DEFORMATION QUANTIZATION - A BRIEF SURVEY

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1. Introduction

Quantization is, most broadly, the process of forming a quantum mechanical system starting from a classical mechanical one. See (Be) for an early attempt to obtain a general definition of quantization. (AbM) also provides an introductory account of the subject. There are various methods of quantization; see (BW) for a general introduction to the geometry of quantization, and a specific geometric method (geometric quantization).

In this survey we will be interested in deformation quantization. Intuitively a deformation of a mathematical object is a family of the same kind of objects depending on some parameter(s). The deformation of algebras is central to our problem, and in particular we are concerned with the deformations of function algebras. We use the Poisson bracket to "deform" the ordinary commutative product of observables in classical mechanics, elements of our function algebra, and obtain a noncommutative product suitable for quantum mechanics.

In the first section of this paper we will give a short overview of the main ideas in deformation quantization. In the second section we will go into more (historical) details about the directions in which the area developed in the last 20 years and in the last section we will try to give a very sketchy summary of recent results by Kontsevich.

2. Deformation Quantization: An Overview

The basic setup in deformation theory is as follows: We start with an algebraic structure, e.g. a Lie algebra or an associative algebra and we ask: Does there exist a one-(or $n-$)parameter family of similar structures such that for an initial value (say zero) of the parameter(s) we get the structure we started with? (Fl) The question also has a purely mathematical interest when considered in the most abstract sense, but here we will be mainly interested in the more specific questions which arise within the subject of quantization.

In general the observables of classical mechanics are identified with smooth real-valued functions on a Poisson manifold, and they form a commutative (Poisson) algebra. The traditional quantum formalism on the other hand interprets observables as certain operators on a Hilbert space (and these do not commute!). Instead of forcing quantization to involve such a radical change in the nature of the observables, the authors of the influential papers (BFFLS1,BFFLS2) suggested that it be understood as a deformation of the structure of the algebra of classical observables. The main idea can be traced back to Dirac (Di), who noted that quantization could be thought of as taking a Poisson manifold and putting a new noncommutative product on the algebra of functions, say $f * g$, such that the commutator $[f, g] = f * g - g * f$ is equal to $-i\hbar\{f, g\}$ plus terms of order $\hbar^2$. [We can safely assume that Dirac actually noticed that the higher order terms in $\hbar$ do come up, but he chose to disregard them. In (Di) we see that the suggested method of quantization with only a linear term in $\hbar$ is almost always taken to be an approximation to the real quantum object.]

2.1. **Some Definitions and Preliminaries:**

Definition 1. A *Poisson algebra* is a real vector space A equipped with a commutative associative algebra structure

$$(f, g) \longrightarrow fg$$

and a Lie algebra structure

$$(f, g) \longrightarrow \{f, g\}$$

which satisfy the compatibility condition

$$\{fg, h\} = f\{g, h\} + \{f, h\}g$$

Definition 2. A *Poisson manifold* is a manifold M whose function space $C^\infty(M)$ is a Poisson algebra with respect to the usual pointwise multiplication of functions and a prescribed Lie algebra structure.

Definition 3. A *formal deformation* of the algebra $A = C^\infty(M)$, or equivalently a *star-product* *, is defined to be a map

$$* : A \times A \longrightarrow A[\hbar]$$

$$(f, g) \longmapsto \sum_{k=0}^{\infty} c_k(f, g)\hbar^k$$

satisfying

(i) formal associativity, i.e. for all $p \geq 0$

$$\sum_{k+l=p} [c_k(c_l(f,g),h)) - c_k(f, c_l(g,h))] = 0$$

(ii) $c_0(f,g) = fg$
(iii) $(1/2)(c_1(f,g) - c_1(g,f)) = \{f,g\}$ where $\{,\}$ is the Poisson bracket.
(iv) Each map $c_k : A \times A \longrightarrow A$ should be a bidifferential operator.

Definition 4. A *formal deformation* of the Poisson bracket is a skew-symmetric map

$$[\ ,\ ] : A \times A \longrightarrow A[\hbar]$$

$$(f,g) \longmapsto \sum_{k=0}^{\infty} T_k(f,g)\hbar^k$$

satisfying:
(i) the formal Jacobi identity, i.e. for all $p \geq 0$

$$\sum \Big(\sum_{k+l=p} T_k(T_l(f,g),h)) = 0$$

where the outer sum is taken over the cyclic permutations of the set $\{f,g,h\}$.
(ii) $T_0(f,g) = \{f,g\}$ where $\{,\}$ is the Poisson bracket.
(iii) Each map $T_k : A \times A \longrightarrow A$ should be a bidifferential operator.

For variations of these definitions see (Br).

In classical mechanics the phase space M is the cotangent bundle of the configuration space which is a smooth manifold. The observables are smooth functions on the phase space, and under the ordinary pointwise multiplication the smooth functions make $C^\infty(M)$ into a commutative algebra. The canonical symplectic structure on M induces a Poisson structure on $C^\infty(M)$. [The standard Poisson structure is:

$$\{f,g\} = X_g \cdot f]$$

Thus the question of quantization becomes: Can we find a formal deformation of the function algebra of an arbitrary Poisson manifold? Equivalently, can we define an associative multiplication operation *, a star-product, depending on the parameter $\hbar$, of two functions so that the function space $C^\infty(M)$ with usual linear operators and this star-product will be a formal deformation of the commutative algebra of functions with the Poisson bracket?

2.2. **The Classical Example - The Moyal Product.** .

[Here we will be following (very closely) the exposition in (Ka).] Let $M = R^d$ and let α be a constant Poisson structure on M. To be more specific, let

$$\alpha = \sum_{i,j} \frac{1}{2} \alpha^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \text{ where } \alpha^{ij} = -\alpha^{ji} \in R$$

where the x_i are the coordinates on R^d. In such a case we would have:

$$\{f, g\} = \sum_{i,j} \alpha^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

The Moyal *-product is then given by exponentiating this Poisson operator:

$$f * g = \left(\exp(-i \frac{\hbar}{2} \sum_{i,j} \alpha^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}) \right) (f, g)$$

This is the first known example of a non-trivial deformation of the Poisson bracket and the idea can be generalized to any Poisson manifold equipped with a flat torsionless Poisson connection. See (BW), (Ka) or (W) for more details.

2.3. **Other Directions in Deformation Quantization.** .

The question of existence of star-products has been extensively studied. We will see in the next section a short account of these developments. For a more detailed account see (Br). Another direction in which research in deformation quantization has developed is strict deformation quantization in which the parameter $\hbar$ is no longer a formal parameter, but a real one. In a way, the deformed algebras $A[\hbar]$ are identified with the original algebra A. For a survey of this research see (Bh).

3. Deformation Quantization: Historical Development

The results about deformation quantization came in gradually, as existence proofs in increasing levels of generality. For instance the generalization of the Moyal *-product, as mentioned above, led to some existence results. Even more general results could be obtained by cohomological methods (G), (GeS). [The following will involve some terminology that will be defined shortly.] In attempting to solve the existence problem for the c_k recursively at each stage (referring to the definition of the formal deformation of an algebra given in 2.1) we obtain an equation of the form

$$\partial c_j = F_j$$

where F is a quadratic expression in the terms determined before. The operator ∂ goes from bilinear to trilinear A-valued functionals on A and is precisely the coboundary operator for Hochschild cohomology with values in A of the algebra A with multiplication given by c_0. (W). It turns out that the obstruction to extending the star-product belongs to the third Hochschild cohomology group of the commutative algebra A, while the obstruction to extending a deformation of the Poisson bracket belongs to the third Chevalley cohomology group of the Lie algebra $(A, \{,\})$.

Here we digress slightly to give some definitions for the terms used above:

Let A be an associative algebra (over some commutative ring K) and for simplicity assume it is a module over itself with the adjoint action (i.e. algebra multiplication). [The generalization to cohomology valued in a general module is intended.]

Definition 5. A *p-cochain* is a p-linear map C from A^p into the module A, and its coboundary ∂C is given by:

$$\partial C(u_0, u_1, ..., u_p) = u_0 C(u_1, u_2, .., u_p) - C(u_0 u_1, u_2, .., u_p)$$
$$+ .. + (-1)^p C(u_0, u_1, .., u_{p-1} u_p) + C(u_0, u_1, .., u_{p-1}) u_p$$

This is a complex, i.e. $\partial^2 = 0$.

Definition 6. A p-cochain C is a *p-cocycle* if $\partial C = 0$.

Definition 7. Let $Z^p(A, A)$ be the space of all p-cocyles and $B^p(A, A)$ the space of those p-cocycles that are coboundaries (of a (p-1)-cochain). The *pth Hochschild cohomology space* (of A valued in A) is defined as

$$H^p(A, A) = Z^p(A, A)/B^p(A, A)$$

The *Chevalley cohomology* is developed in a similar vein. Let A be a Lie algebra, with bracket $\{,\}$. The p-cochains here are skew-symmetric, i.e. they are linear maps $B : \Lambda^p A \longrightarrow A$ and the Chevalley coboundary operator ∂_c is defined on a p-cochain B by:

$$\partial_c C(u_0, u_1, ..u_p) = \sum_{j=0}^{p} (-1)^j \{u_j, C(u_0, .., \hat{u}_j, .., u_p)\}$$
$$+ \sum_{i<j} (-1)^{i+j} C(\{u_i, u_j\}, u_0, .., \hat{u}_i, .., \hat{u}_j, .., u_p)$$

(where $\hat{u}$ means that u has to be omitted.) Again this is a complex, i.e. $\partial_c^2 = 0$. Thus the cocycles space Z_c^p, the coboundaries space B_c^p, and the quotient space $H_c^p(A, A)$ (or $H_c^p(A)$ in short) can be defined, analogously..

We can now return to our short historical account:

With the above mentioned tools, first, in mid 1970s, the existence of star-products for symplectic manifolds whose third cohomology group is trivial was proved, but this restriction turned out to be merely technical. In the early 1980s the existence of star-products for larger and larger classes of symplectic manifolds was proved, and finally it was shown that any symplectic manifold can be "quantized". A further generalization was achieved with (Fe) where Fedosov proved that the results about the canonical star-product on an arbitrary symplectic manifold can be used to prove that all regular Poisson manifolds can be quantized. (See (W) for an account of Fedosov's construction).

However the question at the end of Section 2.1 was still open. In physics we sometimes require manifolds which have a degenerate Poisson bracket and so are not symplectic. The broadest framework for classical mechanics thus involves general Poisson manifolds. Therefore all the results mentioned above provided only a partial answer to the problem of quantization.

In 1993-1994 M. Kontsevich proposed a statement ("Formality Conjecture") which would imply the desired result, i.e. if the Formality Conjecture could be proved this would imply that any finite-dimensional Poisson manifold can be canonically quantized (in the sense of deformation quantization). The Formality Conjecture is proved in (Ko3) thus answering our question in 2.1 in the positive. In the last part of this paper we will try to summarize these results (in a very sketchy manner, I am afraid!).

4. Kontsevich's Results:

The cohomological arguments were introduced before, as we have seen above. In his work about quantization of Poisson manifolds, Kontsevich also made use of cohomology. However, his approach involved further concepts that we will be introducing in 4.1. In 4.2 we briefly discuss the basic conjecture / theorem. Section 4.3 will involve some interpretations, implications, and possible new questions.

4.1. **Some More Definitions and Preliminaries:**

Definition 8. A *differential graded Lie algebra*, or DGLA in short, is a Z-graded Lie superalgebra

$$L = \bigoplus_{i \geq 0} L^i$$

with a map $d : L^i \longrightarrow L^{i+1}$ such that

$$d[a, b] = [da, b] + (-1)^i [a, db] \text{ for a} \in L^i, b \in L$$

(where the bracket is denoted by [,]).

Let $A = C^\infty(M)$ be the algebra of smooth functions on a smooth real manifold M. Let $C^\circ(A, A)$ be the (local) Hochschild complex of the algebra A over M, i.e. for any n,

$$C^n(A, A) = \{\phi \in Hom(A^{\otimes n}, A) | \phi(f_1, f_2, .., f_n) \\ \text{is a differential operator in each entry } f_i\}$$

Then the corresponding Hochschild cohomology satisfies:

$$H^n(A, A) = \Lambda^n TM = \text{ smooth multivector fields on M}$$

Both the Hochschild complex and the cohomology are differential graded Lie algebras (DGLAs).

Definition 9. Two DGLAs L, L' are *quasi-isomorphic* if there is a chain

$$L \longrightarrow L_1 \longleftarrow L_2 \longrightarrow ... \longleftarrow L_n \longrightarrow L'$$

of DGLA homomorphisms all of which induce isomorphisms of cohomology.

4.2. **The Formality Conjecture.** .

In 1993-1994 (see (Ko1) and (Ko2)), Kontsevich proposed the following

Conjecture 1. *(Kontsevich's Formality Conjecture) The Hochschild complex C^n is quasi-isomorphic as a DGLA to its (Hochschild) cohomology H^n.*

and proved that this conjecture would imply the desired result for our question at the end of 2.1. In other words he proved the following

Theorem 2. *The Formality Conjecture for a manifold M implies deformation quantization of any Poisson structure on M.*

[For a short account of a proof of this theorem see (Vo) or (Ya).] Finally in (Ko3) Kontsevich proved the following

Theorem 3. *Let M be a smooth manifold and $A = C^\infty(M)$. Then there is a natural isomorphism between equivalence classes of deformations of the null Poisson structure on M and equivalence classes of smooth deformations of the associative algebra A. In particular any Poisson bracket on M comes from a canonically defined (modulo equivalence) star product.*

[The main part of the proof involves the affine case, i.e. when M is essentially R^d. The formulation of the result allows the "gluing" of charts, and this yields an explicit universal formula for the star-product on M which involves graphs and Stokes' formula.]

Hence we can conclude that classes of star products correspond to classes of deformations of the Poisson structure. Moreover, our problem is solved: any Poisson structure can be deformed. For the theorem implies the conjecture which implies the existence of deformation quantization.

4.3. **Final Results and Implications.** .

(In this part we will be mainly following (St) and (Ko4)).

Kontsevich's result proves that classes of star products correspond to classes of deformations of the Poisson structure, and our problem of 2.1 is finally solved: we can quantize (formally) any Poisson structure. A later result shows that in addition to the existence of a canonical way of quantization, we can define a universal infinite-dimensional manifold parametrizing quantizations.

Now it can be seen that Kontsevich has proved a more general result than the existence of deformation quantization of any Poisson manifold. He has proved that in a suitably defined homotopy category of DGLAs two objects are equivalent. The first object is the Hochschild complex of the algebra of functions on the manifold M, and the second is a Z-graded Lie superalgebra of multivector fields on M. In the course of the proof he constructs an explicit isomorphism for the case $M = R^n$ which under a certain interpretation tells us that our assumptions in deformation quantization include some sort of string theory. Although he has some doubts as to the naturality of the method for quantum mechanics Kontsevich seems to believe that the result shows there is some intrinsic relation with string theory. (See remarks in 1.5 of (Ko3) about this.)

Thus even though the original question that arose at the end of 2.1 is answered, there are new interesting questions that come up. For instance a natural conjecture that could follow the above results could be: The relation with 1-differentiable deformations of the Poisson bracket with Fedosov's construction (i.e. any Fedosov deformation can be obtained by a sequence of successive 1-differentiable deformations of the initial Poisson bracket) extends to general Poisson manifolds. Another relation to be considered could be that of the result with some 2-cohomology on the manifold. A comparison of Kontsevich's proof with the proofs of the previous results, eg. the geometric methods of Fedosov also remains to be done.

In a different direction we might consider a remark of Kontsevich: the isomorphism of the (second and the) main theorem in 3.2 should be taken as one of a family of isomorphisms, and this leads us to a new conjecture: the motivic Galois group acts on deformation quantizations. (Ko4) Another question that remains involves the infinite dimensional case: Kontsevich's results settle the problem in the finite dimensional case, but there are places in physics where we have to deal with infinite dimensional Poisson manifolds, i.e infinite dimensional manifolds with a Poisson structure on them. This case involves new problems and perhaps may shed light on a better mathematical understanding of quantum field theory. (St)

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