Symmetric Submanifolds of Riemannian Symmetric Spaces

Sankaran Viswanath

May 11, 2000

1 Introduction

A symmetric space is a Riemannian manifold that is "symmetric" about each of its points: for each $p \in M$ there is an isometry σ_p of M such that $(\sigma_p)_* = -I$ on T_pM. Symmetric spaces and their local versions were studied and classified by E.Cartan in the 1920's. In 1980 D.Ferus [F2] introduced the concept of *symmetric submanifolds* of Euclidean space: A submanifold M of $\mathbb{R}^n$ is a symmetric submanifold if and only if it is preserved by reflections at each of its normal spaces. Ferus then went on to classify all symmetric submanifolds of $\mathbb{R}^n$. Ferus' notion can be generalized to define a symmetric submanifold of any ambient Riemannian manifold. Backes and Reckziegel [BR] established a criterion that identifies symmetric submanifolds of the "standard" spaces $\mathbb{R}^n$, S^n, H^n of constant curvature.

In this survey article, we shall be concerned with symmetric submanifolds of an ambient manifold which is itself a symmetric space. We'll first outline the basic classification of symmetric spaces and then proceed to describe the work of Ferus, Backes, Reckziegel, Naitoh and others on finding symmetric submanifolds in specific ambient symmetric spaces.

2 Symmetric Spaces: Basic Classification

We briefly review facts about symmetric spaces, referring to Helgason [H] and Jöst [J] for details.

A *locally symmetric space* is a Riemannian manifold in which the geodesic symmetry at each point is an isometry in a normal neighborhood of the point. Symmetric spaces are locally symmetric too; the geodesic symmetries in this case are global isometries.
We now have [H]

Theorem 2.1 (E.Cartan) *Let M be a Riemannian manifold. Then M is locally symmetric $\Leftrightarrow \nabla R = 0$, where R is the curvature tensor of M, and ∇ is the connection induced on 4-tensors on M by the Levi-Civita connection of M.*

Theorem 2.2 *A complete, locally symmetric, simply connected Riemannian manifold is a symmetric space.*

Examples: $\mathbb{E}^n$ (Euclidean space), S^n (the sphere) and H^n (hyperbolic space) are all symmetric spaces. [J]

One can obtain all information about a Riemannian symmetric space from its group of isometries. If M is a symmetric space and G its group of isometries, then G acts transitively on M. Fix $p \in M$ and let H be the isotropy subgroup at p. G has a Lie group structure and H is a closed Lie subgroup of G. The Lie algebra $\mathfrak{g}$ of G is just the space of Killing vector fields on M. The Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$ and has a natural complementary subspace $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. It turns out that the characterization of symmetric spaces is the same as characterization of such triples $(\mathfrak{g}, \mathfrak{h}, \mathfrak{p})$.

Definition 2.3 *Let M be a symmetric space and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as above. Then:*
M is of Euclidean Type *if* $[\mathfrak{p}, \mathfrak{p}] = 0$
M is of Compact Type *if $\mathfrak{g}$ is a semisimple Lie algebra and M is of nonnegative curvature*
M is of Noncompact Type *if $\mathfrak{g}$ is a semisimple Lie algebra and M is of nonpositive curvature*

Theorem 2.4 [H] *Let M be a simply connected symmetric space. Then*

$$M \simeq M_0 \times M_+ \times M_-$$

where M_0, M_+, M_- are symmetric spaces of Euclidean, compact and noncompact types respectively.

Definition 2.5 *The* rank *of a symmetric space M is the dimension of the largest abelian subalgebra of* $\mathfrak{p}$.[Notation as above]

3 Symmetric Submanifolds

Definition 3.1 *Let M be a Riemannian manifold and $S \subset M$ a regular(="embedded") submanifold. S is called a* symmetric submanifold *of M iff $\forall p \in S$, $\exists$ an isometry t_p of M satisfying the following:*

1. $t_p(p) = p$, $t_p(S) = S$
2. $(t_p)_*(\xi) = \xi \; \forall \xi \in (T_pS)^\perp \subset T_pM$
3. $(t_p)_*(X) = -X \; \forall X \in (T_pS)$

Remarks 3.2 Setting $\sigma_p : S \to S$ equal to $t_p|_S$, we get $(\sigma_p)_* = -I$ on T_pS. Hence S is a Riemannian symmetric space in its own right. The t_p's serve as "extrinsic" symmetries of S in M. One could very easily extend this notion to define "symmetric immersions" of S into M. In another direction, by demanding that all conditions hold locally, we can get a notion of a locally symmetric submanifold of M. We refer to Naitoh [N] for the relevant definitions.

Example 3.3 *S^n is a symmetric submanifold of $\mathbb{R}^{n+1}$ $\forall n \geq 1$.*

Ferus [F2] completely classified the locally symmetric submanifolds of $\mathbb{E}^n$. To carry this out he proved an analogue of Theorem 2.1.

Theorem 3.4 *Let $S \subset \mathbb{E}^n$ be a regular submanifold and $l : TS \times TS \to (TS)^{\perp}$ be its second fundamental form. S is a symmetric submanifold of $\mathbb{E}^n \Rightarrow \nabla l = 0$.*

The parallelism of l ($\nabla l = 0$) means that

$$\mathcal{D}_X(l(Y,Z)) = l(\nabla_X Y, Z) + l(Y, \nabla_X Z)$$

for all vector fields X, Y, Z on S. Here ∇ denotes the Levi Civita connection on S, and $\mathcal{D}$ is the connection induced on the *normal bundle* of S by the Levi Civita connection of $\mathbb{E}^n$.

Example 3.5 *A class of submanifolds of $\mathbb{E}^n$ with parallel second fundamental form :* Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Let $\eta \in \mathfrak{g}$ be such that $ad\eta$ is semisimple and $(ad\eta)^3 = (ad\eta)$. Choose a *Cartan Decomposition* (see [H]) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ *with* $\eta \in \mathfrak{p}$. Let H be the maximal compact subgroup of G generated by $\exp(\mathfrak{h})$ and let $H_0 = \{h \in H | Ad\, h(\eta) = \eta\}$. Then $f : M = H/H_0 \to \mathfrak{p}$ sending $[h] \to Ad\, h(\eta)$ is an embedding of M into the Euclidean space $\mathfrak{p}$. Here we choose the metric on $\mathfrak{p}$ to be the restriction of the Killing form of $\mathfrak{g}$.

It turns out [F1] that $f(M)$ has parallel second fundamental form in $\mathfrak{p}$. In a later work [F2], Ferus shows that $f(M)$ is also a symmetric submanifold of $\mathfrak{p}$. Such an M is called a *Symmetric R-space.*

We infact have :

Theorem 3.6 (Ferus) *The symmetric R-spaces are essentially all the submanifolds of $\mathbb{E}^n$ with parallel second fundamental form .* [cf [F2], Theorem 2 for a precise statement]

Theorem 3.6, the observations preceding it, and Theorem 3.4 now combine to tell us that the symmetric R-spaces are all the symmetric submanifolds of Euclidean space.

Symmetric R-spaces include all Riemannian symmetric spaces of compact type except for some exceptional cases. Notice that our earlier example of $S = S^1$, $M = \mathbb{R}^2$ corresponds to the choices $G = SL_2\mathbb{R}$, $H = SO_2\mathbb{R}$
$\mathfrak{h} = \text{Span}\left(\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\right) \subset \mathfrak{sl}_2\mathbb{R}$, $\mathfrak{p} = \text{Span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right\}$ and $\eta = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$.
Theorem 3.6 and the above remarks now imply:
S is a symmetric submanifold $\Leftrightarrow \nabla l = 0$, where l is the second fundamental form of S.
Strübing gave a direct proof of this fact, even when the ambient manifold $\mathbb{E}^n$ was replaced by S^n or H^n. We state this as :

Theorem 3.7 (Ferus, Strübing) *Let $S \subset M$ be a connected, complete, regular submanifold where $M = \mathbb{E}^n$, S^n or H^n. Then the following are equivalent:*

1. *S is a symmetric submanifold of M*
2. *S has parallel second fundamental form*

Finding symmetric submanifolds of the standard spaces $\mathbb{E}^n$, S^n, H^n of constant curvature is thus reduced to finding complete submanifolds with $\nabla l = 0$. (Observe that a symmetric submanifold , being an (intrinsic) Riemannian symmetric space , is complete)

4 Symmetric Submanifolds of Spaces of Constant Curvature

Let N^n_κ denote the standard n-dimensional space of constant curvature κ. So $N^n_0 = \mathbb{R}^n$, $N^n_{\kappa>0} = S^n(\kappa)$ and $N^n_{\kappa<0} = H^n(\kappa)$. The first reduction of the problem of finding symmetric submanifolds of N^n_κ is from the following theorem, which together with Theorem 3.7 says that symmetric submanifolds are determined by their "initial data".

Theorem 4.1 (Reckziegel [R]) *Let S and T be connected, complete, regular submanifolds of a Riemannian manifold M, with parallel second fundamental forms . Further let $p \in S \cap T$ and suppose $T_pS = T_pT$ and $l_S = l_T$ on T_pS $(= T_pT)$, where l_S and l_T are the second fundamental forms of S and T taking values in $T(S)^\perp$ and $T(T)^\perp$ repectively. Then $S = T$.*

For a fixed $p \in N^n_\kappa$, finding symmetric submanifolds S of N^n_κ containing p is now reduced to the characterization of pairs (E, l) of a subspace $E \subset T_p N^n_\kappa$ and a bilinear map $E \times E \xrightarrow{l} E^\perp$ that can occur as "initial data" of symmetric submanifolds.

The characterization turns out to be a neat algebraic condition. To this end, we define :

Definition 4.2 *Let E be a subspace of a Euclidean vector space V, $l : E \times E \to E^\perp$ (in V), a symmetric bilinear map and let $\kappa \in \mathbb{R}$ be arbitrary. The* Triple System *L associated to (E, l) is the bilinear map : $L = S + R$: $E \times E \to End(E)$ where the symmetric and skew-symmetric parts of L are given by :*

$$< S(x,y)v, w >= \kappa < x, y >< v, w > + < l(x,y), l(v,w) >$$

and

$$\begin{aligned} R(x,y)z &= S(y,z)x - S(x,z)y \\ &i.e \\ < R(x,y)z, w > &= \kappa(< x, w >< y, v > - < x, v >< y, w >) \\ &\quad + (< l(x,w), l(y,v) > - < l(x,v), l(y,w) >) \end{aligned}$$

One easily checks that $S = S^*$ and $R = -R^*$ wrt $< .,. >$ and that $S(x,y) = S(y,x)$, $R(x,y) = -R(y,x)$.

Consequently:

$$L(x,y)^* = S(x,y) - R(x,y) = L(y,x) \tag{1}$$

Another easy computation gives

$$L(x,y)z = L(z,y)x \; \forall x, y, z \in E \tag{2}$$

Definition 4.3 *Let E be a Euclidean vector space. A bilinear map $L : E \times E \to End(E)$ is called a* **Euclidean Jordan Triple system** *(EJTS) on E iff the formulas* (1), (2) *and* (3) *below hold :*

$$[L(x,y), L(v,w)] = L(L(x,y)v, w) - L(v, L(y,x)w) \tag{3}$$

Our story of finding symmetric submanifolds is completed by the following theorem, which states that the triple system associated to the initial data $(E, l) = (T_p S, l_p)$ of a symmetric submanifold S of N^n_κ containing p is an EJTS.

Theorem 4.4 (Backes and Reckziegel [BR]) *Let $p \in N^n_\kappa$, $E \subset T_p N^n_\kappa$ be a subspace and $l : E \times E \to E^\perp$ a symmetric bilinear map. Then the following are equivalent :*

1. *There is a symmetric submanifold S of N^n_κ such that $p \in S$, $T_p S = E$ and l = the second fundamental form of S at p*

2. *The triple system L associated to* (E, l) [Definition 4.2] *is a EJTS. Further the S in (1) is unique*

Since the triple system L of Definition 4.2 always satisfies the relations (1) and (2), the real constraint imposed by the fact that S is a symmetric submanifold is the formula (3) for the commutator. Consequently (3) is the algebraic condition that characterizes initial data (E, l) of symmetric submanifolds of N^n_κ.

5 Other ambient symmetric spaces

The key to our earlier results was Theorem 3.7 [Symmetric submanifold $\Leftrightarrow$ $\nabla l = 0$]. Naitoh [N] investigated symmetric submanifolds of compact simply connected symmetric spaces and proved a generalization of Theorem 3.7.

Theorem 5.1 (Naitoh) *Let $S \subset M$ be a connected, complete, regular submanifold of a simply connected Riemannian symmetric space M. Then the following are equivalent:*

1. *S is a symmetric submanifold of M*
2. *S has parallel second fundamental form and the normal spaces $(T_pS)^\perp$ are* **curvature invariant** $\forall p \in S$

Curvature Invariance of a subspace $V \subset T_pM$ means that $R(V, V)V \subset V$, where R is the curvature tensor of M.

For $M = \mathbb{E}^n$, S^n or H^n, given any regular submanifold (=embedded submanifold) $S \subset M$, it turns out that its normal spaces are automatically curvature invariant. So, Theorem 5.1 indeed generalizes Theorem 3.7.

Theorem 5.1 allows us to look only for $S \subset M$ with parallel second fundamental form and curvature invariant normal spaces. When M is also compact, Naitoh reduces the classification of such S's to that of certain algebraic objects associated with a Lie group acting on M [N]. For other isolated results when the ambient spaces are certain rank 1 symmetric spaces , we refer to the list of papers cited by Naitoh in [N].

References

[BR] E. Backes and H. Reckziegel, *On symmetric submanifolds of spaces of constant curvature*, Math Ann. 263 (1983), 419-433.

[F1] D. Ferus, *Immersions with parallel second fundamental form*, Math Z. 140 (1974), 87-93.

[F2] D. Ferus, *Symmetric submanifolds of Euclidean space*, Math Ann. 247 (1980), 81-93.

[H] S. Helgason, *Differential Geometry, Lie groups and Symmetric spaces*, Academic Press (1978).

[J] J. Jöst, *Riemannian geometry and geometric analysis*, Springer (1998).

[N] H. Naitoh, *Symmetric submanifolds of compact symmetric spaces*, Tsukuba J Math 10 (1986), no. 2, 215-242.

[R] H. Reckziegel, *On the problem whether the image of a given differentiable map into a Riemannian manifold is contained in a submanifold with parallel second fundamental form*, J Reine Angew Math 325(1981), 87-103.

[S] W. Strübing, *Symmetric submanifolds of Riemannian manifolds*, Math Ann 245 (1979), 37-44.