# Symplectic Gromov-Witten Invariants

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The theory of Gromov-Witten invariants has its origins in Gromov's pioneering work. Encouraged by conjectures coming from physicists, it took a while until a rigorous mathematical foundation was laid. The aim of this short survey is to present some of the results of the last decade concerning this field.
Gromov-Witten invariants count holomorphic maps satisfying certain requirements from Riemann surfaces into symplectic manifolds $M$ . Usually one wants the domain to have a given form and the image to pass through fixed homology cycles on $M$. When the appropriate number of constraints is imposed there are only finitely many maps satisfying them. This count is the corresponding GW invariant. To define these invariants one introduces an almost complex structure $J$ on $M$ compatible with the symplectic form. Then one can form the moduli spaces of $J$-holomorphic (or, more generally $(J, \nu)$-holomorphic) stable maps from curves into $M$. The construction of a nice compactification and the topology of these spaces are presented in Section 1.
In Section 2 we define GW invariants (also called correspondences by B. Siebert) as homology classes of the moduli spaces of stable curves. A comprehensive description of the recursive relations satisfied by the invariants is far beyond the aim of this survey, but we however talk about the gluing equation, whose main idea is fairly easy to grasp intuitively: a map with reducible domain can be regarded as 2 separate maps with different domains, whose points corresponding to the node have the same image in $M$.
In the last 2 sections we quote more recent results on the theory of relative Gromov-Witten invariants: they count maps which in addition to our usual requirements intersect a given codimension 2 submanifold $V \subset M$ with prescribed multiplicities. To construct a "nice" moduli space parametrizing these types of maps we restrict to some pairs $(J, \nu)$ compatible with $V$ in a sense to be made precise. After defining the connected sum of 2 given $n$-dimensional manifolds, in our last theorem we write the invariants of their connected sum in terms of the relative GW invariants of the initial manifolds.

## 1 Stable holomorphic curves

Let $(M, \omega)$ be a symplectic manifold with an almost complex structure $J$ tamed by $\omega$, that is $\omega(v, Jv) > 0$ for any tangent vector $v$. Let $\Sigma$ be a closed Riemann surface with complex structure $j$. A map $\phi : \Sigma \to M$ is called $J$ holomorphic if it satisfies the homogeneous equation:

$$\frac{1}{2}(d\phi + J \circ d\phi \circ j) = 0.$$

In general the space of $J$- holomorphic maps $\Sigma \to M$ fails to be compact. For example in the case $\Sigma = \mathbf{CP}^1$ there is a non-compact group (namely $Aut(\mathbf{CP}^1) = PGL(2)$) which acts on these spaces by reparametrization. More important, these spaces are not compact because of the appearance of "bubbles" as limits of maps. The classical example of bubbling off in algebraic geometry is the degeneration of the family of (complex) plane quadrics $xy = \varepsilon$ into a pair of lines as $\varepsilon = 0$. The problem of finding an appropriate compactification of these spaces was solved by Gromov in [1]. Essentially we allow singular Riemann surfaces which we call "bubble domains". It is also convenient to introduce marked points on the domain in order to obtain a finite automorphism group of the map.

**Definition 1.1.** $(C, x_1, x_2 \dots x_n, \phi)$ is called *a stable $J$-holomorhpic curve with $n$ marked points* if the following conditions hold:

- $C$ is a reduced connected complete algebraic curve with at most ordinary double points as singularities*;*
- $x_i$ is a regular point of $C$ for every $i$*;*
- $\phi : C \to M$ and for every irreducible component of $C$ the restriction of $\phi$ to that component is $J$-holomorphic;
- $Aut(C, x_1 \dots x_n, \phi) := \{\sigma : C \to C \text{ biregular}/\sigma(x_i) = x_i \textit{ and } \phi \circ \sigma = \phi\}$ is finite*.*

The condition on the finiteness of the automorphism group is called the *stability condition.* In down-to-earth terms it says that every rational component contracted by $\phi$ (i.e. the restriction of $\phi$ to that component is constant) contains at least 3 marked points. If we take $M = \{pt\}$ we get the spaces of stable algebraic curves $\overline{M}_{g,n}$ introduced by Mumford. The universal curve over this space is $\overline{M}_{g,n+1}$ with the morphism given by forgetting the last marked point. We define the genus of $(C, x_1 \dots x_n, \phi)$ to be the arithmetic genus $h^1(C, O_C)$ of $C$.
One can think think of bubble domains as obtained by pinching a set of non-intersecting embedded circles in a smooth 2 manifold. The following definition formalizes this idea:

**Definition 1.2.** *A resolution of a $(g, n)$ bubble domain $B$ with $d$ nodes* is a smooth oriented genus $g$ manifold together with $d$ disjoint circles $\gamma_l$ and $n$ marked points disjoint from $\gamma_l$ and a resolution map:

$$r : \Sigma \to B$$

that respects orientation and marked points, takes each $\gamma_l$ to a node of $B$ and restricts to a diffeomorphism on $\Sigma - \cup\gamma_l$.

Gromov also endowed these spaces with a topology (called Gromov topology). Roughly, a sequence of $J_i$-holomorphic maps $(\Sigma, \phi_i)$ converges to a stable $J$-holomorphic map $(B, \phi)$ with bubble domain $B$ and resolution $r : \Sigma \to B$ if the area of the sequence is bounded, $J_i \to J$, and there are diffeomorphisms $\psi_i$ of $\Sigma$ preserving orientation and marked points such that the modified subsequence $(\Sigma, \phi_i \circ \psi_i)$ converges to a limit

$$\Sigma \to B \to M.$$

This convergence is everywhere $C^0$ and $C^\infty$ on compact sets disjoint from the collapsing circles $\gamma_l$ of the resolution. The "symplectic area" (integral over the image of $\omega$) of the image is preserved in the limit.

We are now able to state the compactness theorem. For this, let $A \in H^2(M;\mathbf{Z})$ and define $\mathcal{C}^{hol}_{g,n}(M,A)$ to be the space of $J$-holomorphic curves of genus $g$ with $n$ marked points such that $\phi_*[C] = A$.

**Theorem 1.3.** *If $J$ is tamed by some symplectic form $\omega$, the space $\mathcal{C}^{hol}_{g,n}(M,A)$ with the Gromov topology is compact and Hausdorff.*

Of course we want to give $\mathcal{C}^{hol}_{g,n}(M,A)$ the structure of a manifold. This is not possible because there are maps $(C,x,\phi)$ with non-trivial automorphism group. However we can locally give it the structure of an oriented orbifold. Specifying charts around a generic point $(C,x,\phi) \in \mathcal{C}^{hol}_{g,n}(M,A)$ is quite technically involved. The main idea is that we can split the deformation of a map $(C,x,\phi)$ into the deformation of the (possibly not stable) underlying curve $(C,x)$ (to a "less singular" one $(C',x)$) and the deformation of a map from $C'$ to $M$. Hence the space of stable maps is locally modeled by the product of $\overline{M}_{g,n}$ and the set of $J$ holomorphic maps from the fibers of the universal curve which was proved in [6] to be a manifold. Moreover, one can prove that the strata $S_k$ consisting of maps whose domains have $k$ double points are suborbifolds of real codimension $2k$.
It is often convenient to look at the solutions of the perturbed equation

$$\partial_J \phi = \frac{1}{2}(d\phi + J \circ d\phi \circ j) = \nu$$

We call the maps satisfying the above equation $(J,\nu)$-holomorphic and the corresponding moduli space $\mathcal{C}^{\nu}_{g,n}(M,A)$.

# 2 Absolute Gromov-Witten invariants

To be able to define Gromov-Witten invariants we need to make sense of a fundamental class $[\mathcal{C}^{hol}_{g,n}(M,A)]$. To get enumerative results one should know the dimension of these spaces.

**Theorem 2.1.** If $g = 0$ then the (real) dimension is:

$$dim(\mathcal{C}^{hol}_{0,n}(M,A)) = 2c_1(TM)\cdot A + (1-g)(dim(M)-6) + 2n$$

where the right hand side is called *the Riemann-Roch dimension.*

The Riemann Roch dimension is what we expect to get just by a naive compuation. However in general the spaces $\mathcal{C}^{hol}_{g,n}(M,A)$ can be very singular and have different dimension that the expected one. It turns out that there is always a well defined virtual fundamental class that lives in the corect dimension. Its rigorous construction was done by several authors ( e.g. Siebert in [7]) using different approaches (and slightly different assumptions on $M$).

Notice that these spaces come with natural morphisms :

$$\begin{array}{ccc} \mathcal{C}^{hol}_{g,n}(M,A) & \xrightarrow{ev} & M^n \\ \downarrow{\scriptstyle p} & & \\ \overline{M}_{g,n} & & \end{array}$$

where $ev((C, x_1 \ldots, x_n, \phi)) = (\phi(x_1), \ldots, \phi(x_n))$ is the evaluation at the $n$ marked points and $p$ is the morphism which sends a point $(C, x_1 \ldots, x_n, \phi)$ to its underlying curve (eventually stabilizing).

**Definition 2.2.** *The maps*

$$\begin{aligned} H^*(M)^{\otimes n} &\to H_*(\overline{M}_{g,n}) \\ (\alpha_1 \otimes \ldots \otimes \alpha_n) &\longmapsto p_*([\mathcal{C}_{g,n}^{hol}(M, A)]^{virt} \cap ev^*(\alpha_1 \times \ldots \times \alpha_n)) \end{aligned}$$

*are called Gromov-Witten correspondences (or Gromov-Witten invariants) of $(M, J)$.*

**Remarc 2.3.** The symplectic approach given above to the Gromov-Witten invariants is more involved than the algebraic one, in which one works with algebraic varieties, stacks and does intersection theory in the Chow ring. However one expects that for complex projective manifolds $M$ (to which both approaches apply) the algebraic and symplectic virtual fundamental class agree. This far from obvious fact was proved by B. Siebert in [8].

Alternatively one can define equivalent objects using Poincare duality. For example they can be seen as a homomorphism:

$$GW_{g,n,A,M} : H^*(M)^{\otimes n} \otimes H_*(\overline{M}_{g,n}; \mathbf{Q}) \to \mathbf{Q}$$

given by:

$$GW_{g,n,A,M}(\alpha_1, \ldots, \alpha_n; \beta) := \int_{[\mathcal{C}_{g,n}^{hol}(M,A)]^{virt}} ev_1^*(\alpha_1) \smile \ldots \smile ev_n^*(\alpha_n) \smile p^* PD(\beta)$$

where $PD(\beta)$ denotes the Poincare dual of $\beta$. The geometric interpretation is as follows: for cycles $\beta \subset \overline{M}_{g,n}$ and $A_1, \ldots, A_n \subset M$ Poincare dual to $\alpha_1, \ldots, \alpha_n$ it counts (with signs) the "expected" number of $J$ holomorphic maps $(C, x, \phi)$ of genus $g$ with $(C, x) \in K$ and $\phi(x_i) \in A_i$. "Expected" means that this agrees with the actual number in nice conditions, for example when $\mathcal{C}_{g,n}^{hol}(M, A)$ has the expected dimension (e.g. when $g = 0$) and is transversal to $K \times A_1 \ldots \times A_n$ under $p \times ev$. Usually these numbers are called *Gromov-Witten invariants*. Unfortunately the constraints imposed on the maps fail to be transversal at points corresponding to multiple covered maps or at constant maps (sometimes called ghosts). Ruan and Tian proved that, for generic $\nu$, the cut down moduli space obtained by imposing the above constraints on $\mathcal{C}_{g,n}^{\nu}(M, A)$ is a manifold, consisting of finitely many points that, counted with signs, give an invariant independent of $\nu$.

**Example 2.4.** The easiest case is $g = 0$ and $A = 0$ (i.e constant maps). Since every stable constant map must have at least 3 marked points $\overline{M}_{0,n} = \emptyset$ for $n < 3$ hence the corresponding invariants are 0. For $n > 3$ the dimension formula from theorem 2.1 says that the cycles $A_i$ Poincare dual to $\alpha_i$ have the sum of their codimensions bigger than dim $M$, hence they do not intersect when in general position. The only non-zero invariant are for $n = 3$ and they count triple intersections of Poincare duals:

$$GW_{0,3,0,M}(\alpha_1, \alpha_2, \alpha_3) = \int_M \alpha_1 \smile \alpha_2 \smile \alpha_3.$$

The GW invariants depend on $J$. However it turns out their nature is symplectic. Recall that $(M, \omega)$ and $(M', \omega')$ are symplectic deformation equivalent if there is a diffeomorphism $\rho : M \to M'$ and a one-parameter family of symplectic forms $\omega_t$ on $M$such that $\omega_0 = \omega$ and $\omega_1 = \rho^*\omega'$. Siebert showed in [7] that the GW invariants are independent of changes $J$ inside the symplectic deformations class:

**Theorem 2.5.** *The GW-correspondences are invariants of the symplectic deformation type of* $(M, \omega)$.

The GW invariants satisfy some recursion relations which allow us to reduce their computation to some easier cases. We will not elaborate on the whole set of axioms, but just sketch the ideas lying behind the so-called gluing equation, which counts contributions coming from maps with reducible domains in the genus 0 case. Given 2 curves $C_1 \in \overline{M}_{0,n_1+1}$ and $C_2 \in \overline{M}_{0,n_2+1}$ with $n_1 + n_2 = n$ we construct a new curve in the boundary stratum of $\overline{M}_{0,n}$ by identifying the last marked points. So for every partition $I = I_1 \cup I_2$ of the $n$ marked points with $|I_i| \geq 2$ we get a well defined map:

$$\xi_I : \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0,n}.$$

To be able to glue 2 points in the corresponding product of the moduli spaces of maps we need to require that the last marked points in $C_1$ and $C_2$ to be sent to the same point in $M$. This can be rephrased by saying that $ev_{n_1+1} \times ev_{n_2+1} \in \Delta \subset M \times M$, where $\Delta$ is the diagonal in $M \times M$. The cohomology class of $\Delta$ can be expressed as follows: fix a basis $e_0, \ldots e_K$ of $H^*(M)$, let

$$g_{\mu\sigma} := \int_M e_\mu \smile e_\sigma$$

and denote by $g^{\mu\sigma}$ the inverse matrix ; then we have:

$$[\Delta] = \sum g^{\mu\sigma} e_\mu \otimes e_\sigma.$$

If we fix 2 homology classes $\beta_i \in H_*(\overline{M}_{0,n_i+1})$ the above considerations yield the formula:

$$GW_{0,n,A,M}\Big(\alpha_1, \ldots, \alpha_n; \xi_{I*}(\beta_1 \otimes \beta_2\Big) = \varepsilon(I, \alpha) \sum_{A_1+A_2=A} \sum_{e_\mu, e_\sigma}$$
$$GW_{0,n_1+1,A_1,M}\Big(\{\alpha_i\}_{i\in I_1}; \beta_1\Big) g^{\mu\sigma} GW_{0,n_2+1,A_2,M}\Big(\{\alpha_i\}_{i\in I_2}; \beta_2\Big)$$

where we define:

$$\varepsilon(I, \alpha) = (-1)^{\sharp\{j<i | i\in I_1, j\in I_2, deg(\alpha_i)deg(\alpha_j)\in 2\mathbf{Z}+1\}}$$

and we throw it in to fix the sign coming from permuting odd degree cycles in the integrals. Moreover let's consider the morphism $\pi : \overline{M}_{0,n} \to \overline{M}_{0,4} \cong \mathbf{CP}^1$ that forgets all but the first 4 marked points and define $\beta := PD(\pi^* PD([pt])$. $M_{0,4}$ contains 3 points corresponding to curves with 2 reducible components: they are distinguished by the pairs of marked points on each component i.e. one of them has the pair $\{1, 2\}$ lying on one component and $\{3, 4\}$ on the other component etc. Since any 2 points are equivalent in the homology of $\mathbf{CP}^1$ we can deduce that the corresponding cycles $\beta_1, \beta_2, \beta_3$ obtained by pull-back with $\pi$ are equivalent. Fortunately the homology of $\overline{M}_{0,n}$ is well understood; for example $\beta_1$ consists of cycles whose points are curves with (at least) 2 irreducible components, such that the pairs $\{1,2\}$ and $\{3,4\}$ lie on separate components and the other marked points are distributed anyway on the coponents etc. One can write:

$$\beta_1 = \sum_{\substack{1,2\in I_1\\3,4\in I_2}} \xi_{I*}([\overline{M}_{0,n_1+1}] \otimes [\overline{M}_{0,n_2+1}])$$

where $I$ is a partition of the marked points. Carrying the equivalence $\beta_1 = \beta_2$ to the definition of GW invariants and combining it with the splitting formula stated above we get the gluing formula:

$$GW_{0,n,A,M}\big(\alpha_1, \ldots, \alpha_n; \beta_1\big) = GW_{0,n,A,M}\big(\alpha_1, \ldots, \alpha_n; \beta_2\big)$$

or if we unravel it:

$$\begin{aligned}
&\sum_{\substack{1,2\in I_1\\3,4\in I_2}} \varepsilon(I,\alpha) \sum_{A_1+A_2=A} \sum_{\mu,\sigma} GW_{0,n_1+1,A_1,M}\Big(\{\alpha_i\}_{i\in I_1}, e_\mu; [\overline{M}_{0,I_1+1}]\Big) \cdot \\
&\qquad\qquad\qquad\qquad \cdot g^{\mu\sigma} GW_{0,n_2+1,A_2,M}\Big(\{\alpha_i\}_{i\in I_2}, e_\sigma; [\overline{M}_{0,I_2+1}]\Big) \\
&= \sum_{\substack{1,3\in I_1\\2,4\in I_2}} \varepsilon(I,\alpha) \sum_{A_1+A_2=A} \sum_{\mu,\sigma} GW_{0,n_1+1,A_1,M}\Big(\{\alpha_i\}_{i\in I_1}, e_\mu; [\overline{M}_{0,I_1+1}]\Big) \cdot \\
&\qquad\qquad\qquad\qquad \cdot g^{\mu\sigma} GW_{0,n_2+1,A_2,M}\Big(\{\alpha_i\}_{i\in I_2}, e_\sigma; [\overline{M}_{0,I_2+1}]\Big)
\end{aligned}$$

where we've denoted by $\overline{M}_{0,I_1+1}$ the moduli space of curves containing the obvious marked points. Although these formulas look ugly, most of the summands vanish for dimensional reasons. They were used to prove Kontsevich's nice recursive formula for the number $N_d$ of degree $d$ (complex) curves in $\mathbf{CP}^2$ through $3d-1$ points in general position:

$$N_d = \sum_{d_1+d_2=d} \Big[ \binom{3d-1}{3d_1-1} d_1^2 d_2^2 - \binom{3d-1}{3d_1-2} d_1^3 d_2 \Big] N_{d_1} N_{d_2}$$

Ruan and Tian managed in [6] to extend the formulas for the numbers $N_d$ in an arbitrary dimensional projective space $\mathbf{P}^N$.
It is often convenient to encode all the invariants in a generating series; for this let $NH_2(M)$ denote the Novikov ring of $M$; the elements of $NH_2(M)$ are linear combinations $\sum c_A t_A$ over $A \in H_2(M, \mathbf{Z})$ where $c_A \in \mathbf{Q}$, $t_A$ are formal variables satisfying $t_A t_B = t_{A+B}$ and $c_A = 0$ if $\omega(A) < 0$. Dualizing in the definition 2.2 and summing after all $A$ we get maps:

$$GW_{g,n} : H^*(\overline{M}_{g,n}) \otimes H^*(M^n) \to NH_2(M)$$

We can also set $\overline{M} := \cup_{g,n} \overline{M}_{g,n}$ and let $\mathbf{T}(M)$ denote the total tensor algebra $\mathbf{T}(H^*(M))$. Now introduce a new variable $\lambda$ to get a map:

$$GW_M : H^*(\overline{M}) \otimes \mathbf{T}(M) \to NH_2(M)[\lambda]$$

defined by the series:

$$GW_M = \sum_{A,g,n} \frac{1}{n!} GW_{g,n,A,M} t_A \lambda^{2g-2}.$$

This turns out to be extremely useful when counting maps with disconnected domains. Such maps occur for example in [9]. To extend GW invariants to this more general case, define $\widetilde{\mathcal{M}}_{\chi,n}$ to be the space of all compact Riemann surfaces of Euler characteristic $\chi$ with finitely many unordered components and with a total of $n$ ordered marked points. For each surface, after fixing an ordering of its components the location of the marked points defines an ordered partition $\pi = (\pi_1, \ldots, \pi_l)$ of $n$. Hence

$$\widetilde{\mathcal{M}}_{\chi,n} = \bigsqcup_{\pi} \bigsqcup_{g_i} (\overline{M}_{g_1,\pi_1} \times \ldots \times \overline{M}_{g_l,\pi_l})/S_l$$

where the sums are taken over all partitions of $n$, respectively over all $g_i$ such that $\sum(2-2g_i) = \chi$. The symmetric group acts by permuting the components. Then one can define the "Gromov-Taubes" invariant

$$GT_M : H^*(\widetilde{\mathcal{M}}) \otimes \mathbf{T}(M) \to NH_2(M)[\lambda]$$

by setting:

$$GT_M := e^{GW_M}.$$

# 3 Relative Gromov-Witten invariants

Ionel and Parker managed to extend the above definitions to invariants of $(M, \omega)$ relative to a codimension 2 submanifold $V$ ([2]). Curves in general position will intersect $V$ in a finite collection of points. The relative invariants still count $(J,\nu)$-holomorphic curves but also keep track of how those curves intersect $V$. The construction can not be done for generic $(J,\nu)$ but rather for some special pairs "compatible" with $V$ in the sense of the definition 3.1. Let $\mathcal{I}$ be the set of such $(J,\nu)$. The universal moduli space of stable maps $\overline{\mathcal{UC}}_{g,n}(M) \to \mathcal{I}$ is the set of all stable $(g,n)$ maps who are $(J,\nu)$-holomorphic for some $(J,\nu) \in \mathcal{I}$. Denote the orthogonal projection onto the normal bundle $N_V$ by $\xi \mapsto \xi^N$. Also let $\nabla$ be the pull-back connection on $\phi^*TM$ and define $\nabla^J = \nabla + \frac{1}{2}(\nabla J)J$.

**Definition 3.1.** *Let $\mathcal{I}^V$ be the submanifold of $\mathcal{I}$ of pairs $(J,\nu)$ satisfying the following conditions:*

- *$J$ preserves $TV$ and $\nu^N|_V = 0$ ;*
- *for all $\xi \in N_V$, $v \in TV$ and $w \in TC$:*

$$[(\nabla_\xi J + J\nabla_{J\xi} J)(v)]^N = [(\nabla_v J)\xi + J(\nabla_{Jv} J)\xi]^N$$

$$[\nabla_\xi \nu + J\nabla_{J\xi}\nu)(w)]^N = [(J\nabla_{\nu(w)} J)\xi]^N$$

The last 2 conditions are required for technical reasons related to the variation of these maps. The first condition means that $V$ is a $J$-holomorphic submanifold and that $(J,\nu)$-holomorphic curves in $V$ are also $(J,\nu)$-holomorphic curves in $M$. In particular with this definition any $(J,\nu)$-holomorphic map into $V$ is holomorphic as a map into $M$. The first question to ask is whether there exist such "good pairs" $(J,\nu)$. The answer is given in the following

**Proposition 3.2.** The space $\mathcal{I}^V$ is non-empty and path-connected.

We need to exclude from the definition of our invariant maps which have components mapping entirely to $V$ :

**Definition 3.3.** *A stable $(J,\nu)$-holomorphic map is called $V$ regular if no component of its domain is mapped entirely into $V$ and if none of its special points (i.e. marked or singular) is mapped into $V$.*

The $V$-regular maps of class $A \in H_2(M,\mathbf{Z})$ form an open subset of the space of stable maps. We denote it by $\mathcal{C}^V(M,A)$. For each $V$-regular map $\phi$ the inverse image $\phi^{-1}(V)$ consists of $l$ distinct isolated points $p_i$. Denote by $s_i$ their multimplicity, i.e. their order of contact of the image of $\phi$ with $V$ at $p_i$. We consider $s=(s_i)_{i=\overline{1,l}}$ and define the degree, length and order of $s$ by:

$$deg(s)=\sum s_i \qquad l(s)=l \qquad |s|=s_1s_2\dots s_l.$$

These vectors $s$ label the components of $\mathcal{C}^V(M,A)$. Associated to each $s$ such that $deg\ s = A\cdot V$ is the space

$$\mathcal{C}^V_{g,n,s}(M,A)\subset \mathcal{C}_{g+l(s)}(M,A)$$

of all $V$- regular maps such that $\phi^{-1}(V)$ consists of exactly the points $p_i$, each with multiplicity $s_i$. Forgetting these last $s$ points defines a projection

$$\mathcal{C}^V_{g,n,s}(M,A)\to \mathcal{C}^V_{g,n}(M,A)$$

onto one component of $\mathcal{C}^V_{g,n}(M,A)$, who is a disjoint union of such components. Notice that the images $C_1$ and $C_2$ of 2 regular maps can be distinguished by their intersection points with $V$, their homology class $A$ and moreover by the class $[C_1\sharp(-C_2)]\in H_2(M\backslash V)$. The next construction is meant to give a space which keeps track of all this data.

**Construction 3.4.** Recall that the domain of a map has $n+l(s)$ marked points, the last $l(s)$ of which are mapped into $V$.

- Let $i_v$ be the intersection map

$$\begin{array}{rcl} i_v:\mathcal{C}^V_{g,n,s}(M) & \to & V_s \\ (C,x_1\dots x_n,p_1\dots p_n,\phi) & \mapsto & ((\phi(p_1),s_1),\dots,(\phi(p_l),s_l)) \end{array}$$

  Here $V_s$ is the space of all pairs $((v_1,s_1)\dots(v_l,s_l))$ such that $v_i\in V$. Obviously $V_s$ is diffeomorphic with $V^{l(s)}$. If we take the union over all sequences $s$ we get a map

$$i_v:\mathcal{C}^V_{g,n}(M,A)\to \mathcal{S}V$$

  where

$$\mathcal{C}^V_{g,n}(M,A)=\coprod_A\coprod_s \mathcal{C}^V_{g,n,s}(M,A) \quad and \quad \mathcal{S}V=\coprod_s V_s$$

- Choose $D(\varepsilon)$ the $\varepsilon$ disk bundle in the normal bundle of $V$ regarded as a tubular neighbourhood. Then choose a diffeomorphism of $M\backslash\overline{D(\varepsilon)}$ with $M\backslash V$. Set:
$$S := \partial\overline{D(\varepsilon)} \quad and \quad \hat{M} := [M\backslash\overline{D(\varepsilon)}] \cup S$$
Then $\hat{M}$ is compact with $\partial\hat{M} = S$.

- The appropriate homology theory (which is built, roughly speaking from chains which intersect $V$ at finitely many points) gives us a map
$$\rho : H_2(\hat{M}, S^*) \to \mathcal{D}$$
where $\mathcal{D}$ is the space of divisors on $V$ (i.e. finite collections of points, counted with signs and multiplicities) and $S^*$ is $S$ with the disjoint union topology of its fiber circles.

We are now able to define $\mathcal{H}_M^V := H_2(\hat{M}, S^*) \times_{\mathcal{D}} \mathcal{S}V$ where $\mathcal{S}V \to \mathcal{D}$ is the obvious map.

The space $\mathcal{H}_M^V$ comes with a well-defined map
$$h : \mathcal{C}_{g,n}^V(M) \to \mathcal{H}_M^V$$
which lifts the intersection map $i_v$.
Of course $\mathcal{H}_M^V$ has components labeled by $A$ and $s$ corresponding to the restriction of the maps:
$$h : \mathcal{C}_{g,n,s}^V(M, A) \to \mathcal{H}_{M,A,s}^V.$$
We want to compacitfy the space $\mathcal{C}_{g,n,s}^V(M, A)$. One way to do this is to take its closure $\overline{\mathcal{C}}_{g,n,s}^V(M, A) \subset \mathcal{C}_{g,n+l(s)}^V(M, A)$ in the space of stable maps. Basically there are 3 types of maps that can occur in the boundary stratum:

- stable maps with no components or special points in $V$;
- stable maps with smooth domain which is mapped entirely into $V$;
- maps with some components on $V$ and with some components off $V$.

By analyzing each type of map, Ionel and Parker managed to show that the irreducible parts of the frontier $\overline{\mathcal{C}}_{g,n,s}^V(M, A)\backslash\mathcal{C}_{g,n,s}^V(M, A)$ have codimension at least 2. This proves that the compactification space carries a well-defined virtual fundamental class. They call $\overline{\mathcal{C}}_{g,n,s}^V(M, A)$ the space of $V$-stable maps. Their main result is the following:

**Theorem 3.5.** *For a generic $(J, \nu) \in \mathcal{I}^V$ the image of $\overline{\mathcal{C}}_{g,n,s}^V(M, A)$ under the map*
$$p \times ev \times h : \overline{M}_{g,n,s}^V(M, A) \to \overline{M}_{g,n+l(s)} \times M^n \times \mathcal{H}_{M,A,s}^V$$
*defines an element*
$$GW_{M,A,g,n,s}^V \in H_*(\overline{M}_{g,n+l(s)} \times M^n \times \mathcal{H}_M^V; \mathbf{Q})$$
*of real dimension*
$$2c_1(TM) \cdot A + (dim M - 6)(1 - g) + 2(n + l(s) - deg s).$$
*This homology class is independent of generic $(J, \nu) \in \mathcal{I}^V$. For each closed symplectic manifold $(M, \omega)$ with a codimension 2 submanifold $V$, and for each $g, n$ we call the above homology class the relative invariant of $(M, V, \omega)$.*

Again, as it happens for the absolute case, the relative invariants are unchanged under symplectic isotopies. Generaly we say that $(M, V, \omega)$ is deformation equivalent with $(M', V', \omega')$ if there is a diffeomorphism $\varphi : M' \to M$ such that $((M', V', \omega')$ is isotopic to $(M', \varphi^{-1}(V), \varphi^*\omega)$.

**Proposition 3.6.** *The relative invariants depend only on the symplectic deformation class of* $(M, V, \omega)$.

One can encode all the relative invariants in a series; the total relative invariant is a map:

$$GW_M^V : H^*(\overline{M}) \otimes \mathbf{T}^*(M) \to H_*(\mathcal{H}_M^V; \mathbf{Q}[\lambda]);$$

if we think of $GW_{M,A,g,n,s}^V \in H_*(\mathcal{H}_{M,A,s}^V)$ then the we have the expansion:

$$GW_M^V = \sum_{n,g} \frac{1}{n!} \sum_{\substack{A,s \\ degs = A \cdot V}} \frac{1}{l(s)!} GW_{M,A,g,n,s}^V t_A \lambda^{2g-2}.$$

The corresponding Gromov-Taubes invariant is again defined by the exponential $GT_M^V := e^{GW_M^V}$.

**Example 3.7.** The Hurwitz numbers are examples of GW invariants of $\mathbf{CP}^1$ relative to several points in $\mathbf{CP}^1$.

The classical Hurwitz number $N_{g,d}$ counts the number of smooth genus $g$ curves with a fixed ramification divisor in general position. More generally, if $\alpha$ is an unordered partition of $d$ then one can use relative GW invariants techniques to compute the number of smooth degree $d$ maps from a genus $g$ curve to $\mathbf{CP}^1$ with the ramification above a fixed point $P_0 \in \mathbf{CP}^1$ as specified by the partition $\alpha$ and simple branching at other $d + l(\alpha) + 2g - 2$ points in general position.

# 4 The connected sum formula

Ionel and Parker continue their remarkable work in [3]. They deduce a formula for the GW invariants of the symplectic sum of 2 manifolds $M_1$ and $M_2$ in terms of the relative GW invariants of $M_1$ and $M_2$. Let's first define the symplectic sum of 2 manifolds:

**Construction 4.1.** Assume that $M_1$ ad $M_2$ are $2n$-dimensional symplectic manifolds each containing symplectomorphic copies of a $(2n-2)$-dimensioanl submanifold $(V, \omega_V)$. assume there exists a symplectic bundle isomorphism $\psi : (N_{M_1}V)^* \to N_{M_2}V$.
*Then, given the above data, there exists a symplectic manifold* $Z$ *of dimension* $2n+2$ *and a fibration* $\lambda : Z \to D$ *over a disk* $D \subset \mathbf{C}$ *such that for all* $\lambda \neq 0$ *the fibers* $Z_\lambda$ *are smooth compact symplectic submanifolds- the symplectic connect sums and the central fiber* $Z_0$ *is the singular symplectic manifold* $M_1 \cup_V M_2$.

Now, one can study limits of sequences of $(J, \nu)$-holomorphic maps into $Z_\lambda$ as $\lambda \to 0$. Since the limit map needn't be connected, we expect to deduce a formula involving the Gromov-taubes invariants $GT_{M_i}$ rather than the GW invariants.

To write down the explicit formula we need to construct more objects.
Notice that given bubble domains $C_1$ and $C_2$, not necessarily connected or stable, with Euler characteristic $\chi_i$ and $n_i + l$ marked points we can construct a new curve by identifying the last $l$ marked points and then forgetting their marking and considering them nodes. We get this way a map:

$$\xi_l : \widetilde{\mathcal{M}}_{\chi_1,n_1+l} \times \widetilde{\mathcal{M}}_{\chi_2,n_2+l} \to \widetilde{\mathcal{M}}_{\chi_1+\chi_2,n_1+n_2}.$$

Taking the union for each $\chi_i, n_i$ we get an attaching map $\xi_l : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ for each $l$.
One can glue maps $\phi_1$ into $M_1$ and $\phi_2$ into $M_2$ provided that the images meet $V$ at the same points with the same multiplicity. The domains glue according to the attaching map $\xi$ defined above, while the images determine elements of the intersection homology spaces $\mathcal{H}^V_{M_1,A,s}$ and $\mathcal{H}^V_{M_2,A,s}$. The following map records the effects of this gluing on the homology level:

**Definition 4.2.** *The convolution operator*

$$* : H_*(\widetilde{\mathcal{M}} \times \mathcal{H}^V_{M_1}; \mathbf{Q}[\lambda]) \otimes H_*(\widetilde{\mathcal{M}} \times \mathcal{H}^V_{M_2}; \mathbf{Q}[\lambda]) \to H_*(\widetilde{\mathcal{M}}; NH_2(Z)[\lambda]$$

*is given by*

$$(k \otimes h) * (k' \otimes h') = \sum_s \frac{|s|}{l(s)!} \lambda^{2l(s)} (\xi_{l(s)})_* (k \otimes k') \langle h, h' \rangle_s.$$

The last ingredient needed in the formula is the count of the contribution coming from maps having components who lie entirely in $V$ as $\lambda \to 0$. These maps are not counted in the relative invariants of $M_1$ or of $M_2$. The analysis of this difficulty shows that the contribution is related to a certain GT invariant of the projective bundle $\mathbf{P}_V := \mathbf{P}(N_X V \oplus \mathbf{C})$, which we will denote $S_V$. For details we refer the reader to [3], definition 11.3. Before stating the main result we need to make sense of a definition:

**Definition 4.3.** A constraint $\alpha \in \mathbf{T}^*(Z)$ is said to separate as $(\alpha_{M_1}, \alpha_{M_2})$ if there exists $\alpha_0 \in \mathbf{T}^*(Z_0)$ such that $\pi^*\alpha_0 = \alpha$ and $\pi_0^*(\alpha_0) = (\alpha_{M_1}, \alpha_{M_2}) \in \mathbf{T}(H^*(M_1) \oplus H^*(M_2))$, where $\pi$ and $\pi_0$ are the projections $Z \to Z_0$ and $M_1 \oplus M_2 \to Z_0$.

**Theorem 4.4.** *Let $Z$ be the symplectic sum of $(M_1, V)$ and $(M_2, V)$ and suppose $\alpha \in \mathbf{T}^*(Z)$ separates as $(\alpha_{M_1}, \alpha_{M_2})$. Then the GT invariant of $Z$ is given in terms of relative invariants of $M_1$ and $M_2$ by:*

$$GT_Z(\alpha) = GT_(M_1)^V(\alpha_{M_1}) * S_V * GT^V_{M_2}(\alpha_{M_2})$$

*where * is the operator from definiton 4.2.*

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