Contact Geometry and 3-manifolds

James Otterson
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Abstract: *The aim of this survey is to present current results on contact geometry of 3-manifolds. We are particularly interested in the interaction of contact geometry with foliation theory. Care has being taken to make this survey accessible to as wide an audience as possible. We hope that, after reading this article, the reader will be able to refer to current research.*

At first glance, contact geometry and foliation theory belong to entirely different worlds. Given a manifold M^n and a (n-1)-distribution ξ on it, we would like this distribution to be integrable in order to obtain a foliation. In contact geometry the opposite is true, we want ξ to be nowhere integrable. Confoliations are the middle ground of these two situations; we allow a bit of foliation and a bit of contact geometry. This seems to be the worst case scenario: we would not be able to use the techniques of foliation theory nor of contact geometry. Quite the opposite happens, however. The interactions of these two theories is a fertile ground where nice results of foliation theory (Reeb Stability theorem) can be extended to confoliations and where a duality of tight contact structures and taut foliations is present. Our first step will be to make the above notions precise.

1 Foliation Theory

The idea of foliation theory is to break down a manifold M into immersed submanifolds that fit together nicely; i.e. locally they look like a piled ream of paper. It is hoped to get new insights about the manifold by looking at such assembled subpieces.

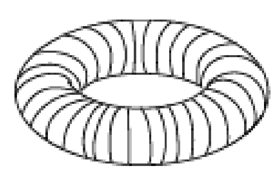
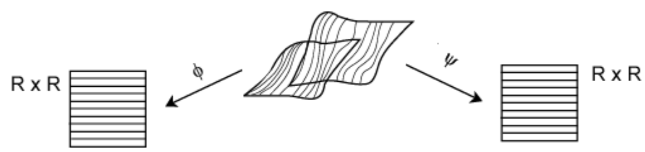


Figure 1: A Foliated Torus

Definition: A foliation F of dimension d (or codimension n) on a smooth manifold M^{d+n} is an maximal atlas compatible with the differential structure of M such that the change of coordinates map from a chart (U, ϕ) to (V, ψ) is given by a formula of the form

$$\phi \cdot \psi^{-1}(x, y) = (f(x, y), g(y)) \qquad x \in R^d,\ y \in R^n$$

where by a compatible atlas one means that $F \subset A$ if A is the differential structure of M



1.1 Examples

(a) Given a nowhere vanishing vector field X on a compact manifold M, its integral curves produce a foliation of the manifold

(b) More generally, by a <u>k-dimensional distribution</u> ξ on a manifold M, we mean a function $\xi : M \to TM$ that assigns to each point $p \in M$ a k-dimensional subspace of T_pM such that for each $p \in M$ $\exists$ a neighborhood U of p and k vector fields $X_1, ..., X_k$ that span ξ_p.

We say that two vector fields X and Y are sections of ξ if $\forall p \in M, X_p$ and $Y_p \in \xi_p$. A theorem of Frobenius states that a k-distribution ξ produces a k-dimensional foliation F if and only if for any two sections X and Y of ξ, $[X, Y]$ is also a section of ξ

(c) Another important example for us is the <u>Reeb foliation</u> of the solid torus. We will provide only a sketch of this foliation (*Refer to* [CN], [CC]):

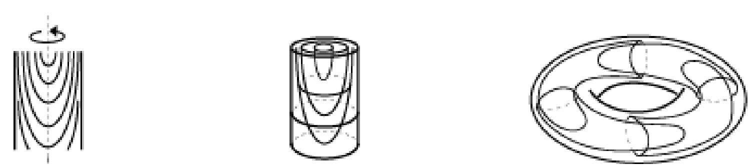


Figure 2: Start with a foliation of a unbounded strip as above. Rotate to get a foliation of the solid cylinder. Identify the bottom and the top.

A <u>Reeb Component</u> of a codimension one foliated 3-manifold is an embedded solid torus foliated in the above manner. This concept plays a essential role on the theory that follows.

1.2 Important Notions in Foliation Theory

Definition: A Leaf L of a foliation F is an equivalence class of points of the manifold M where two points p and q are considered equivalent if there are finite many charts of F, say $(U_1, \phi_1), ..., (U_n, \phi_n)$, and points $p = p_1, ..., p_n = q$ where $p_i \in U_i$ for $i = 1, ...n$ such that p_i and p_{i-1} belong to $\phi_i(R^d \times r_i)$ for some $r_i \in R^n$.[1]
Example: The circles of Figure 1 above are the leaves of the foliation of the torus. The twisted hyperboloids of the Reeb foliation (Figure 2) are leaves of this foliation. The boundary torus is another leaf of the Reeb foliation.

Definition: A foliation F of codimension one on a closed manifold is called taut if one can embed into it a transverse circle that intersects each leaf.

Theorem (Goodman [GO]): A codimension one foliation F of a closed 3-manifold is taut if and only if it does not have a Reeb Component.

Definition: Note that if we have a d-foliation F of a $(n+d)$-manifold M then, by considering the tangent spaces to each leaf, we get a d-distribution on M. A transversely orientable foliation is a foliation such that its d-distribution is transversely orientable.[2]

Theorem (Reeb Stability Theorem): Suppose that F is a transversely oriented codimension one foliation of a compact connected manifold M. If F has a compact leaf L with finite fundamental group then all leaves are diffeomorphic to L. Furthermore, there is a submersion $f : M \to S^1$ such that the leaves of F are the level sets $f^{-1}(\theta)$. *(refer to* [CN] *page 72)*

The theorem bellow is a weak version of the one found in [ET] page 49. We state it here to remark that the same result holds for tight contact structures. Other properties that hold for both structures are discussed on Section 3. The equivalent form for tight structures is discussed in the next section

Theorem (Thurston): Let F be a taut foliation on an oriented 3-manifold M(we may view F as a distribution that generates the foliation). Let N be a closed embedded orientable 2-surface $N \subset M$ then:

$$\begin{array}{ll} For\ N \neq S^2, & |e(F)[N]| \leq -\chi(N) \\ For\ N = S^2, & e(F)[N] = 0 \end{array}$$

Where $\chi(N)$ denotes the Euler characteristic of N and $e(F)[N]$ the value of the Euler class $e(F) \in H^2(M)$ evaluated on N.

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[1] The leaves are immersed (sometimes embedded) submanifolds. [CN]

[2] Recall that a distribution ξ is called *transversely orientable* if there exists a distribution η complementary to ξ which is orientable

2 Contact Geometry

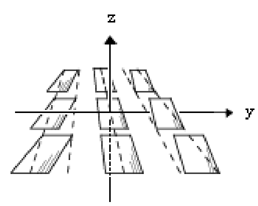
Contact geometry is essentially the opposite of foliation theory. It is concerned with $(n-1)$-distributions on n-manifolds ("hyperplane distributions") that fail to be integrable even locally.

In contrast with 1-distributions, where the theorem of existence of solutions of differential equations asserts that locally one can always integrate a vector field X, there exists 2-distributions that fail to be locally integrable as is shown in the next example.

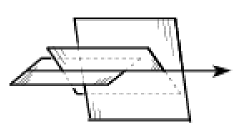
Example ([A], [ET], [S1], [S2]): Consider the vector fields

$$X_p = \frac{\partial}{\partial x}\Big|_p - b\frac{\partial}{\partial z}\Big|_p \qquad and \qquad Y_p = \frac{\partial}{\partial y}\Big|_p \qquad p = (a, b, c) \qquad p \in R^3$$

They generate the following plane distribution:



We claim that this distribution cannot be integrated at any point. Just notice the behavior a surface S tangent to ξ would have along the y-axis and how fast it has to rotate on the xz-plane as we move along the y-axis. For example, suppose $0 \in S$. The reader should convince herself that in this case S intersects the x-axis in a line segment. Let k be any point of S on the x-axis besides 0. As in the drawing bellow, notice that we could travel along the y-axis direction to the left of 0 and k through a straight line on $S \cap xy$-plane. This contradicts the fact that the plane distribution starts twisting as we move. The reader is strongly encourage to work through the details of this argument.

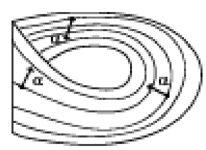


Refer to [ET] for a light geometric study of such distributions and to [S2] for a generalization of this example to a family of badly behaved plane distributions. One can also refer to [E2] for the definition of two plane distributions on R^2 that are locally equivalent (in a sense made precise on the next section) but for which there is no diffeomorphism $f : R^3 \to R^3$ that sends one plane distribution to another.

2.1 An Algebraic Definition of Contact Structures

As in much of differential geometry, we will translate the geometrical notion of contact geometry in algebraic terms using the cotangent bundle of the manifold. This will also make our ideas more precise.

A tangent hyperplane $H_p \subset T_pM$ of a manifold M can also be defined by specifying an element $\alpha_p \in T_p^*M$ for which $ker\, \alpha_p = H_p$. For C^∞ distributions we can find for all $p \in M$ a neighborhood U of p and a 1-form α on U so that for any $q \in U$ $ker\, \alpha_q = H_q$. Such an α is clearly not unique. It is easy to see that a local 1-form may not be extended to the entire manifold.



On what follows, for simplicity of exposition more then for mathematical necessity we will assume that given a hyperplane distribution ξ of M we can always find a 1-form α of M such that $\xi_p = ker\, \alpha_p \quad \forall p \in M$. We will call such 1-form a <u>defining 1-form</u> of ξ

Suppose that ξ is a hyperplane distribution of M with defining 1-form α. If for every two sections X and Y of ξ, $[X, Y]$ is also a section of ξ then by Frobenius theorem ξ is integrable. In terms of the 1-form α we have

$$0 = \alpha(X) = \alpha(Y) = \alpha([X, Y])$$

in view of the identity

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

ξ is integrable if and only if $d\alpha(X, Y) \equiv 0$ for all sections X and Y of ξ

As stated above this is the opposite of what we would like to have for a contact structure. Hence we impose that $d\alpha$ be nondegenerate (symplectic) on ξ (i.e. $\forall X_p \in \xi_p \; \exists Y_p \in \xi_p$ such that $d\alpha_p(X_p, Y_p) \neq 0$)

Definition: Given a hyperplane distribution ξ of M, (M, ξ) is a <u>contact manifold</u> if there exists a defining 1-form α such that $d\alpha|_{\xi_p}$ is non-degenerate for all $p \in M$. Such a 1-form is called a <u>contact form</u>.

Theorem (Darboux's Local Form of Contact Structures): All contact manifolds of the same dimension are locally diffeomorphic. Namely, given two contact manifolds (M, ξ) and (N, η) and points $x \in M$ and $y \in N$ there exist neighborhoods U of x, V of y and a diffeomorphism $f : U \to V$ that sends $\xi|_U$ to $\eta|_V$.

Hence, to understand the local behavior of contact structures in 3-manifolds, is enough to understand the contact structure of the previously discussed example.

It is easy to see that if $\alpha \wedge d\alpha \neq 0$ then ξ is not integrable (just use the definition of wedge product, argue by contradiction). Later we will need a stronger result:

Theorem [MS pg 105]: Let (M^{2n+1}, ξ) be a hyperplane distribution with defining 1-form α. Then $d\alpha$ is nondegenerate if and only if $\alpha \wedge (d\alpha)^n \neq 0$

Notice that for 3*-manifolds the above equation becomes* $\alpha \wedge d\alpha \neq 0$. *This will be essential for the algebraic formulation of confoliations on* 3*-manifolds.*

2.2 Tight and Overtwisted Contact Structures

The first step to understand contact 3-manifolds is Martinet's theorem:

Theorem: Every 3-manifold admits a contact structure.

After this, contact 3-manifolds are classified as either overtwisted or tight for a more detailed analysis. Overtwisted manifolds are fairly well understood, but there are still many open questions about tight structures.

Definition: A 3-dimensional contact manifold (M, ξ) is called <u>overtwisted</u> if we can embed a disk D (an <u>overtwisted disk</u>) in it in such a way that its interior is transversal to ξ everywhere except at one point and its boundary is tangent to ξ. If (M, ξ) is not overtwisted then it is called <u>tight</u>.

The standard picture for such an embedded disk on an overtwisted manifold is the following:

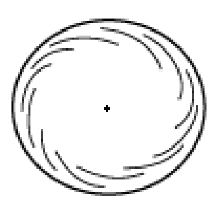


Figure 3: The curves represent the foliation obtained by intersecting the 2-distribution with the overtwisted disk. The dot represents the point where the disk fails to be transversal to the distribution.

2.2.2 Overtwisted Manifolds

There is a stronger form of Martinet's theorem due to Bennequim that states that every contact manifold can be deformed to an overtwisted structure ([B],[G]). Overtwisted contact structures have been completely classified on 1989 by Eliashberg. We follow below his arguments of [E2].

Definition: Fix a point p and an embedded disk $p \in D$ on a oriented connected 3-manifold. Let $\underline{Distr(M)}$ be the space of all 2-distributions on M fixed at point p provided with the C^∞ topology. Let $\underline{Cont(M)}$ be the subspace of $Distr(M)$ consisting of contact structures of M and let $\underline{Cont^{ot}(M)}$ be the subspace of $Cont(M)$ consisting of overtwisted structures which have the disk D as an overtwisted disk.

Theorem (Gray [Gr]): If M is a compact manifold then any two contact structures belonging to the same component of $Cont(M)$ are isotopic

Theorem [E2]: The inclusion map of $Cont^{ot}(M)$ on $Distr(M)$ is a homotopy equivalence. In particular, two overtwisted structures on a closed manifold M are isotopic if they are homotopic as plane distributions.

2.2.3 Tight Manifolds

Much of the current research on tight manifolds is concerned with the existence and construction of tight structures on a given manifold. The first 3-manifold that does not admit a tight contact structure was discovered in 2001

Theorem [EH]: The connected sum of the Poincaré homology sphere P with $-P$ does not admit a tight contact structure.

The classification of tight structures is known for some manifolds such as the Lens Spaces ([H], [G]), S^3, R^3, RP^3, and $S^2 \times S^1$ ([E3]) and 2-torus bundles over S^1 ([H], [G]).

The following theorem of Eliashberg [E2] extends Thruston's theorem from taut foliation to tight contact manifolds.

Theorem (Eliashberg): Let ξ be a tight contact structure on an oriented 3-manifold M. Let N be a closed embedded orientable 2-surface $N \subset M$ then:

$$\begin{array}{ll} For\ N \neq S^2, & |e(\xi)[N]| \leq -\chi(N) \\ For\ N = S^2, & e(\xi)[N] = 0 \end{array}$$

Where $\chi(N)$ denotes the Euler characteristic of N and $e(\xi)[N]$ the value of the Euler class $e(\xi) \in H^2(M)$ evaluated on N.

3 Confoliations

Recalling section 2.1, a 2-distribution ξ of a 3-manifold is a contact structure if its defining 1-form α is such that $\alpha \wedge d\alpha \neq 0$, and it is a foliation when $\alpha \wedge d\alpha = 0$.

Notice that a contact manifold is orientable since the sign of $\alpha \wedge d\alpha$ is independent of choice of α.

Definition: A positive confoliation of an oriented 3-manifold M is a 2-distribution ξ with a defining 1-form α such that $\alpha \wedge d\alpha \geq 0$. Negative confoliations are defined in the obvious way.

Example: The distribution ξ generated by $X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$ *and* $Y = \frac{\partial}{\partial y}$, as considered on Section 2 above, has defining 1-form $\alpha = y\,dx + dz$ (just take the "cross product" of X_p with Y_p). One easily calculates that $\alpha \wedge d\alpha = -dx \wedge dy \wedge dz$ and hence this is a negative confoliation.

Example: For a confoliation that is not a contact structure nor a foliation, consider the "smooth step" function $s : R \to [0,1]$, i.e. a C^∞ function that is zero on $b \leq 0$, one for $b \geq 1$ and strictly increasing between zero and one. Let $S(a,b,c) = s(b)$ and define a 1-form α by $\alpha_p = bS(p)\,dx_p + dz_p$ for $p = (a,b,c)$. Then $\alpha \wedge d\alpha = -(b\frac{\partial S}{\partial y} + S)\,dx \wedge dy \wedge dz$, a negative confoliation.

This confoliation is a foliation with horizontal leaves on the region $\{(a,b,c) : b < 0\}$ and the contact structure on the previous example on the region $\{(a,b,c) : b > 0\}$. One would say that this confoliation is a lamination on the set $\{(a,b,c) : b \leq 0\}$; a lamination is roughly speaking a foliation on a closed subset of a manifold. In general the set $\{p : (\alpha \wedge d\alpha)|_p = 0\}$ is not laminated by our confoliation since it may have empty interior. Hence, a confoliation is not necessarily a way to break a 3-manifold into a lamination and a contact structure.

As stated on the introduction, we can extend Reeb stability theorem to confoliations

Theorem (Reeb Stability Theorem for Confoliations): Suppose that a confoliation ξ on a closed oriented manifold M has an integral embedded 2-sphere S. Then ξ is a foliation and (M, ξ) is diffeomorphic to $S^2 \times S^1$ where the leaves are the $S^2 \times x_0$ slices. *Reference* [ET]

An essential ingredient to prove the Reeb Stability Theorem is the holonomy. The holonomy also plays a role on the geometrical interpretation of confoliations. To learn more about the holonomy refer to [CC] or [CN]. To learn how the holonomy is used on confoliations refer to [ET].

3.1 Relations between Taut and Tight Structures

First we will discuss a method employed for the creation of large quantities of taut foliations and tight contact structures.

Definition: Let (M, ξ) be a positive confoliation such that $\partial W = M$ for some compact symplectic manifold (W, ω). We call (W, ω) a symplectic filling of (M, ξ) if M is oriented as the boundary of the canonically oriented symplectic manifold (W, ω) and $(\omega|_M)|_\xi$ does not vanish. (We say that M is semi-fillable if it just one of the connected components of ∂W)

Theorem; IF (M, ξ) is a integrable 2-distribution with a symplectic filling then the foliation generated by ξ is taut. The converse is partially true. A taut manifold is semi-fillable (it is not known if we could replace semi-fillable by fillable in this case). *Refer to [ET]*

Theorem: If (M, ξ) is a contact manifold with a symplectic filling then (M, ξ) is a tight contact manifold.

Until recently it was not known if the converse of this theorem would hold. In 2002 [EH] proved:
Theorem: There are tight manifolds that are not symplectically fillable.

The relation between Taut and Tight structures is even stronger. Refer to [HKM]

Theorem: If a 3-manifold carries a taut foliation, then it also supports a tight contact structure

Theorem: If M is a 3-manifold with boundary then the converse of the above theorem also holds. (the statement is not true in general, S^3 supports a tight contact structure but has no taut foliation)

These theorems together with the inequality involving the Euler characteristic provide strong evidence that notions from taut foliations and tight structures may be extended to confoliations. However, at this point even the definition of tight/contact confoliation is an open problem. The reader is referred to [ET] for a discussion of this matter together with more properties shared by tight structures and taut foliations.

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[1] Illustrations where made with Adobe Illustrator

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