QUIVER VARIETIES FROM A SYMPLECTIC VIEWPOINT

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1. Introduction

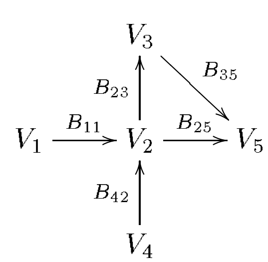
We examine an interesting class of symplectic manifolds called quiver varieties which were recently introduced by Nakajima [6]. These manifolds are produced from directed graphs, which we call quivers. There are a number of reason for the introduction of quiver varieties. They provide an interesting class of spaces which includes many previously studied symplectic manifolds. Also, they have connections with the study of moduli spaces of vector bundles over projective varieties. Finally, there is an interesting construction by Nakajima of representations of Kac-Moody Lie algebras and certain quantum affine algebras using the homology of these quiver varieties (see [7]).

Like many spaces arising naturally in representation theory, quiver varieties can be studied using symplectic geometry or algebraic geometry. In this paper, we focus on the symplectic geometry approach. We give a definition of two flavours of quiver varieties using symplectic reduction. We present polygon spaces and partial flag manifolds as particular examples of these two flavours.

2. Quiver Varieties

Definition 1. Let $\Gamma = (I, E)$ be an oriented graph, where $I = \{1, \ldots, n\}$ is the set of vertices and $E \subset I \times I$ is the set of edges. If Γ has no loop edges, then we call Γ a *quiver*.

A *representation* of a quiver Γ is a choice of finite dimensional complex vector space V_i for each vertex $i \in I$ and a linear map $B_{ij} : V_i \to V_j$ for each edge $(i, j) \in E$. For example:



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In some sense, this generalizes the notion of a representation of a group. Recall that a representation of a group G is a choice of a vector space V along with a linear map $\rho(g)$ for each element of $g \in G$, such that multiplication in the group corresponds to composition of the linear maps. So a representation of G is equivalent to a functor from the one object category $\mathcal{G}$, where the morphisms are group elements and composition of morphisms is multiplication of group elements, to the category $\mathcal{V}ec$ of finite dimensional complex vector spaces.

From our quiver Γ we can freely generate a category $\mathcal{Q}_\Gamma$ where the set of objects is I and $\mathrm{Hom}(i, j)$ is the set of paths in Γ from i to j. A representation of Γ as defined above then corresponds to a functor from $\mathcal{Q}_\Gamma$ to $\mathcal{V}ec$. Such a functor is made by assigning to some path the composition of the linear maps assigned to the edges of the path.

For a finite group G, the set of all representations of G up to isomorphism naturally has the discrete topology of a lattice generated by the irreducible representations. (This is because we have a correspondence between representations and their characters and the characters form a discrete subgroup of the inner product space of class functions). For quivers, the space of representations of a quiver up to isomorphism is an interesting space which we call a quiver variety.

Definition 2. Two representations $((V_i)_{i\in I}, (B_h)_{h\in E}), ((W_i)_{i\in I}, (C_h)_{h\in E})$ of Γ are said to be isomorphic if there exist isomorphisms $U_i : V_i \to W_i$ for each $i \in I$ such that $C_{ij}U_i = U_j B_{ij}$ for all $(i, j) \in E$.

Note that given two different collections $(V_1, \dots, V_n), (W_1, \dots, W_n)$, a representation of Γ on $(V_1, \dots, V_n)$ can be isomorphic to some representation of Γ on $(W_1, \dots, W_n)$ only if the dimensions of V_i and W_i agree for all i. So we may as well fix dimensions and then choose fixed vector spaces of those dimensions.

Let $v = (v_1, \dots, v_n) \in \mathbf{N}^n$. Let $V = (V_1, \dots, V_n)$ be a collection of complex Hermitian vector spaces with $\dim V_i = v_i$.

Let $\mathrm{Hom}(V) = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j)$.

Let $\mathrm{U}(V) = \mathrm{U}(V_1) \times \cdots \times \mathrm{U}(V_n)$.

Define an action of $\mathrm{U}(V)$ on $\mathrm{Hom}(V)$ by:

$$(g_1, \dots, g_n) \cdot (B_{ij})_{i,j} = (g_j B_{ij} g_i^{-1})_{i,j}$$

Note that $\mathrm{Hom}(V)$ is exactly the set of all representations of Γ on (V_i) and that two representations are isomorphic if and only if they are in the same orbit under the action of $\mathrm{GL}(V)$. Hence the set of isomorphism classes of representations of Γ of dimension v is in bijection with the $\mathrm{GL}(V)$ orbits on $\mathrm{Hom}(V)$. This suggests we should consider the quotient space $\mathrm{Hom}(V)/\mathrm{GL}(V)$. However, in general this will be unsatisfactory as this quotient will not possess as much geometrical structure as our original space.

Perhaps the most natural step at this point is to take the Geometric Invariant Theory quotient of $\mathrm{Hom}(V)$ by $\mathrm{GL}(V)$ (see [5] for this approach).

However by the Kirwan-Ness theorem we obtain a diffeomorphic space if we instead proceed to define a symplectic structure and then form a symplectic quotient by $\mathrm{U}(V)$ (see [7]). We will choose the symplectic approach here.

Since V_i, V_j are Hermitian vector spaces, $\mathrm{Hom}(V_i, V_j)$ has a Hermitian form given by $\langle A, B\rangle = \mathrm{tr}(AB^*)$ where $B^* : V_j \to V_i$ is the Hermitian adjoint of B. Hence $\mathrm{Hom}(V_i, V_j)$ has a natural symplectic vector space structure given by $\omega(A, B) = 2\mathrm{Im}(\mathrm{tr}(AB^*))$. So $\mathrm{Hom}(V) = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j)$ is a symplectic vector space.

We wish to show that the action of $\mathrm{U}(V)$ on $\mathrm{Hom}(V)$ is Hamiltonian and we wish to find its momentum map. This task is made simple by the following lemmas:

Lemma 1. *Let V, W be complex Hermitian vector spaces. Then the action of $\mathrm{U}(V)$ on $\mathrm{Hom}(V, W)$ is Hamiltonian with momentum map:*

$$\mu(B) = -B^*B$$

where we have identified $\mathfrak{u}(V)^\star$ with Hermitian operators on V via the trace form.

The action of $\mathrm{U}(W)$ on $\mathrm{Hom}(V, W)$ is also Hamiltonian with momentum map:

$$\mu(B) = BB^*$$

Moreover the two actions commute.

The proof of this result is a straightforward computation using the definition of a momentum map and our definition of the symplectic form on $\mathrm{Hom}(V, W)$.

Lemma 2. *Let M be a symplectic manifold. Suppose that we have Hamiltonian G, H actions on M with momentum maps μ_1, μ_2 respectively. Suppose also that the actions of G and H commute and that μ_1 is invariant under the action of H and μ_2 is invariant under the action of G.*
Then $G \times H$ acts on M with momentum map $\mu(p) = (\mu_1(p), \mu_2(p)) \in \mathfrak{g}^\star \oplus \mathfrak{h}^\star \cong \mathrm{Lie}(G \times H)^\star$.

Lemma 3. *Let M_1, M_2 be symplectic manifolds. Suppose that we have Hamiltonian G actions on each of M_1, M_2 with corresponding momentum maps μ_1, μ_2.*
Then the diagonal action of G on $M_1 \times M_2$ is Hamiltonian with momentum map $\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2)$.

These lemmas follow easily from the definition and basic properties of momentum maps.

Using these lemmas we can see that the $\mathrm{U}(V)$ action on $\mathrm{Hom}(V)$ is made from Hamiltonian $\mathrm{U}(V_i)$ actions on each $\mathrm{Hom}(V_j, V_k)$ as described in Lemma 1 (this action will be trivial if $i \neq j$ and $i \neq k$). Then we use the third lemma to produce for each i a Hamiltonian $\mathrm{U}(V_i)$ action on $\mathrm{Hom}(V)$ and then use the second lemma to get a Hamiltonian $\mathrm{U}(V)$ action on $\mathrm{Hom}(V)$.

Tracking what happens to the momentum map during this process shows that the overall momentum map is given by:

$$\mu((B_{ij})_{i,j}) = \Big(\sum_{(j,1)\in E} B_{j1}B_{j1}^* - \sum_{(1,i)\in E} B_{1i}^* B_{1i}, \\ \dots, \sum_{(j,n)\in E} B_{jn}B_{jn}^* - \sum_{(n,i)\in E} B_{ni}^* B_{ni} \Big)$$

Note that the normal subgroup $\{(t\mathbf{1}, \dots, t\mathbf{1}) : t \in \mathrm{U}(1)\} \subset \mathrm{U}(V)$ acts trivially on $\mathrm{Hom}(V)$ so we get an action of the quotient group $G = \mathrm{U}(V)/\mathrm{U}(1)$ on $\mathrm{Hom}(V)$. Note the quotient map $\mathrm{U}(V) \to G$ induces an inclusion of $\mathfrak{g}^\star$ into $\mathfrak{u}(V)^\star$ as the subspace annihilating $(\mathbf{1}, \dots, \mathbf{1}) \in \mathfrak{u}(V)$.

Since our momentum map lands in this subspace, we have a Hamiltonian G action on $\mathrm{Hom}(V)$ with the same momentum map.

Given $(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$ with $\lambda_1 + \dots + \lambda_n = 0$ we may perform reduction at $\lambda = (\lambda_1\mathbf{1}, \dots, \lambda_n\mathbf{1}) \in \mathfrak{g}^\star$ (as this is invariant under the coadjoint action).

Definition 3. The *quiver variety* for Γ with dimensions v at level λ is:

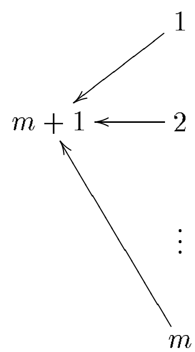
$$R_\lambda(v) = \mathrm{Hom}(V) \mathbin{/\!/}_{\lambda} G = \mu^{-1}(\lambda)/G$$

We now consider the dimension of $R_\lambda(v)$. Note that when we perform symplectic reduction at a coadjoint invariant momentum level we decrease the dimension of our space by twice the dimension of the group acting. In our case, we see that, over $\mathbf{R}$, $\dim \mathrm{Hom}(V_i, V_j) = 2v_iv_j$ and $\dim \mathrm{U}(V_i) = v_i^2$. Hence:

$$\dim R_\lambda(v) = 2 \sum_{(i,j)\in E} v_i v_j - 2(\sum_i v_i^2 - 1) = -\langle v, Cv\rangle - 2$$

where C is the generalized Cartan matrix for Γ defined by $C_{ii} = 2$ and $-C_{ij}$ equals the number of edges joining i and j in the underlying graph of Γ (where we ignore orientation). It is known from theory of root systems that the underlying graph is a simply-laced Dynkin diagram iff $\langle v, Cv\rangle > 0$ for all $(v_1, \dots, v_n) \neq 0$ and that the underlying graph for Γ is an simply-laced affine Dynkin diagram iff $\langle v, Cv\rangle \geq 0$ for all choices $(v_1, \dots, v_n)$. This is the first suggestion that there is a relationship between quiver varieties and Lie algebras. For now, just note that we should start with a graph that is not a Dynkin diagram if we are to get anything interesting.

2.1. **Polygon Spaces.** Consider the graph:



We choose dimensions $v_i = 1$ for $1 \leq i \leq m$ and $v_{m+1} = 2$. Hence

$$\mathrm{Hom}(V) = \bigoplus_{i=1,\ldots,m} \mathrm{Hom}(\mathbf{C}, \mathbf{C}^2) \cong (\mathbf{C}^2)^m$$

We have $\mathrm{U}(V) = \mathrm{T}^m \times \mathrm{U}(2)$ and hence $G = \mathrm{T}^m \times \mathrm{U}(2)/\mathrm{U}(1) \cong \mathrm{T}^m \times \mathrm{SO}(3)$. By our prescription above, the momentum map is the function from $(\mathbf{C}^2)^m$ to $\mathbf{R}^m \oplus \mathfrak{u}(2)$ given by:

$$(v_1, \ldots, v_m) \mapsto (|v_1|^2, \ldots, |v_m|^2, v_1 v_1^* + \cdots + v_m v_m^*)$$

We reduce at the point $\lambda = (\lambda_1, \ldots, \lambda_m, 0)$ with $\lambda_1 + \cdots + \lambda_m = 0$, applying the reduction in stages starting with the T^m action.

The T^m on $(\mathbf{C}^2)^m$ is just a product of copies of the standard $\mathrm{U}(1)$ action on $\mathbf{C}^2$, so reducing at the level $(\lambda_1, \ldots, \lambda_m)$ gives us a product of spheres: $S^2_{\lambda_1} \times \cdots \times S^2_{\lambda_m}$ of radii $\lambda_1, \ldots, \lambda_m$.

Now we must consider a $\mathrm{SO}(3)$ action on this product of spheres. It can be shown that this is the standard diagonal action of $\mathrm{SO}(3)$ and that the momentum map $S^2_{\lambda_1} \times \cdots \times S^2_{\lambda_m} \to \mathbf{R}^3$ is given by regarding each S_{λ_i} inside $\mathbf{R}^3 \cong \mathfrak{so}(3)$ and then summing the points.

If we think of an element of $S^2_{\lambda_1} \times \cdots \times S^2_{\lambda_m}$ as a polygonal path in $\mathbf{R}^3$, then the momentum map takes the endpoint of the path. So our momentum map condition forces the path to retun to the origin. Hence the quiver variety $R_\lambda(1, \ldots, 1, 2)$ is simply the space of all polygons in $\mathbf{R}^3$ of side lengths $\lambda_1, \ldots, \lambda_m$, up to rotation.

Hausmann and Knutson [1] have studied the topology and geometry of polygon spaces. Knutson [2] has also studied more general star-shaped quiver varieties where the single outlying vertices above are replaced by strings of vertices all of which have dimension 1. He relates these quiver varieties to partial flag manifolds and the combinatorics of honeycombs.

3. Framed Quiver Varieties

We now introduce a modification of our notion of quiver variety.

Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ be elements of $\mathbf{N}^n$.

Let $V = (V_1, \ldots, V_n), W = (W_1, \ldots, W_n)$ be two collections of complex Hermitian vector spaces with $\dim V_i = v_i$ and $\dim W_i = w_i$. One thinks of V_i as the vector space assigned to vertex i and W_i and as a vector space shadowing V_i .

Let:

$$\mathrm{Hom}(V,W) = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, W_i)$$

So we have assigned maps to each arrow in the quiver as well as a map from each V_i to its shadow vector space.

$$\begin{array}{ccc} V_1 & \xrightarrow{B_{12}} & V_2 \\ \big\downarrow{\scriptstyle D_1} & & \big\downarrow{\scriptstyle D_2} \\ W_1 & & W_2 \end{array}$$

The reason for introducing these shadow vector spaces originated in the work of Kronheimer and Nakajima [3] on the the moduli space of vector bundles over certain spaces. We will not pursue this connection any farther.

Let $\mathrm{U}(V) = \mathrm{U}(V_1) \times \cdots \times \mathrm{U}(V_n)$ as before.

Define an action of $\mathrm{U}(V)$ on $\mathrm{Hom}(V, W)$ by:

$$(g_1, \ldots, g_n) \cdot ((B_{ij})_{i,j}, (D_k)_k) = ((g_j B_{ij} g_i^{-1})_{i,j}, (D_k g_k^{-1})_k)$$

As above, the $\mathrm{U}(V)$ action is Hamiltonian and its momentum map can be built using the lemmas above. We find:

$$\mu((B_{ij})_{i,j}, (D_k)_k) = \Big(\sum_{(j,1)\in E} B_{j1} B_{j1}^* - \sum_{(1,i)\in E} B_{1i}^* B_{1i} - D_n^* D_n, \\ \ldots, \sum_{(j,n)\in E} B_{jn} B_{jn}^* - \sum_{(n,i)\in E} B_{ni}^* B_{ni} - D_n^* D_n \Big)$$

As long as $w \neq 0$ the $\mathrm{U}(V)$ action is effective and so there is no need to quotient by an S^1. Hence given $(\lambda_1, \ldots, \lambda_n)$ we can take the symplectic reduction at $\lambda = (\lambda_1 \mathbf{1}, \ldots, \lambda_n \mathbf{1}) \in \mathfrak{u}(V)^\star$.

Definition 4. We define the *framed quiver variety* for Γ with dimensions v, w at level λ to be:

$$R_\lambda(v, w) = \mathrm{Hom}(V, W) /\!\!/_{\lambda}\, \mathrm{U}(V)$$

As before, we may calculate $\dim R_\lambda(v, w)$. We find:

$$\begin{aligned} \dim R_\lambda(v,w) &= \dim \mathrm{Hom}(V,W) - 2 \dim(\mathrm{U}(V)) \\ &= 2 \sum_{(i,j)\in E} v_i v_j + 2 \sum_i v_i w_i - 2(\sum_i v_i^2 - 1) \\ &= 2\langle v, w\rangle - \langle v, Cv\rangle \end{aligned}$$

Note that we have a Hamiltonian $\mathrm{U}(W) = \mathrm{U}(W_1) \times \cdots \times \mathrm{U}(W_n)$ action on $\mathrm{Hom}(V, W)$ commuting with the $\mathrm{U}(V)$ action by the lemmas above. Hence $R_\lambda(v, w)$ carries a residual Hamiltonian $\mathrm{U}(W)$ action.

3.1. **Partial Flag Manifolds.** We now examine a framed quiver variety which we can identify as a $\mathrm{U}W$ coadjoint orbit. The presentation of this example from the symplectic point of view is original. For this example from the algebraic viewpoint see Nakajima [5] and Mirkovic and Vybornov [4] for more general work on A_n quiver varieties.

Consider the A_n quiver, that is a quiver whose underlying graph is the A_n Dynkin diagram and whose arrows all point in the same direction. We consider the case where (v, w) satisfy $v_1 < \cdots < v_n < w_n$ and $w_i = 0$ for $i < n$.

As there is only one W_i with positive dimension we will write W for W_n, $k =$ for $w_n = \dim W_n$, D for $D_n : V_n \to W_n$, and B_i for $B_{i,i+1} : V_i \to V_{i+1}$. Also for convenience let $v_0 = 0$. Then an element of $\mathrm{Hom}(V, W)$ looks like:

$$\begin{array}{ccccccc} V_1 & \xrightarrow{B_1} & V_2 & \xrightarrow{B_2} \cdots \xrightarrow{B_n} & V_n \\ & & & & \downarrow{\scriptstyle D} \\ & & & & W \end{array}$$

We wish to identify $R_\lambda(v, w)$ with the partial flag manifold in W of type $(v_1, \ldots v_n)$ which is defined as:

$$\begin{aligned} F_{v_1,\ldots,v_n}(W) &= \{(A_1 \subset A_2 \subset \ldots \subset A_n \subset W) : \dim A_i = v_i\} \\ &\cong \{(U_1, \ldots, U_n) : U_1 \oplus \ldots \oplus U_n = W \text{ is an orthogonal} \\ &\quad \text{decomposition and} \dim U_i = v_i - v_{i-1} \text{ for } i \leq n\} \end{aligned}$$

More specifically we have the following result:

Theorem 1. *Assume that $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbf{Q}$. Then:*

$$\begin{aligned} \psi : R_\lambda(v, w) &\to \mathfrak{u}(W)^* \\ [B_1, \ldots, B_n, D] &\mapsto DD^* \end{aligned}$$

is a $\mathrm{U}(W)$-equivariant symplectomorphism of $R_\lambda(v, w)$ with the $\mathrm{U}(W)$ coadjoint orbit O_ν, where:

$$\nu = \{(\lambda_n - \lambda_{n-1} + \cdots \pm \lambda_1)^{\oplus v_1}, \ldots, \lambda_n^{\oplus v_n - v_{n-1}}, 0^{\oplus w - v_n}\}$$

In particular it is diffeomorphic to $F_{v_1,\ldots,v_n}(W)$.

When we regard $\mathfrak{u}(W)^*$ as Hermitian operators on W, then coadjoint $\mathrm{U}(W)$ orbits are parametrized by multisets $\nu = \{\nu_1^{\oplus m_1}, \ldots, \nu_n^{\oplus m_n}\}$. The O_ν coadjoint orbit is the set of Hermitian operators having distinct eigenvalues $\nu_1, \ldots, \nu_n$ which occur with multiplicities $m_1, \ldots, m_n$. The coadjoint orbit O_ν is diffeomorphic to a partial flag manifold via the map taking a Hermitian matrix to its eigenspaces (which is neccesarily an orthogonal decomposition of W). Note that the type of the resulting flag manifold is determined by the multiplicities. So the last statement of the theorem follows from the main statement.

To illustrate the methods, we give a complete proof starting with a couple of lemmas in linear algebra:

Lemma 4. *Let V, W be Hermitian vector spaces and let $V = U_1 \oplus \ldots \oplus U_p$ be an orthogonal decomposition.*

Suppose that $B : V \to W$ is a linear map such that if $v \in U_i$ and $v' \in U_j$ then:

$$\langle Bv, Bv' \rangle = 0 \quad \text{if } i \neq j$$
$$\langle Bv, Bv' \rangle = \nu_i \langle v, v' \rangle \quad \text{if } i = j$$

with $\nu_i \neq 0$ for all i.

Then we have the following orthogonal decomposition: $W = \tilde{U}_1 \oplus \ldots \oplus \tilde{U}_p \oplus Z$ where $\tilde{U}_i = BU_i$ and $Z = (BU)^\perp$.

With respect to this decomposition $BB^* : W \to W$ is given by:

$$BB^*(\tilde{u}_1 + \cdots + \tilde{u}_p + z) = \nu_1 \tilde{u}_1 + \cdots + \nu_p \tilde{u}_p$$

Proof. We have that B is injective on each U_i and hence injective. For each i let u_i be such that $Bu_i = \tilde{u}_i$.

Let $x \in V$. Then:

$$\langle B^*(\tilde{u}_1 + \cdots + \tilde{u}_p + z), x \rangle = \langle Bu_1 + \cdots + Bu_p, Bx \rangle$$

Hence $B^*(\tilde{u}_1 + \cdots + \tilde{u}_p + z) = \nu_1 u_1 + \cdots + \nu_p u_p$ and the result follows. □

Lemma 5. *Let V, W be Hermitian vector spaces and let $V = U_1 \oplus \ldots \oplus U_p$ be an orthogonal decomposition.*

Let $H : V \to V$ be a linear map such that $H|_{U_i} = \nu_i \mathbf{1}$.

Let $C : V \to W$ be a linear map such that $C^*C - H = \gamma \mathbf{1}$.

Then if $v \in U_i$ and $v' \in U_j$ then:

$$\langle Cv, Cv' \rangle = 0 \quad \text{if } i \neq j$$
$$\langle Cv, Cv' \rangle = (\gamma + \nu_i) \langle v, v' \rangle \quad \text{if } i = j$$

Proof. Let $v \in U_i, v' \in U_j$. Then:

$$\langle C^*Cv, v' \rangle - \langle Hv, v' \rangle = \langle \gamma v, v' \rangle$$
$$\implies \langle Cv, Cv' \rangle = \gamma \langle v, v' \rangle + \nu_i \langle v, v' \rangle$$

as so the result follows. □

Armed with these lemmas we proceed to the proof of our theorem:

Proof. The first step is to show that DD^* has the advertised eigenvalues and multiplicities.

Note that the momentum map condition $\mu = (\lambda_1, \ldots, \lambda_n)$ becomes:

$$B_1^* B_1 = \lambda_1 \mathbf{1}, \quad B_i^* B_i - B_{i-1} B_{i-1}^* = \lambda_i \mathbf{1} \ \text{ for } 1 < i < n,$$
$$D^* D - B_{n-1} B_{n-1}^* = \lambda_n \mathbf{1}$$

We proceed through the quiver. Note that the B_1 satisfies the conditions of Lemma 5 with $H = 0$. Hence we deduce that B_1 satisfies the conditions of Lemma 4 with $p = 1$ and $\nu_1 = \lambda_1$. Hence $B_1 B_1^*$ has eigenvalues $\lambda_1, 0$ occuring with multiplicities $v_1, v_2 - v_1$.

Now apply Lemma 5 to $C = B_2, H = B_1 B_1^*$. This shows that B_2 statisfies the conditions of the Lemma 4 with $p = 2$, $\dim U_1 = v_1, \dim U_2 = v_2 - v_1$ and $\nu_1 = \lambda_2 - \lambda_1, \nu_2 = \lambda_2$. Hence by Lemma 4 $B_2 B_2^*$ has eigenvalues $\lambda_2 - \lambda_1, \lambda_2, 0$ with multiplicities $v_1, v_2 - v_1, v_3 - v_2$.

Hence, it follows inductively that DD^* has eigenvalues $\lambda_n - \lambda_{n-1} + \cdots \pm \lambda_1, \lambda_n - \lambda_{n-1} + \cdots \pm \lambda_2, \ldots, \lambda_n, 0$ and that they occur with multiplicities $v_1, v_2 - v_1, \ldots, w - v_n$.

So we see that ψ does indeed land in the desired coadjoint orbit. Note that the action of $\mathrm{U}(V)$ on $\mathrm{Hom}(V, W)$ does not affect DD^*, so ψ is well defined.

Also, ψ is a $\mathrm{U}(W)$-equivariant map since the action of $\mathrm{U}(W)$ on $\mathrm{Hom}(V, W)$ conjugates DD^*.

To show that the map is 1-1, suppose that we have $(B_1, \ldots, B_n, D)$ and $(B_1', \ldots, B_n', D') \in \mu^{-1}(\lambda)$ with $DD^* = D'D'^*$.

We need to show that these two elements are related by the action of $\mathrm{U}(V)$. But note that the orthogonal decomposition of W is fixed by the eigenspace decomposition of $DD^* = D'D'^*$. So to turn D' into D, we need only to adjust the orthogonal decomposition of V_n and then precompose by a unitary map on each U_i. This can be accomplished by an element of $\mathrm{U}(V_n)$. Now the relation between B_n and B_n' is the same as it was for D and D' so we can similarly transform on into the other using $\mathrm{U}(V_{n-1})$. Proceeding, we see that $(B_1, \ldots, B_n, D)$ and $(B_1', \ldots, B_n', D')$ are in fact related by an element of $\mathrm{U}(V)$.

Hence ψ is 1-1.

Now, since ψ is $\mathrm{U}(W)$ equivariant it must map onto a union of $\mathrm{U}(W)$ orbits. Hence the map is onto.

A calculation involving the symplectic form (which we omit) completes the proof. □

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