{\phi^*(x_{(1)})}$ is the hamiltonian vector field corresponding to the function $\phi^*(x_{(1)})$.

We see that in this way, each element x of $H^* = Fun(G^*)$ defines a vector field σ_x on P. The fact that D^Φ is a left action of H^* on V implies that the map $x \mapsto \sigma_x$ is a Lie algebra homomorphism from $(Fun(G^*), \{\cdot,\cdot\})$ to the Lie algebra of vector fields on P with the commutator bracket. Lu proves:

Theorem 4. *The infinitesimal action $x\mapsto\sigma_x$ of $(Fun(G^*),\{\cdot,\cdot\})$ on P is the same as the right infinitesimal action $X\mapsto{\sigma'}_X$ of $\mathfrak{g}$ on P induced by the Poisson map $\phi : P \to G^*$ as given by ${\sigma'}_X = -{\pi_P}^{\#}(\phi^*(X^{right}))$;*

where X^{right} is the right-invariant 1*-form corresponding to* X*. Therefore, the action* D^Φ *of* H^* *on* V *is a quantization of the Poisson action of* G *on* P *induced by* ϕ*.*

Thus we can see why Lu's definition for the quantum moment map is reasonable: The quantum moment map for the quantum action defined as above is related in a natural way to (more precisely is the pullback of) the moment map of the limiting Poisson action.

Thus having justified her choice for the quantum moment map, Lu goes on to discuss a few of the properties of the action D^Φ and of the map Φ. One such is the following:

Proposition 3. *The action* D^Φ *of* H^* *on* V *leaves every two-sided ideal of* V *invariant.*

This follows readily from the definition of D^Φ. The semi-classical result corresponding to this is the fact that a Poisson action on a Poisson manifold P with a moment map leaves the symplectic leaves of P invariant.

Let's consider a basic example. Recall, from Section 2.2, the definition of adjoint actions for a Hopf algebra (Example 7). One can see actually that the left (resp. right) action of H^* on itself induced by the identity map $id_{H^*} : H^* \to H^*$ turns out to be precisely the left (resp. right) adjoint action of H^* on itself.

In Section 2.2, the adjoint actions were defined for an arbitrary Hopf algebra. However if we again look at the special case which we are interested in, that is, if we assume that H is a quantum group associated to a Poisson-Lie group, then we can see that the above generalizes the notion of the dressing action. In fact the above theorem (Theorem 4) implies that the left (resp. right) adjoint action of H^* on itself is a quantization of the right (resp. left) dressing action G on G^*. Thus the adjoint actions defined above may be called *quantum dressing actions* as well.

In the Poisson case, the dressing actions came up in the study of G-equivariance. So it is natural to expect the adjoint actions to have some relevance in the corresponding notion for the quantum setting. Indeed in Lu's Proposition 3.13, we see that this expectation is justified:

Proposition 4. *The map* $\Phi : H^* \to V$ *is* H^**-equivariant with respect to the left adjoint action of* H^* *on itself and the left action* D^Φ *of* H^* *on* V*.*

To see the analogy, one needs only to review Theorem 3 of Section 3.2, and to note that the quantum moment map Φ corresponds to a quantum action which is the quantization of a Poisson action with Poisson moment map ϕ.

5. Final Remarks- Other Directions

Here we have discussed one method of quantizing the notion of moment maps. This method seems fruitful as it leads naturally to a generalization of the reduction construction; see [L2] for more on this. It might be interesting to compare the moment map defined here with the map described in [B], as both methods clearly involve deformations. We have mostly discussed the results in the special case where our Hopf algebras were associated to certain Poisson objects; clearly the constructions here were developed with the Poisson picture in mind. However it may also be of some interest to see how far one can get for the general Hopf algebra case.

6. BIBLIOGRAPHY:

[B] Bursztyn, H., "Momentum Maps and Dual Pairs in Deformation Quantization," Paper for Math 277, UC Berkeley, 2000.

[CP] Chari, V., and Pressley, A., *A Guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1995.

[CW] Cannas da Silva, A. and Weinstein, A., *Geometric Models for Noncommutative Algebras*, Berkeley Math. Lecture Notes, AMS 1999.

[D] Drinfeld, V. G., "Quantum Groups," Proc. ICM Berkeley 1986, Vol. 1 AMS 1986, pp. 789-820.

[L1] Lu, J. H., "Momentum Mappings and Reductions of Poisson Lie Group Actions", Proc. of the Séminaire Sud-Rhodanien de Géométrie à Berkeley, 1989, 1991 Springer-MSRI Series.

[L2] Lu, J. H., "Moment Maps at the Quantum Level", Commun. Math. Phys. 157 (1993), pp. 389-404.

[M] Montgomery, S., *Hopf Algebras and Their Actions on Rings*, AMS Reg. Conf. Ser. Math. no. 82, 1993.

[R] Reyman, A., "Poisson Structures Related to Quantum Groups", *Quantum Groups and their Applications in Physics*, Proc. International School of Physics, Course CXXVII (1994), 1996, pp. 407-445.

[STS] Semenov-Tian-Shansky, M. A., "Poisson Lie groups, Quantum Duality Principle, and the Quantum Double", *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups*, Contemp. Math., 175, AMS., Providence, RI, 1994, pp. 219-248.

[V] Vaisman, I., *Lectures on the Geometry of Poisson Manifolds*, Progr. in Math. 118, Birkhauser, Boston, 1994.