RIEMANNIAN MANIFOLDS WITH INTEGRABLE GEODESIC FLOWS

ANDREW MILLER

1. Introduction

In this paper we will survey some recent results on the Hamiltonian dynamics of the geodesic flow of a Riemannian manifold. More specifically, we are interested in those manifolds which admit a Riemannian metric for which the geodesic flow is integrable.

In Section 2, we introduce the necessary topics from symplectic geometry and Hamiltonian dynamics (and, in particular, we defined the terms *geodesic flow* and *integrable*). In Section 3, we discuss several examples of manifolds which admit metrics whose geodesic flows are integrable, and in Section 4 we consider the topology of manifolds with integrable geodesic flows.

2. Geometric Preliminaries

2.1. **Symplectic Manifolds and Hamiltonian Dynamics.**

Details on topics in this section can be found in the classic textbook [1]

A *symplectic manifold* (M, ω) is a smooth manifold M together with a smooth, closed, nondegenerate 2-form ω, called the *symplectic form*. Note that the nondegeneracy condition on ω requires that M is even-dimensional. A *Hamiltonian* on M is a smooth function $H: M \to \mathbb{R}$. To every Hamiltonian H on M there is associated a vector field ξ_H on M, defined by the condition

$$dH(v) = \omega(v, \xi_h) \quad \text{for all } v \in TM.$$

The vector field ξ_H is called the *Hamiltonian vector field* of the Hamiltonian H, or sometimes the *symplectic gradient* $\operatorname{sgrad} H$. The *Hamiltonian flow* is then defined as the flow of this vector field on H.

The Hamiltonian flow behaves very nicely in relation to the symplectic form. In the first place, we have

Theorem 2.1. *A Hamiltonian phase flow preserves the symplectic form ω; i.e., if g^t denotes the flow, we have*

$$(g^t)^*\omega = \omega.$$

Recall that an *integral* of a flow on M is a smooth, real-valued function on M which is constant on every orbit of the flow. Another property of a Hamiltonian flow is then the following

Theorem 2.2. *The function H is an integral of the Hamiltonian phase flow with Hamiltonian H.*

Proof.

$$dH(\xi_H) = \omega(\xi_H, \xi_H) = 0.$$ □

Theorem 2.2 is the mathematical formulation of the mechanical principle of the conservation of energy.

From the symplectic form, we get an additional structure on the algebra of Hamiltonians on M, i.e. the algebra $C^\infty(M, \mathbb{R})$.

Definition 2.3. Let F, G be Hamiltonians on the symplectic manifold (M, ω), and let ξ_G denote the Hamiltonian vector field of G. We define the *Poisson bracket* $\{F, G\}$ of F and G as

$$\{F, G\} = dF(\xi_G).$$

Then $\{F, G\}$ is the derivative of F in the direction of the Hamiltonian flow of G.

The Poisson bracket is, in fact, a Lie bracket for the algebra of Hamiltonians on M. That is, it is bilinear, skew-symmetric, and it satisfies the *Jacobi identity*

$$\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0.$$

Two functions f_1, f_2 on M are said to be *in involution* if they Poisson-commute ($\{f_1, f_2\} = 0$). It is immediate from the definition of the Poisson bracket that a function f on M is an integral for the Hamiltonian flow of H if and only if f and H are in involution.

Definition 2.4. Suppose that $\{f_1, f_2, \ldots, f_k\}$ is a set of integrals of a flow on a manifold M. We say that these integrals are *independent at* $x \in M$ if their differentials $\{df_1, df_2, \ldots, df_k\}$ at x form a linearly independent subset of T_x^*M.

With this terminology in hand, we are finally ready to state the definition of integrability in Hamiltonian dynamics:

Definition 2.5. The flow of a Hamiltonian H on a symplectic manifold M^{2n} is said to be *integrable*, or *completely integrable* if there exist n everywhere independent integrals $f_1 = H, f_2, \ldots, f_n$ of the flow which are in involution.

A classical theorem of Liouville states that when a Hamiltonian flow is integrable, the flow itself is geometrically very simple. Liouville's theorem asserts the existence of *action-angle coordinates* on M in which the flow behaves as quasiperiodic flows on tori. Thus, the phase space of an integrable Hamiltonian system is foliated by invariant tori.

Note also that the independence condition in Definition 2.5 is often weakened to require only that the integrals are independent almost everywhere, or on an open dense subset of M.

2.2. **The Geometry of the Geodesic Flow.** Let (M^n, g) be a Riemannian manifold with metric $g = (g_{ij})$.

One way to place the geodesic equations of M into the context of Hamiltonian dynamics is to look at the cotangent bundle T^*M. We put the canonical symplectic structure on T^*M and define the Hamiltonian H on T^*M in local coordinates by

$$H(x, p) = \frac{1}{2} \sum g^{ij} p_i p_j.$$

That is, we use the metric-induced diffeomorphism $TM \cong T^*M$ to pull-back the energy function on TM to T^*M. The Hamiltonian flow of H is called the *cogeodesic flow*; the projections of cogeodesic trajectories onto M are geodesics. This approach is taken, for example, in [5].

An equivalent approach is to put a symplectic structure directly on TM, i.e. a closed, nondegenerate, smooth 2-form ω on TTM. For more details on the constructions to follow, see [17].

Let $\pi : TM \to M$ be the projection onto M and let ∇ be the Levi-Civita connection on M given by the metric g . Using π and ∇, we decompose $T_\theta TM$ into vertical and horizontal subbundles. Namely, the *vertical subbundle* is the subbundle of TTM whose fiber $V(\theta)$ at $\theta \in TM$ is given by

$$V(\theta) = \ker d_\theta \pi.$$

The *horizontal subbundle* is the subbundle whose fiber $H(\theta)$ at $\theta = (x, v)$ is given by

$$H(\theta) = \ker K_\theta,$$

where $K_\theta : T_\theta TM \to T_x M$ is a linear map defined in terms of ∇. Then

$$T_\theta TM = H(\theta) \oplus V(\theta),$$

and we get an isomorphism

$$T_\theta TM \to T_x M \times T_x M$$

given by

$$\xi \mapsto (d_\theta \pi(\xi), K_\theta(\xi)).$$

.

Finally, we define the form ω on TM by setting

$$\omega_\theta(\xi, \eta) = \langle K_\theta(\xi), d_\theta \pi(\eta) \rangle - \langle d_\theta \pi(\xi), K_\theta(\eta) \rangle$$

for all $\xi, \eta \in T_\theta TM$. It is clear from the definition that ω is skew-symmetric. In fact, ω is also nondegenerate and exact, and hence closed, so (TM, ω) is a symplectic manifold.

The *geodesic flow* g^t on TM is defined as follows: Let $(x, v) \in TM$, and let c_v denote the unique geodesic passing through x with initial velocity v. Then we set

$$g^t(x, v) = (c_v(t), \dot{c}_v(t)).$$

Now define the function $H : TM \to \mathbb{R}$ by $H(x, v) = \frac{1}{2}\langle v, v \rangle_x$, and let G denote the vector field on TM obtained from the geodesic flow.

Theorem 2.6. *G is the Hamiltonian vector field for H. That is,*

$$dH(\zeta) = \omega(\zeta, G) \quad \text{for all } \zeta \in TTM.$$

Thus, the geodesic flow is the Hamiltonian flow of H on (TM, ω), so it makes sense to speak of the integrability or nonintegrability of the geodesic flow on a Riemannian manifold.

3. Examples of Manifolds with Integrable Geodesic Flow

3.1. **The Classical Examples.** Prior to the papers of Mishchenko and Fomenko [13] and Thimm [22], only four examples of integrable geodesic flows were known: flat metrics on $\mathbb{R}^n$ and T^n, surfaces of revolution, left-invariant metrics on $SO(3)$, and n-dimensional ellipsoids with different principal axes.

The integrability of surfaces of revolution is readily seen by using another mechanical formalism: Lagrangian dynamics. Let N be a surface of revolution endowed with the metric induced from $\mathbb{R}^3$. Then geodesics on N are extremal paths for the Lagrangian $L : TN \to \mathbb{R}$

$$L(x, v) = \langle v, v \rangle.$$

(For readers unfamiliar with Lagrangian mechanics, in this case it is nothing but the variational calculus of geodesics.) If we use local cylindrical coordinates (r, ϕ) on N, then ϕ is easily seen to be a cyclic coordinate for the Euler-Lagrange equations, and hence the corresponding generalized momentum $P = r^2\dot{\phi}$ is conserved. Therefore, P and L form a complete set of integrals of the geodesic flow of N. For more details, and an interesting geometric discussion of P, see [1, §19]. This case was first integrated by Clairaut (18th c.), and so P is called "Clairaut's integral."

The geodesics of a left-invariant metric on $SO(3)$ can also be viewed in the context of classical mechanics. In this case, the geodesics are trajectories of the mechanical system of a freely rotating rigid body. That this system is integrable is well-known to any student of mechanics. Euler was the first to integrate this system, in 1765; for details, consult [1, Ch. 6 and App. 2].

The case of the n-dimensional ellipsoid is more difficult than those above. In 1838, Jacobi integrated the triaxial ellipsoid. There are many proofs in the literature of the integrability of the geodesic flow of the ellipsoid; see [20] for one such proof, as well as references to several others.

3.2. **Recent Examples.** Thimm's paper [22] sparked a flurry of recent activity in this field. In this paper, the author developed a technique, now called the *Thimm method*, for obtaining integrals in involution of the geodesic flow on certain homogeneous spaces.

The essence of his method is to use extra information obtainable when we have a Lie group G of isometries of a Riemannian manifold M which generates a *Hamiltonian action* on TM. (For the definition of a Hamiltonian action, and of the *moment map* discussed below, see [1, Appendix 5]; note that Arnold calls these actions *Poisson actions*.)

Given such a Hamiltonian action by G on TM, we get a natural map

$$\Phi : TM \to \mathfrak{g}^*,$$

called the moment map (here, as usual, $\mathfrak{g}^*$ denotes the dual to the Lie algebra of G). Then we have

Theorem 3.1. *For any $f : \mathfrak{g}^* \to \mathbb{R}$, the pullback $\Phi^* f = f \circ \Phi$ commutes with any G-invariant Hamiltonian H on TM (in particular, it commutes with the Hamiltonian induced by the metric). Moreover, the pullback is a homomorphism of Poisson algebras; in particular, if f and g are in involution, then $\Phi^* f$ and $\Phi^* g$ are in involution.*

(The Poisson structure on $C^\infty(\mathfrak{g}^*)$ is defined by

$$\{f, g\}(\alpha) = \alpha([df_\alpha, dg_\alpha]) \quad).$$

Thus, the problem of finding functions in involution on TM is reduced to finding functions in involution on $\mathfrak{g}^*$. The rest of Thimm's method consists of a general construction by which one can find such functions. For details, see Thimm's paper or the excellent discussion in [18].

Thimm then applies his method to specific examples. In each case, and in most applications of Thimm's method, the real difficulty lies in showing that the functions one obtains are independent.

Theorem 3.2 (Thimm)**.** *The following manifolds admit Riemannian metrics with integrable geodesic flows:*

- *Real and complex Grassmannians*
- *Distance spheres in $\mathbb{C}P^{n+1}$*
- $SU(n+1)/SO(n+1)$

- $SO(n+1)/SO(n-1)$
- $\mathbb{C}H^n = U(n,1)/U(n) \times U(1)$
- $U(n+1)/O(n+1)$, *which can be viewed as the Lagrangian subspaces of a symplectic vector space*

Bloch, Brocket, and Crouch [3] give an alternative proof of the independence of the integrals for the Grassmannians which is more conceptual than Thimm's.

Many authors have applied or extended Thimm's method to obtain new examples of manifolds with integrable geodesic flows. One such particularly interesting application is due to R. P. Gomez in [9], in which he proved that the geodesic flow on infinite-dimensional Hilbert Grassmannians is integrable.

G. P. Paternain and R. J. Spatzier [18] have done considerable work in extending Thimm's method. One of the first results in the paper is that the geodesic flow of a left-invariant metric on $SU(3)/T^2$, where T^2 is a maximal torus, is integrable provided that not every geodesic is a 1-parameter subgroup of $SU(3)$.

In the remainder of the paper, the authors are concerned with manifolds which the original Thimm method is unable to analyze. One limitation of Thimm's method is that it only produces example of homogeneous manifolds with integrable geodesic flows. To remove this limitation, Paternain and Spatzier paper use Riemannian submersions in conjunction with the Thimm method, constructing examples of inhomogeneous manifolds with integrable geodesic flows.

Riemannian submersions are defined as follows: Consider a submersion $f : M \to N$ between Riemannian manifolds. Suppose $x \in N$ is such that $f(p) = x$ for some $p \in M$, and let $K_p = \ker d_p f$. Since f is a submersion, every $v \in T_x N$ can be lifted uniquely to a vector $f^* v \in K^\perp \subset T_p M$.

Definition 3.3. The map f is called a *Riemannian submersion* if the isomorphism

$$f^* : T_x N \to K_p^\perp$$

is an isometry.

A particularly nice example of a Riemannian submersion arises when we have a group Γ acting freely and properly on a Riemannian manifold M. Then $B = M/\Gamma$ is a manifold, and we can define a metric on B by asserting that the projection $\pi : M \to B$ is a Riemannian submersion. This metric is called the *submersion metric* on the quotient manifold B.

It is well-known that in this case TB is symplectomorphic with the symplectic reduction ([1, Appendix 5]) of TM by Γ. Thus, Γ-invariant functions in involution on TM descend to functions in involution on TB. Again, the difficulty arises in showing that these functions on TB are independent.

Using this method, the authors consider the following example: Suppose X is a Riemannian manifold with integrable geodesic flow such that S^1 acts freely by isometries on X. Suppose further that S^1 leaves the integrals invariant. If N is a complete surface of revolution, then there is a diagonal action by S^1 on $X \times N$.

Theorem 3.4. *The geodesic flow on the quotient manifold $X \times_{S^1} N$, endowed with the submersion metric, is integrable.*

By using appropriate X and N, the authors obtain

Corollary 3.5. $\mathbb{C}P^n \# \overline{\mathbb{C}P^n}$ *admits a metric with integrable geodesic flow.*

Theorem 3.4 also applies to certain inhomogeneous manifolds constructed by J.-H. Eschenburg [8]. The authors prove that surface bundles over some of these manifolds have integrable geodesic flows. Furthermore, they prove by more general arguments that some of the Eschenburg examples themselves have integrable geodesic flows. The interest in these manifolds lies in the fact that, in contrast to all of Thimm's examples, they are *strongly inhomogeneous*; this means that they do not have the homotopy type of any compact homogeneous manifold.

Paternain and Spatzier also prove that an exotic 7-sphere of Gromoll and Meyer, and $\mathbb{C}P^n \# \mathbb{C}P^n$ for n odd, admit integrable geodesic flows.

The methods listed so far depend in part on the existence of a large group of isometries. C. E. Durán [7] has obtained a very general result which covers many manifolds which do not have this property.

Definition 3.6. A Riemannian manifold M is a *Z-manifold* if for all $p \in M$, all geodesics emanating from p return to p

Examples of Z-manifolds include the compact rank one symmetric spaces (CROSSes): $S^2, \mathbb{K}P^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and the Cayley plane.

Theorem 3.7 (Durán)**.** *The geodesic flow on any Z-manifold is integrable.*

Corollary 3.8. *The geodesic flow on a space form M/Γ, where M is a CROSS and Γ is a finite group acting freely by isometries, is integrable.*

In particular, we see that the lens spaces have integrable geodesic flows.

Durán's paper is fascinating not only for the results but also for the proofs. He proves Theorem 3.7 by considering the space $\mathrm{Geod}(M)$ of geodesics of M. For general Z-manifolds, $\mathrm{Geod}(M)$ is not a manifold. However, it does have the structure of a *symplectic V-manifold*, a sort of symplectic "manifold with singularities." (Nowadays, V-manifolds are referred to as *orbifolds*.) In fact, using the V-manifold structure, Durán is able to prove a stronger result than that stated above:

Definition 3.9. Let $\{H, f_1, \ldots, f_{n-1}\}$ be an integrable system. The integrals are *tame* in a given energy level $H^{-1}(c)$ if the singular set of the map $x \mapsto (f_1(x), \ldots, f_{n-1}(x))$ is a polyhedron.

Theorem 3.10. *The geodesic flow of a Z-manifold is integrable with tame integrals.*

Note that Definition 3.9 and Theorem 3.10 are only of interest when we allow the integrals to be dependent on a set of measure zero (or on the complement of an open dense set). Since such a set may be quite pathological, it is desirable to have stronger conditions on the behavior of the integrals.

4. The Topology of Manifolds with Integrable Geodesic Flow

As well as constructing new examples of manifolds with integrable geodesic flow, G. P. Paternain has also done considerable work in studying the topological properties of such manifolds. Here we are asking what obstructions there might be to the existence of a metric with integrable geodesic flow on a given manifold.

Before proceeding, we need to state a couple of definitions.

Definition 4.1. Let M be a compact Riemannian manifold with integrable geodesic flow and integrals $f_1 = H, f_2, \ldots, f_n$. We say that the geodesic flow is integrable with *periodic integrals* if the Hamiltonian vector fields of $f_i, 2 \leq i \leq n$, generate circle actions.

Definition 4.2. A compact manifold M with finite fundamental group is said to be *rationally elliptic* if the total rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ of M is finite-dimensional.

With these definitions in hand, Paternain proves in [15]:

Theorem 4.3. *Let M be a compact Riemannian manifold whose geodesic flow is integrable with periodic integrals. Then $\pi_1(M)$ has sub-exponential growth, and if $\pi_1(M)$ is finite, M is rationally elliptic.*

(The growth of a finitely-generated group G is defined to be the growth of the function $n(k) = \{$the number of elements $g \in G$ whose minimal word has length $k\}$.)

By considering Hamiltonian systems which succumb to a generalized Thimm method, Paternain also proves:

Theorem 4.4. *Let M be a compact Riemannian manifold whose geodesic flow is integrable by the Thimm method. Then $\pi_1(M)$ has sub-exponential growth, and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Homogeneous spaces are known to be rationally elliptic, and rational ellipticity imposes some strong restrictions on the topology of a manifold. ([15] and references therein). For example, if a manifold M is rationally elliptic, then the Euler characteristic $\chi(M) \geq 0$, and $\chi(M) > 0$ if and only if $H_p(M, \mathbb{Q}) = 0$ for all odd p. Thus, Theorems 4.3 and 4.4 imply in particular that if M is a simply connected compact manifold with negative Euler characteristic, then M does not admit a metric whose geodesic flow is integrable with periodic integrals, nor a metric whose geodesic flow is integrable by the Thimm method.

An example of a manifold whose fundamental group has exponential growth is any manifold which admits a Riemannian metric of negative sectional curvature. Paternain notes that in all known examples of manifolds with integrable geodesic flow, $\pi_1(M)$ actually has polynomial growth. This fact and the above results lead him to propose:

Conjecture 4.5. *Let M be a compact Riemannian manifold whose geodesic flow is integrable. Then $\pi_1(M)$ has polynomial growth. Moreover, if $\pi_1(M)$ is finite, then M is rationally elliptic.*

Paternain proves Theorems 4.3 and 4.4 by considering a quantity associated to the geodesic flow called the *topological entropy*. Very roughly speaking, the topological entropy of a flow on a manifold is a number which measures the exponential complexity of the orbits of the flow. (For more on the topological entropy of the geodesic flow, including a definition, see [2, Chapter V] and [17].) Paternain proves that under the hypotheses of his theorems, the topological entropy h_{top} of the geodesic flow is zero (that is, in a sense, the dynamics of these flows are simple). As he also proves, $h_{top} = 0$ implies for compact Riemannian manifolds M with finite fundamental group that M is rationally elliptic. Hence the following is a stronger statement than Conjecture 4.5, at least in the analytic category.

Conjecture 4.6. *Let M be a Riemannian manifold with integrable geodesic flow such that the integrals are real analytic. Then the topological entropy of the geodesic flow is zero.*

A theorem of Kozlov's states that Conjecture 4.6 is true if M is two-dimensional. In [14], Paternain gives an alternate proof of this theorem and extends it to the case when the connected components of the set of critical points of the non-metric integral form sub-manifolds.

More can be shown if we add other restrictions to the integrals of the geodesic flow on M. Paternain proves in [16]

Theorem 4.7. *Let M be a compact Riemannian manifold whose geodesic flow is integrable with "nondegenerate" first integrals. Then $\pi_1(M)$ has sub-exponential growth. If $\pi_1(M)$ is finite, then the loop space homology of M, $\sum_{i=1}^{k} \dim H_i(\Omega M, K)$, grows sub-exponentially for any coefficient field K.*

Paternain's nondegeneracy condition is a bit too involved to describe here; the interested reader is referred to his paper. Suffice it to say that most of the well-known examples of integrable Hamiltonian systems satisfy his condition, so it is not too restrictive.

Returning to the real analytic category, Taimanov has proved [21]

Theorem 4.8. *If M admits a metric with integrable geodesic flow such that the integrals are real analytic, then*

- *$\pi_1(M)$ is almost abelian.*
- *$\dim H_1(M;\mathbb{Q}) > \dim M$.*
- *There is an injection of algebras $H^*(\mathbb{T}^d,\mathbb{Q}) \hookrightarrow H^*(M,\mathbb{Q})$, where $d = \dim H^1(M,\mathbb{Q})$.*

A group G is *almost abelian* if G contains an abelian subgroup with finite index. Topologically, $\pi_1(M)$ is almost abelian if and only if there is a finite-sheeted cover $\tilde{M} \to M$ such that $\pi_1(\tilde{M})$ is abelian. Moreover, if $\pi_1(M)$ is almost abelian, then $\pi_1(M)$ has polynomial growth of degree less than or equal to $\dim M$.

Examples of manifolds whose fundamental group is *not* almost abelian include compact manifolds which admit a metric with negative curvature. Thus, we have:

Corollary 4.9. *If a compact manifold M admits a metric of negative curvature, then for any real-analytic structure on M and any real-analytic metric on M, the geodesic flow of M is not analytically integrable.*

L. Butler [6] has constructed examples showing that Theorem 4.8 is not true for smooth integrals.

5. Closing Remarks

In this paper, we were interested in which manifolds admit metrics whose geodesic flows are integrable. A closely related question is this: Suppose that M admits metrics with integrable geodesic flows. Can we classify such metrics? The interested reader is referred to [4], [5], [10], [11], [12], [23]. [5] in particular has a very complete bibliography.

One can also study other aspects of the dynamics of the geodesic flow. One such topic is the ergodic theory of the geodesic flow. For an entry point into this field, see [2, Chapter V] or [17].

References

[1] V. I. Arnold. *Mathematical methods of classical mechanics.* Springer-Verlag, New York, 1989.

[2] M. Berger. *Riemannian geometry during the second half of the twentieth century.* American Mathematical Society, Providence, RI, 2000.

[3] A. M. Bloch, R. W. Brockett, and P. E. Crouch. Double bracket equations and geodesic flows on symmetric spaces. *Comm. Math. Phys.*, 187(2):357–373, 1997.

[4] A. V. Bolsinov and A. T. Fomenko. Application of classification theory for integrable Hamiltonian systems to geodesic flows on 2-sphere and 2-torus and to the description of the topological structure of momentum mapping near singular points. *J. Math. Sci.*, 78(5):542–555, 1996.

[5] A. V. Bolsinov, V. S. Matveev, and A. T. Fomenko. Two-dimensional Riemannian metrics with integrable geodesic flows. local and global geometry. *Sb. Mat.*, 189(10):1441–1466, 1998.

[6] L. Butler. A new class of homogeneous manifolds with Liouville-integrable geodesic flows. *C. R. Math. Acad. Sci. Soc. R. Can.*, 21(4):127–131, 1999.
[7] C. E. Durán. On the complete integrability of the geodesic flow of manifolds all of whose geodesics are closed. *Ergodic Theory Dynam. Systems*, 17(6):1359–1370, 1997.
[8] J.-H. Eschenburg. New examples of manifolds with strictly positive curvature. *Invent. Math.*, 66(3):469–480, 1982.
[9] R. P. Gomez. An integrable flow on a family of Hilbert Grassmannians. *New York J. Math.*, 2:69–85 (electronic), 1996.
[10] K. Kiyohara. Compact Liouville surfaces. *J. Math. Soc. Japan*, 43(3):555–591, 1991.
[11] K. Kiyohara. Two classes of Riemannian manifolds whose geodesic flows are integrable. *Mem. Amer. Math. Soc.*, 130(619):viii+143, 1997.
[12] V. S. Matveev and P. Topalov. Conjugate points of hyperbolic geodesics of square-integrable geodesic flows on closed surfaces. *Moscow Univ. Math. Bull.*, 53(1):39–41, 1998.
[13] A. S. Mishchenko and A. T. Fomenko. Euler equations on finite-dimensional Lie groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42:396–415, 1978.
[14] G. P. Paternain. Entropy and completely integrable Hamiltonian systems. *Proc. Amer. Math. Soc.*, 113(3):871–873, 1991.
[15] G. P. Paternain. On the topology of manifolds with completely integrable geodesic flows. *Ergodic Theory Dynam. Systems*, 12(1):109–121, 1992.
[16] G. P. Paternain. On the topology of manifolds with completely integrable geodesic flows. II. *J. Geom. Phys.*, 13(3):289–298, 1994.
[17] G. P. Paternain. *Geodesic flows.* Birkhäuser Boston Inc., Boston, MA, 1999.
[18] G. P. Paternain and R. J. Spatzier. New examples of manifolds with completely integrable geodesic flows. *Adv. Math.*, 108(2):346–366, 1994.
[19] R. J. Spatzier. Riemannian manifolds with completely integrable geodesic flows. In *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990)*, pages 599–608. Amer. Math. Soc., Providence, RI, 1993.
[20] S. Tabachnikov. Projectively equivalent metrics, exact transverse line fields and the geodesic flow on the ellipsoid. *Comment. Math. Helv.*, 74(2):306–321, 1999.
[21] I. A. Taĭmanov. Topological obstructions to integrability of geodesic flows on non-simply-connected manifolds. *Math. USSR-Izv.*, 30(2):403–409, 1988.
[22] A. Thimm. Integrable geodesic flows on homogeneous spaces. *Ergodic Theory Dynam. Systems*, 1(4):495–517 (1982), 1981.
[23] K. Yamato. A class of Riemannian manifolds with integrable geodesic flows. *J. Math. Soc. Japan*, 47(4):719–733, 1995.