# Momentum Maps, Dual Pairs and Reduction in Deformation Quantization*

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**Abstract**

This paper is a brief survey of momentum maps, dual pairs and reduction in deformation quantization. We recall the classical theory of momentum maps in Poisson geometry and present its quantum counterpart. We also discuss quantization of momentum maps and applications of quantum momentum maps to quantum versions of Marsden-Weinstein reduction.

This paper is organized as follows. We recall the classical notions of momentum map, hamiltonian action and symplectic dual pair in Section 1. In Section 2.2 we discuss quantum momentum maps and show how they produce examples of quantum dual pairs in Section 2.4. The problem of quantizing momentum maps is briefly discussed in Section 2.5. We mention in Section 2.6 how quantum momentum maps play a role in quantum versions of the classical Marsden-Weinstein reduction procedure.

## 1 Momentum Maps, Symplectic Dual Pairs and Reduction

Let $(M, \omega)$ be a symplectic manifold and $G$ a Lie group. We will assume for simplicity that both $G$ and $M$ are connected. Let $\psi : G \times M \longrightarrow M$ be an action of $G$ on $M$ satisfying

$$\psi_g^* \omega = \omega.$$

Such an action is called *symplectic*. We will refer to the triple $(M, \omega, \psi)$ as a *symplectic G-space*. A *momentum map* for a symplectic action is a $C^\infty$ map

$$J : M \longrightarrow \mathfrak{g}^* \tag{1.1}$$

such that

$$i_{v_M} \omega = dJ^v, \tag{1.2}$$

where $v_M$ denotes the infinitesimal generator of the action corresponding to $v \in \mathfrak{g}$ and $J^v \in C^\infty(M)$ is defined by $J^v(x) = \langle J(x), v \rangle$. Note that $J$ naturally defines a linear map $\mathcal{J} : \mathfrak{g} \longrightarrow C^\infty(M)$ by $\mathcal{J}(v) = J^v$, called a *comomentum map*. Conversely, any linear map $\mathcal{J} : \mathfrak{g} \longrightarrow C^\infty(M)$

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with $i_{v_M}\omega = d\mathcal{J}(v)$ defines a momentum map. An action $\psi$ admitting a momentum map is called *weakly hamiltonian*.

Any symplectic action naturally defines a map $\psi_* : \mathfrak{g} \longrightarrow \mathsf{sp}(M) \subset \mathfrak{X}(M)$, $v \mapsto v_M$, where $\mathsf{sp}(M)$ denotes the set of symplectic vector fields on $M$. This map is a Lie algebra anti-homomorphism. Since $[\mathsf{sp}(M), \mathsf{sp}(M)] \subseteq \mathsf{ham}(M)$, where $\mathsf{ham}$ denotes the set of hamiltonian vector fields on $M$, it follows that $\psi_*([\mathfrak{g},\mathfrak{g}]) \subseteq \mathsf{ham}(M)$. Hence the map $\Psi : \mathfrak{g} \longrightarrow H^1(M,\mathbb{R})$, $\Psi(v) = [i_{\psi_*(v)}\omega]$ is a homomorphism of Lie algebras ($H^1(M,\mathbb{R})$ endowed with the zero bracket), with kernel containing $[\mathfrak{g},\mathfrak{g}]$. We observe that the existence of a momentum map corresponding to a symplectic action $\psi$ is equivalent to $\Psi = 0$. We can then state

**Proposition 1.1** *Let $\psi$ be a symplectic action of $G$ on $(M,\omega)$. If $H^1(M,\mathbb{R}) = 0$ or $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ (i.e. $H^1(\mathfrak{g},\mathbb{R}) = 0$), then $\psi$ is automatically weakly hamiltonian.*

Let $\psi$ be a weakly hamiltonian action. A momentum map $J : M \longrightarrow \mathfrak{g}^*$ is called *equivariant* if it is equivariant with respect to the coadjoint action of $G$ on $\mathfrak{g}^*$:

$$J \circ \psi_g = \mathsf{Ad}^*_g \circ J. \tag{1.3}$$

Under our connectedness assumptions, a momentum map is equivariant if and only if the corresponding comomentum map $\mathcal{J} : \mathfrak{g} \longrightarrow C^\infty(M)$ defines a Lie algebra homomorphism. This is also equivalent to $J : M \longrightarrow \mathfrak{g}^*$ being a Poisson map, where $\mathfrak{g}^*$ is endowed with the so-called Lie-Poisson structure [8, Sect. 3]. A symplectic action with an equivariant momentum map is called a *hamiltonian action* and the set $(M,\omega,\psi,J)$ is a *hamiltonian G-space.*

If $J$ is a momentum map for a weakly hamiltonian action $\psi$, then $\mathcal{J}([v,w])$ and $\{\mathcal{J}(v),\mathcal{J}(w)\}$ are both hamiltonians corresponding to the vector field $[v,w]_M$. Thus, since we are assuming $M$ connected, we have $c(v,w) = \mathcal{J}([v,w]) - \{\mathcal{J}(v),\mathcal{J}(w)\} \in \mathbb{R}$. Note that $c$ defines a skew-symmetric bilinear form on $\mathfrak{g}$, i.e. a 2-cochain. The Jacobi identity implies that $\delta c = 0$, where $\delta$ is the usual Chevalley differential of $\mathfrak{g}$. Moreover, any other choice of momentum map for $\psi$ only changes $c$ by a coboundary. So any weakly hamiltonian action $\psi$ defines an element $[c] \in H^2(\mathfrak{g},\mathbb{R})$ and $\psi$ is hamiltonian if and only if $[c] = 0$. So we have

**Proposition 1.2** *If $H^2(\mathfrak{g},\mathbb{R}) = 0$, then any weakly hamiltonian action is hamiltonian.*

We finally observe that equivariant momentum maps are unique up to addition of elements in $[\mathfrak{g},\mathfrak{g}]^\perp \subseteq \mathfrak{g}^*$.

**Proposition 1.3** *If $H^1(\mathfrak{g}) = 0$, then momentum maps for hamiltonian actions are unique.*

The discussion above can be carried out in the more general setting of actions on Poisson manifolds [8, Sect. 7].

Equivariant momentum maps play a fundamental role in the theory of symplectic reduction.

**Theorem 1.4 (Marsden-Weinstein-Meyer [23, 24])** *Let $\psi : G \times M \longrightarrow M$ be a hamiltonian action of $G$ on $M$ with momentum map $J$. Let $0 \in \mathfrak{g}^*$ be a regular value of $J$, and suppose that $G$ acts freely and properly on $J^{-1}(0)$. Then the orbit space $M_{red} = J^{-1}(0)/G$ is a manifold and there is a symplectic structure $\omega_{red}$ on $M_{red}$ uniquely determined by $i^*\omega = \pi^*\omega_{red}$, where $i : J^{-1}(0) \hookrightarrow M$ is the natural inclusion and $\pi$ is the projection of $J^{-1}(0)$ onto $M_{red}$.*

Let $\psi$ be a hamiltonian action of $G$ on $(M, \omega)$ with momentum map $J : M \longrightarrow \mathfrak{g}^*$. Suppose that $J$ has constant rank, so that its level sets form a foliation of $M$, and assume that the orbits of $\psi$ form a fibration. We note that these foliations are symplectically orthogonal to each other. Under these assumptions the quotient $M/G$ exists as a manifold, has a natural Poisson bracket inherited from $M$ and the projection $p : M \longrightarrow M/G$ is a Poisson map. We can hence consider the pair of Poisson maps

$$\mathfrak{g}^* \xleftarrow{J} M \xrightarrow{p} M/G. \tag{1.4}$$

In this case, the subalgebras $J^*(\mathfrak{g}^*)$ and $p^*(M/G) = C^\infty(M)^G$ of $C^\infty(M)$ are commutants of one another (with respect to $\{\cdot, \cdot\}$). It is shown in [8, 30] that, under certain regularity conditions, there is a one to one correspondence between the symplectic leaves of $\mathfrak{g}^*$ and $M/G$ and the transverse Poisson structures to corresponding leaves are anti-isomorphic. Hence $\mathfrak{g}^*$ and $M/G$ have closely related Poisson geometry. This motivates the following definition [30]. Let $M$ be a symplectic manifold and let $P_1, P_2$ be Poisson manifolds.

**Definition 1.5** *A symplectic dual pair is a diagram*

$$P_1 \xleftarrow{J_1} M \xrightarrow{J_2} P_2 \tag{1.5}$$

*of Poisson maps with symplectically orthogonal fibers.*

Orthogonality implies that $\{J_1^*(C^\infty(P_1)), J_2^*(C^\infty(P_2))\} = 0$. When the maps $J_i$ are complete surjective submersions with constant rank and each $J_i$ has connected fibers, the fibers of $J_1, J_2$ are symplectically orthogonal if and only if $J_1^*(C^\infty(P_1))$ and $J_2^*(C^\infty(P_2))$ are commutants of one another. If all these conditions are satisfied, and each $J_i$ has simply connected fibers, $P_1$ and $P_2$ are called *Morita equivalent* [32].

# 2 The Quantum Picture

## 2.1 Deformation quantization

In classical mechanics, phase spaces are represented by symplectic (or Poisson) manifolds $M$ and classical observables are elements in the Poisson algebra $C^\infty(M)$. In quantum mechanics, observables form a noncommutative associative algebra. *Deformation quantization* [3] provides a mathematical framework for the problem of quantizing classical mechanical systems. Its goal is to construct and analyze noncommutative algebras corresponding to Poisson algebras by means of *formal deformations* [11].

A formal algebraic deformation of $C^\infty(M)$, or a *star product*, is an associative algebra structure on $C^\infty(M)[[\hbar]]$, the space of formal power series with coefficients in $C^\infty(M)$, of the form

$$f \star g = \sum_{r=0}^{\infty} C_r(f, r)\hbar^r, \quad f, g \in C^\infty(M), \tag{2.1}$$

where $C_r : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ are $\mathbb{C}$-bilinear (Hochschild cochains) and $C_0(f, g) = f \cdot g$ (pointwise multiplication). We also assume that each $C_r$ is a differential operator in each entry. The existence and classification of star products have been established through the joint effort of many authors (see [13, 27, 31] for surveys), culminating with Kontsevich's Formality

Theorem [17]. We note that the associativity of $\star$ implies that $\{f,g\} := \frac{1}{\mathrm{i}}(C_1(f,g) - C_1(g,f))$ automatically defines a Poisson bracket on $C^\infty(M)$ [8, Sect. 19]. The deformation quantization problem is to construct and classify star products on a Poisson manifold $M$ deforming the pointwise multiplication of functions in the direction of the given Poisson bracket. We recall that two star products $\star$ and $\star'$ are called *equivalent* if there exist linear maps $T_r : C^\infty(M) \longrightarrow C^\infty(M)$ such that $\boldsymbol{T} = \mathsf{Id} + \sum_{r=1}^\infty T_r \lambda^r$ satisfies $\boldsymbol{T}(f \star g) = \boldsymbol{T}(f) \star' \boldsymbol{T}(g)$, for all $f, g \in C^\infty(M)$.

### 2.2 Quantum Momentum Maps

As discussed in Section 1, an equivariant momentum map is a a Poisson map $J : M \longrightarrow \mathfrak{g}^*$. This definition admits a dual formulation through a Lie algebra homomorphism $\mathcal{J} : \mathfrak{g} \longrightarrow C^\infty(M)$, where the Lie algebra structure on $C^\infty(M)$ is given by the Poisson bracket $\{\cdot,\cdot\}$. The $\mathfrak{g}$-action on $M$ corresponding to $\mathcal{J}$ is $v \mapsto \{\mathcal{J}(v), \cdot\}$, $v \in \mathfrak{g}$.

Let $\star$ be a star product on $M$, and let $C^\infty(M)_\hbar = (C^\infty(M)[[\hbar]], \star)$ be the corresponding quantized algebra. We denote by $[\cdot,\cdot]_\star$ the commutator with respect to $\star$ and define $[\cdot,\cdot]_\star^\hbar = \frac{\mathrm{i}}{\hbar}[\cdot,\cdot]_\star$.

**Definition 2.1 ([6, 9, 18])** *A quantum (co)momentum map is a Lie algebra homomorphism*

$$\boldsymbol{\mathcal{J}} : \mathfrak{g} \longrightarrow (C^\infty(M)_\hbar, [\cdot,\cdot]_\star^\hbar). \tag{2.2}$$

*The corresponding quantum $\mathfrak{g}$-action is given by*

$$\hat{v} f = [\boldsymbol{\mathcal{J}}(v), f]_\star^\hbar, \;\; v \in \mathfrak{g}, \;\; f \in C^\infty(M). \tag{2.3}$$

Note that we can write $\boldsymbol{\mathcal{J}} = \sum_{r=0}^\infty \mathcal{J}_r \hbar^r$, for linear maps $\mathcal{J}_r : \mathfrak{g} \longrightarrow C^\infty(M)$. An easy computation shows that $\mathcal{J}_0$ is a Lie algebra homomorphism, and hence the classical limit of a quantum momentum map is a momentum map in the classical sense.

Observe that a quantum momentum map as in (2.2) is equivalent to a Lie algebra homomorphism, also denoted by $\boldsymbol{\mathcal{J}}$,

$$\boldsymbol{\mathcal{J}} : \mathfrak{g}_\hbar \longrightarrow C^\infty(M)_\hbar^{Lie}, \tag{2.4}$$

where $\mathfrak{g}_\hbar$ is the deformed Lie algebra $(\mathfrak{g}[[\hbar]], -\mathrm{i}\hbar[\cdot,\cdot])$, and $C^\infty(M)_\hbar^{Lie} = (C^\infty(M)[[\hbar]], [\cdot,\cdot]_\star)$. Using the universal property of universal enveloping algebras (see [8, Sect. 1]), we get the following equivalent definition of quantum momentum maps.

**Definition 2.2 ([33])** *A quantum (co)momentum map is an associative algebra homomorphism*

$$\boldsymbol{\mathcal{J}} : U_{\mathfrak{g}_\hbar} \longrightarrow C^\infty(M)_\hbar, \tag{2.5}$$

*where $U_{\mathfrak{g}_\hbar}$ is the universal enveloping algebra of $\mathfrak{g}_\hbar$. The corresponding quantum $\mathfrak{g}$-action is given by*

$$\hat{v} f = \frac{\mathrm{i}}{\hbar}[\boldsymbol{\mathcal{J}}(v), f]_\star, \;\; v \in \mathfrak{g}, \;\; f \in C^\infty(M).$$

Note that we have a natural isomorphism (of modules) $U_{\mathfrak{g}_\hbar} \cong U_{\mathfrak{g}}[[\hbar]]$ and a vector space isomorphism $pol(\mathfrak{g}^*) \cong U_{\mathfrak{g}}$ given by symmetrization [8]. Hence the algebra structure of $U_{\mathfrak{g}_\hbar}$ induces a star product on $pol(\mathfrak{g}^*)[[\hbar]]$, which extends to a star product on $C^\infty(\mathfrak{g}^*)[[\hbar]]$. This star product is a deformation quantization of the Lie-Poisson structure on $\mathfrak{g}^*$. It turns out that quantum momentum maps admit natural extensions from $pol(\mathfrak{g}^*)[[\hbar]]$ to $C^\infty(\mathfrak{g}^*)[[\hbar]]$.

**Lemma 2.3 ([33])** *Let $\mathfrak{J} : pol(\mathfrak{g}^*)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$ be a quantum momentum map. Then it naturally extends to an algebra homomorphism $\mathfrak{J} : C^\infty(\mathfrak{g}^*)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$.*

Lemma 2.3 shows that Definition 2.2 provides a quantization of the Poisson map $J^* : C^\infty(\mathfrak{g}^*) \longrightarrow C^\infty(M)$ corresponding to a classical equivariant momentum map $J : M \longrightarrow \mathfrak{g}^*$.

The formulation of quantum momentum maps in Definition 2.1 appeared originally in the literature in connection with quantum symplectic reduction [6, 9]. The formulation in Definition 2.2 seems to be better suited for the treatment of quantum dual pairs [33]. We will discuss these two topics in the next sections.

## 2.3 Quantum $G$-Actions

Let $(M, \omega, \psi)$ be a symplectic $G$-space. A star product $\star$ on $M$ is called *$G$-equivariant* if

$$\psi_g^*(f \star h) = \psi_g^*(f) \star \psi_g^*(h), \quad f, h \in C^\infty(M), \;\; g \in G. \tag{2.6}$$

$G$-equivariant star products need not exist in general (see [2] and references therein). By means of Fedosov's construction [10], one can show that $G$-equivariant star products exist whenever $M$ admits a $G$-invariant symplectic connection [9] (see [4] for a classification of $G$-invariant star products) . This is the case, for example, when $G$ is compact. Throughout this section, we will always assume star products $\star$ to be $G$-equivariant. The set $(M, \omega, \psi, \star)$, consisting of a symplectic $G$-space and a $G$-equivariant star product, will be called a *quantum symplectic $G$-space*.

If $(M, \omega, \psi, \star)$ is a quantum symplectic $G$-space, then the corresponding infinitesimal action defines a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathsf{Der}(C^\infty(M)_\hbar)$, $v \mapsto \hat{v}$, where $\mathsf{Der}$ denotes the derivations of the star-product algebra $C^\infty(M)_\hbar := (C^\infty(M)[[\hbar]], \star)$. If this $\mathfrak{g}$-action corresponds to a quantum momentum map $\mathfrak{J} = \mathcal{J}$, we call $(M, \omega, \psi, \star)$ a *quantum hamiltonian $G$-space*.

It is clear that a necessary condition for the existence of a quantum momentum map is that all the derivations $\hat{v}$ arising from the infinitesimal action are inner. We recall that the set of equivalence classes of derivations of a star product $\star$ on a symplectic manifold $M$ modulo inner derivations is in bijection with $H^1(M)[[\hbar]]$ [5, Sect. 4]. Hence if $H^1(M) = 0$, any derivation of $\star$ is inner.

Let us assume for the moment that all derivations of the form $\hat{v}$ are inner. We have a linear map $\mathfrak{g} \longrightarrow C^\infty(M)[[\hbar]]$, $v \mapsto a_v$, so that

$$\hat{v} f = \frac{\mathrm{i}}{\hbar}[a_v, f]_\star. \tag{2.7}$$

A simple computation shows that $[a_{[u,v]} - \frac{\mathrm{i}}{\hbar}[a_u, a_v]_\star, f]_\star = 0$ for all $f \in C^\infty(M)$ and hence $a_{[u,v]} - \frac{\mathrm{i}}{\hbar}[a_u, a_v]_\star \in \mathbb{C}[[\hbar]]$. We can then define a map $\lambda : \bigwedge^2 \mathfrak{g} \longrightarrow \mathbb{C}[[\hbar]]$ by

$$\lambda(u, v) = a_{[u,v]} - \frac{\mathrm{i}}{\hbar}[a_u, a_v]_\star, \;\; u, v \in \mathfrak{g}. \tag{2.8}$$

It is not hard to check that in fact $\lambda$ is a Lie algebra 2-cocycle and that its cohomology class $[\lambda] \in H^2(\mathfrak{g}, \mathbb{C}[[\hbar]])$ is independent of the choice of linear map $v \mapsto a_v$.

**Proposition 2.4 ( [33])** *Suppose all the derivations of the form $\hat{v}$ are inner. Then a quantum momentum map for $(M, \omega, \psi, \star)$ exists if and only if $[\lambda] = 0 \in H^2(\mathfrak{g}, \mathbb{C}[[\hbar]])$.*

We then have the following sufficient conditions for the existence of quantum momentum maps.

**Theorem 2.5 (Xu, [33])** *Let $(M, \omega, \psi, \star)$ be a quantum symplectic $G$-space. If $H^1(M) = 0$ and $H^2(\mathfrak{g}) = 0$, then there exists a quantum momentum map.*

Observe that these are exactly the same sufficient conditions for the existence of classical equivariant momentum maps in Propositions 1.1 and 1.2.

As in the classical case, when quantum momentum maps exist, they are not necessarily unique. If $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are two quantum momentum maps, then the map $\tau : \mathfrak{g} \longrightarrow C^\infty(M)[[\hbar]]$ defined by $\tau(v) = \mathfrak{J}_1(v) - \mathfrak{J}_2(v)$ satisfies $[\tau(v), f]_\star = 0$ for all $f \in C^\infty(M)$. Thus $\tau(v) \in \mathbb{C}[[\hbar]]$, $\forall v \in \mathfrak{g}$. It is not hard to see that $\tau : \mathfrak{g} \longrightarrow \mathbb{C}[[\hbar]]$ is in fact a 1-cocycle. We then have an analog of Prop. 1.3.

**Proposition 2.6 (Xu, [33])** *If $H^1(\mathfrak{g}) = 0$, then quantum momentum maps are unique.*

The similarity between Theorem 2.5 and Propositions 1.1 and 1.2 suggests that the existence of quantum momentum maps should be related to the existence of classical momentum maps. On one hand, a simple computation shows that if $(M, \omega, \psi, \star)$ is a quantum hamiltonian $G$-space with quantum momentum map $\mathfrak{J} = \mathcal{J} + o(\hbar)$, then the action of $G$ on $M$ is hamiltonian, with momentum map $\mathcal{J}$. We will discuss the converse, i.e. the problem of quantizing momentum maps, in Section 2.5.

## 2.4 An Example of a Quantum Dual Pair

If $A$ is an associative algebra and $B \subset A$ is a subalgebra, we denote the commutant of $B$ by $B'$. Let $C^\infty(M)^G$ be the space of $G$-invariant functions on $M$. Then the following holds.

**Theorem 2.7 (Xu, [33])** *Let $(M, \omega, \psi, \star)$ be a quantum hamiltonian $G$-space with quantum momentum map $\mathfrak{J}$. Suppose that $\psi$ is free and proper. Then*

$$(C^\infty(M)^G[[\hbar]])' \cong \mathfrak{J}(C^\infty(\mathfrak{g}^*)[[\hbar]]) \quad \textit{and} \quad (\mathfrak{J}(C^\infty(\mathfrak{g}^*)[[\hbar]])' \cong (C^\infty(M)^G[[\hbar]]). \tag{2.9}$$

Hence the subalgebras $C^\infty(M)^G[[\hbar]]$ and $\mathfrak{J}(C^\infty(\mathfrak{g}^*)[[\hbar]])$ of $C^\infty(M)[[\hbar]]$ are mutual commutants.

Recall that, classically, when the $G$-action $\psi$ is free and proper, the Poisson manifolds $\mathfrak{g}^*$ and $M/G$ form a dual pair and the Poisson subalgebras $J^*(C^\infty(\mathfrak{g}^*))$ and $p^*(C^\infty(M/G)) \cong C^\infty(M)^G$ of $C^\infty(M)$ are mutual (Poisson) commutants. Since $C^\infty(M)^G[[\hbar]]$ provides a quantization of $M/G$ and $C^\infty(\mathfrak{g}^*)[[\hbar]]$ quantizes the Lie-Poisson structure on $\mathfrak{g}^*$, it is natural to call the pair $C^\infty(M)^G[[\hbar]]$ and $C^\infty(\mathfrak{g}^*)[[\hbar]]$ a *quantum dual pair*. In general, we have

**Definition 2.8** *Let $P_1, P_2$ be two Poisson manifolds. Consider the algebras $(C^\infty(P_1)[[\hbar]], \star_1)$, $(C^\infty(P_2)[[\hbar]], \star_2)$. We call them a quantum dual pair if there exists a symplectic manifold $M$, a star product $\star$ on $C^\infty(M)[[\hbar]]$ and algebra homomorphisms $\rho_1 : C^\infty(P_1)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$, $\rho_2 : C^\infty(P_2)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$ so that $\rho_1(C^\infty(P_1)[[\hbar]])$ and $\rho_2(C^\infty(P_2)[[\hbar]])$ are mutual commutants.*

It is not known in general whether a classical dual pair (or Morita equivalent Poisson manifolds [32]) can be quantized to a quantum dual pair. It would be interesting to phrase the previous results in terms of bimodules and investigate geometric conditions under which a quantum dual pair is Morita equivalent in the algebraic sense.

## 2.5 Quantization of Momentum maps

As mentioned in the end of Section 2.3, if $(M, \omega, \psi, \star)$ is a quantum hamiltonian $G$-space, then $M$ is a hamiltonian $G$-space, with momentum map given by the classical limit of the quantum momentum map. We will discuss here the converse problem: starting with a hamiltonian $G$-space $(M, \omega, \psi, \mathcal{J})$, can we find a $G$-equivariant star product $\star$ on $M$ so that $(M, \omega, \psi, \star)$ is a quantum hamiltonian $G$-space with quantum momentum map $\boldsymbol{\mathcal{J}} = \mathcal{J} + o(\hbar)$? In other words, we need a Lie algebra homomorphism $\boldsymbol{\mathcal{J}} = \mathcal{J} + o(\hbar) : \mathfrak{g} \longrightarrow (C^\infty(M)[[\hbar]], \frac{\mathsf{i}}{\hbar}[\cdot,\cdot]_\star)$ so that

$$\{\mathcal{J}(v), f\} = \frac{\mathsf{i}}{\hbar}[\boldsymbol{\mathcal{J}}(v), f]_\star, \;\; v \in \mathfrak{g}, \; f \in C^\infty(M).$$

An even stronger condition on $\star$ is the following.

**Definition 2.9** *Let $(M, \omega, \psi, \mathcal{J})$ be a hamiltonian $G$-space. A star product $\star$ is strongly invariant if $\frac{\mathsf{i}}{\hbar}[\mathcal{J}(v), f]_\star = \{\mathcal{J}(v), f\}$, for all $v \in \mathfrak{g}$ and $f \in C^\infty(M)$.*

In this case, the quantum $\mathfrak{g}$-action has a quantum momentum map given by $\mathcal{J}$ itself.

Recall that given a symplectic connection on $M$ and a series $\Omega = \sum_i \omega_i \hbar^i$, $\omega_i$ closed 2-forms on $M$, we can define a corresponding star product $\star$ by means of Fedosov's contruction [10]. The equivalence class of this star product does not depend upon the choice of symplectic connection. It does depend though on the deRham cohomology classes $[\omega_i] \in H^2(M)$. In fact, one can show that any star product on $M$ is equivalent to a Fedosov star product, and the set of equivalence classes of star products on $M$ is in one to one correspondence with $H^2(M)[[\hbar]]$ [5, 25, 26]. For a star product $\star$, the corresponding element in $H^2(M)[[\hbar]]$ is called its *characteristic class*. The above discussion shows, in particular, that there is (up to isomorphism) only one deformation quantization of $M$ whose characteristic class is independent of $\hbar$. Such a deformation quantization is called *canonical* in [9].

Let $(M, \omega, \psi, J)$ be a hamiltonian $G$-space and suppose we have a $G$-invariant symplectic connection $\nabla$ on $M$.

**Theorem 2.10 ([9, 18])** *Under the assumptions of the classical Marsden-Weinstein Theorem (Theorem 1.4), the Fedosov star product corresponding to $(\nabla, \omega)$ is strongly invariant.*

Hence if $(M, \omega, \psi, \mathcal{J})$ is a classical hamiltonian $G$-space (with, for example, a free and proper action), and $M$ admits a $G$-invariant connection, any canonical star product on $M$ defines a hamiltonian quantum $G$-space, with quantum momentum map $\boldsymbol{\mathcal{J}} = \mathcal{J} + o(\hbar)$. This follows since any canonical star product is equivalent to the star product constructed in Theorem 2.10. However, "quantum corrections" will be necessary in general (see [6] for an example).

A generalization of Theorem 2.10 for star products with more general characteristic classes can be found in [18, Prop. 4.3].

## 2.6 Quantum Symplectic Reduction

The symplectic reduction procedure has been intensively studied in the framework of geometric quantization. We will briefly discuss in this section analogs in deformation quantization of the so-called Guillemin-Sternberg conjecture ("Quantization commutes with reduction") [12], which is now completely proven in its original context of geometric quantization [22, 28].

Let $(M, \omega, \psi, J)$ be a hamiltonian $G$-space, and suppose that $0 \in \mathfrak{g}^*$ is a regular value of $J$, so that $M_0 = J^{-1}(0)$ is a smooth submanifold of $M$. Assume that the action of $G$ on $M_0$ is proper and free. The Marsden-Weinstein reduction theorem (see Theorem 1.4) states that the orbit space $M_{red} = M_0/G$ inherits a natural symplectic structure $\omega_{red}$.

This reduction procedure admits a description at the level of algebra of functions as follows. Fix $e_1, \dots, e_n$ a basis of $\mathfrak{g}$, and let $J_i = \langle J, e_i \rangle$, $i = 1, \dots, n$. Let $A = C^\infty(M)$ be the usual Poisson algebra of functions on $M$, and let

$$A_0 = \{f \in A \mid \{J_i, f\} = 0\}.$$

This is a subalgebra of the $G$-invariant functions $C^\infty(M)^G$. Define

$$I = \{f \in A_0 \mid f = \sum_i g_i J_i,\ g_i \in A\},$$

which is an ideal in $A_0$. Then the quotient $R = A_0/I$ naturally inherits a Poisson algebra structure. One can check the following algebraic version of Theorem 1.4.

**Proposition 2.11** *The Poisson algebra $R$ is isomorphic to $C^\infty(M_{red})$ (with Poisson structure induced by $\omega_{red}$).*

Let $(M, \omega, \psi, J)$ be a hamiltonian $G$-space, and suppose that we can find a $G$-invariant symplectic connection $\nabla$ on $M$. Then we know by Theorem 2.10 that the corresponding canonical Fedosov star product $\star$ is strongly invariant. Let $\boldsymbol{A} = (C^\infty(M)[[\hbar]], \star)$. We consider the algebra

$$\boldsymbol{A}_0 = \{\boldsymbol{f} \in \boldsymbol{A} \mid [J_i, \boldsymbol{f}]_\star = 0\}$$

and the left ideal $\boldsymbol{I} \subset \boldsymbol{A}$ generated by $J_i$,

$$\boldsymbol{I} = \{\boldsymbol{f} \in \boldsymbol{A}_0 \mid \boldsymbol{f} = \sum_i \boldsymbol{g}_i \star J_i,\ \boldsymbol{g}_i \in \boldsymbol{A}\}.$$

We can then define the *quantum reduced algebra* $\boldsymbol{R} = \boldsymbol{A}_0/\boldsymbol{I}$.

**Theorem 2.12 (Fedosov, [9])** *Under the assumptions of the classical Marsden-Weinstein Theorem, the quantum reduced algebra $\boldsymbol{R}$ is isomorphic to the algebra obtained by canonical deformation quantization of $R = C^\infty(M_{red})$.*

Another way of describing the classical Marsden-Weinstein reduction procedure at the level of algebras is through the method of BRST cohomology. This method is frequently used in physics for the quantization of constrained systems [14], and it was described in terms of classical symplectic reduction in [16].

The method of BRST cohomology was first introduced in the context of deformation quantization in [7]. Starting with an arbitrary strongly-invariant star product on $M$ and assuming usual regularity conditions on the $G$-action on $M$, this method provides a way of constructing star products on reduced spaces [7, Thm. 32]. This method is still to be compared with Fedosov's approach (for instance, starting with the canonical star product on $M$, it would be interesting to compute the characterisitic class of the corresponding reduced star product given by the BRST method).

Explicit formulas for star products on $\mathbb{C}\mathrm{P}^n$ were obtained in [6] by means of quantum Marsden-Weinstein reduction. It was observed in this example [29] that starting with equivalent star products, one can end up with non-equivalent ones after phase-space reduction (for that, however, reduction is carried out at a nonzero momentum level).

## 2.7 Other Notions of Quantum Momentum Maps

A generalization of the notions of momentum map and symplectic reduction (as in Section 1) for Poisson actions of Poisson-Lie groups on symplectic (or Poisson) manifolds was developed by Lu in [20]. A momentum map for a Poisson action of $G$ on $M$ is, in this case, a Poisson map from $M$ to the dual group $G^*$. When $G$ has the trivial Poisson structure, $G^*$ coincides with $\mathfrak{g}^*$ (with group operation given by addition), and (1.1) is recovered.

Poisson-Lie groups arise as semi-classical limits of quantum groups, and Poisson actions are semi-classical limits of quantum group actions on module algebras (see [21, Sect. 2] for the definitions). In [21], Lu proposes a definition of quantum momentum maps corresponding to Hopf-algebra actions and shows that the semi-classical limit of this notion coincides with the aforementioned notion of momentum map for Poisson-Lie group actions. This is the subject of Karaali's survey paper [15]. We will just briefly sketch how Lu's quantum momentum maps are related to Definition 2.1.

Let $A$ be a Hopf algebra and let $A^*$ be its dual Hopf algebra. If $A$ is a quantization of a Poisson-Lie group $G$, then $A^* \cong U_\hbar\mathfrak{g}$ (up to completion), and $A^*$-module algebras are quantum analogs of Poisson actions of $G$. Let $V$ be an algebra and suppose $\mathfrak{J} : A^* \longrightarrow V$ is an algebra homomorphism. Let $\Delta$ and $S$ denote the co-product and the antipode map in $A^*$, respectively. One can show that the map

$$D : A^* \otimes V \longrightarrow V, \;\; x \otimes v \mapsto \mathfrak{J}(x_{(1)})v\mathfrak{J}(S(x_{(2)})),$$

where $\Delta(x) = x_{(1)} \otimes x_{(2)}$, makes $V$ into a left $A^*$-module algebra. Lu defines in [21] the map $\mathfrak{J}$ to be the *quantum momentum map* for the action $D$ of $A^*$ on $V$. In the case $V = C^\infty(M)_\hbar$ is a $G$-equivariant quantization of a symplectic (or Poisson) manifold $M$ (we are considering the action of the trivial quantum group $Fun(G)$ by pull-back), then a quantum momentum map in Lu's sense is an algebra homomorphism $\mathfrak{J} : U_\mathfrak{g} \longrightarrow C^\infty(M)_\hbar$. Note that, in this case, $\Delta(x) = 1 \otimes x + x \otimes 1$ and $S(x) = -x$, for $x \in \mathfrak{g}$. Thus the corresponding action $D$ is just $D(x \otimes f) = \mathfrak{J}(x) \star f - f \star \mathfrak{J}(x) = [\mathfrak{J}(x), f]_\star$, which agrees with the action in Definition 2.1 (up to the $\frac{i}{\hbar}$ factor). It would be very interesting to related Lu's quantum Marsden-Weinstein reduction [21] to the results described in Section 2.6.

Finally, we mention that another notion of quantum momentum map is discussed by Landsman in [19], based on analogies between the classical Marsden-Weinstein reduction and Rieffel's induction of representations of $C^*$-algebras; in this setting, Hilbert $C^*$-modules play the role of momentum maps (see [1] for a survey).

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