Collapsing Riemannian Manifolds

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1 Introduction

In this survey paper we investigate the phenomenon of a changing metric on a differentiable manifold. This can be done inside the differentiable category and, apart from the intrinsic beauty of the question, it can be a strong tool of proof (see [H]). However, it appears that the natural setting for the problem is Gromov's metric structure on the class of all metric spaces. This facilitates examining asymptotic behaviour more systematically, allowing for singular limits of sequences of metrics.

In his ICM address [G1] Gromov launched a whole program of synthetic Riemannian geometry, meaning the systematic study of the structure of the space of all Riemannian structures. We will attempt to summarize the most important contributions so far to certain questions that arise here, concentrating on the collapse of manifolds for the most part.

2 Distance between metric spaces

The following concept is a classical one: Let A and B be subspaces of the metric space (Z, d). The *Hausdorff distance* between A and B is defined as

$$d_H^Z(A,B) = \inf\{\, \varepsilon > 0 : A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\,\},$$

where $U_\varepsilon(A)$ is the ε-neighborhood of A. Based on this, Gromov [G2] introduced

Definition 2.1. The *Hausdorff distance* $d_H(X,Y)$ between the metric spaces X and Y is the infimum of $d_H^Z(f(X), g(Y))$ over all metric spaces Z and isometric embeddings $f\colon X \to Z$, $g\colon Y \to Z$.

The distance d_H is defined for all pairs of metric spaces, and when diameters are finite it is guaranteed to be finite. Its symmetry is obvious, and the triangle inequality is not hard to show. For nondegeneracy, see Corollary 2.5 below. This will be our basic concept of distance, and for the most part, we will try to understand the metric structure resulting from this definition. However, there are other possibilities too. We mention at least one more, which has a more limited domain of definition. It uses the concept of *dilatation*, which for a map f from the metric space (X, d^X) to the metric space (Y, d^Y) is defined as follows:

$$\mathrm{dil}(f) = \sup_{x_1 \neq x_2} \frac{d^Y(f(x_1), f(x_2))}{d^X(x_1, x_2)}.$$

Definition 2.2. The *Lipschitz distance* between the metric spaces (X, d^X) and (Y, d^Y), denoted $d_L(X, Y)$, is the infimum of the numbers

$$|\ln(\mathrm{dil}(f))| + |\ln(\mathrm{dil}(f^{-1}))|$$

as f varies over the set of bi-Lipschitz homeomorphisms $f\colon X \to Y$. If there is no such map, the distance is ∞.

Proposition 2.3. *If two compact metric spaces X, Y satisfy $d_L(X, Y) = 0$ then they are isometric.*

Proof. For each integer $n > 0$, there exists a bi-Lipschitz homeomorphism $f_n\colon X \to Y$ such that $1-1/n \leq \mathrm{dil}(f_n) \leq 1+1/n$. Since the sequence (f_n) is equicontinuous, it contains a uniformly convergent subsequence, whose limit mapping is necessarily an isometry. □

The following is a characterization of convergence in the Hausdorff distance, with an important corollary. The proof is elementary. By an *ε-net*, we mean a subset of a metric space whose ε-neighborhood is the whole space; it has *strictly positive separation* if distances between its elements have a positive lower bound.

Proposition 2.4. *If $X_i \to X$ in the Hausdorff metric, then for each pair of positive numbers $\varepsilon' > \varepsilon$, every ε'-net $N \subset X$ with strictly positive separation is the limit of a sequence of ε-nets $N_i \subset X_i$ with respect to the Lipschitz distance.*

Conversely, if the X_i and X have finite diameters and if for each $\varepsilon > 0$, there exists an ε-net of X which is the Lipschitz limit of a sequence of ε-nets $N_i \subset X_i$, then $X_i \to X$ in the Hausdorff metric.

Corollary 2.5. *If for two compact metric spaces $d_H(X,Y)=0$ then X and Y are isometric.*

Proof. If $d_H(X,Y)=0$ then obviously, the constant sequence $X_i=X$ converges to Y in the Hausdorff metric. Fix a positive integer n. For each $\frac{1}{n}$-net $N(\frac{1}{n})\subset Y$, there exists a sequence of $\frac{2}{n}$-nets $N_i(\frac{2}{n})\subset X$ that Lipschitz converges to $N(\frac{1}{n})$. In particular, for each $\varepsilon>0$ there is an integer i_n and a map $f_n\colon N(\frac{1}{n})\to N_{i_n}(\frac{2}{n})$ such that both f_n and f_n^{-1} have dilatation less than $1+\varepsilon$.

Now keep ε fixed and vary n. By choosing the $N(\frac{1}{n})$ nested, we can use a diagonal procedure to obtain a homeomorphism $f_\varepsilon\colon Y\to X$ with the same dilatation properties. Thus we reduced the claim to Proposition 2.3. □

3 Examples of collapsing

There are several possible meanings of *collapsing* a Riemannian manifold M. The most geometrical definition is to assert that a 1-parameter family of Riemannian metrics on M (starting with the original one) produces a lower-dimensional Riemannian manifold as limit (the induced metrics and the Hausdorff metric convergence are to be understood). It is easy to see that in this weak sense all manifolds with a finite diameter collapse to a point, just by rescaling the metric, so further constraints will be necessary to make the notion of collapse more interesting. One natural choice is to require the sectional curvature to be bounded. This indeed rules out the example above (except for the flat case) because if we change the metric g to λg, the Levi-Civita connection and thus the tensor $R(X,Y)Z$ remains the same but

$$K(X\wedge Y)=\frac{\langle R(X,Y)Y,X\rangle}{|X\wedge Y|^2}$$

gets multiplied by $\frac{\lambda}{\lambda^2}=\frac{1}{\lambda}$.

The major result in this direction appeared in two successive papers by Cheeger and Gromov [CG1], [CG2]. Their definition of collapse is the following (note that although it is also geometrical, it has nothing to do with Hausdorff distance):

Definition 3.1. A family of Riemannian metrics, g_δ, on the smooth manifold M is said to *collapse*, if the injectivity radii i_δ at all points uniformly approach

0 as $\delta \to 0$, but $|K_\delta| \leq 1$ for all δ and all tangent 2-planes. We also say that the Riemannian manifold (M, g) is *ε-collapsed* if its injectivity radius is less than ε at all points.

We would expect that such a family of metrics has some nice limit in the Hausdorff metric. The latter condition indeed means that M looks like a smaller dimensional manifold when viewed on a scale much larger than ε. But we have to emphasize that this "smaller dimension" may vary. Thus, collapsing in this sense *does not* imply that our family of Riemannian manifolds converges to a smaller dimensional manifold in the Hausdorff distance (see Example 3.4). We will return to this point in Section 5. As to what collapse in the sense of Definition 3.1. does imply, see [F2].

Examples 3.2. Any flat torus collapses to any smaller dimensional torus, by rescaling the metric on some of the S^1 factors.

Let $M^2 \cong S^1 \times I$ be a surface of revolution, obtained by revolving an arc in the upper half plane about the x-axis. The isometric S^1-action on M^2 lifts to an isometric $\mathbb{R}$-action on the infinite cyclic covering $\tilde{M}^2 \cong \mathbb{R} \times I$. Let $\delta\mathbb{Z} \subset \mathbb{R}$ be the subgroup generated by a translation of length δ. Then the family $\tilde{M}^2/\delta\mathbb{Z}$ collapses (to the interval I), whereas the curvature of M^2, $\tilde{M}^2$ and $\tilde{M}^2/\delta\mathbb{Z}$ for all δ is described by the same function $I \to \mathbb{R}$, therefore it stays bounded (we have unrolled M^2 and then rolled it up tighter).

Berger observed that if one shrinks the circles of the Hopf fibration $S^3 \to S^2$, the curvature stays bounded. The limit of this collapse is S^2, in fact with a metric of constant curvature 4.

The first of these examples turns out to be basic. Cheeger and Gromov prove that the necessary and sufficient condition for a manifold to collapse is the existence of a generalized torus action, which they call an *F-structure of positive rank* (F stands for "flat" in this terminology). Intuitively, an F-structure will mean that different tori of varying dimensions act locally on finite covering spaces of subsets of the manifold M. A certain compatibility condition will insure that M is partitioned into disjoint orbits. Positive rank will just mean that these all have positive dimension.

Examples 3.3. Take a finite collection of surfaces Σ_i with boundaries $\partial\Sigma_i = \bigcup_{j=1}^{N(i)} S^1_{i,j}$. The product manifolds $\Sigma_i \times S^1_i$ have boundaries which consist of tori $S^1_{i,j} \times S^1_i$. Form the closed *graph manifold* Y^3 by identifying these tori in pairs (by elements of $SL(2, \mathbb{Z})$). On each piece $\Sigma_i \times S^1_i \subset Y^3$, S^1 acts by a rotation. On the identified boundary components these actions may not agree but they generate an action of the torus $S^1 \times S^1$ instead, which extends both of them. So the torus that acts locally on Y^3 is of dimension 2 near those former boundary components and of dimension 1 elsewhere.

In all three of our Examples 3.2. there are global torus actions on the collapsing manifolds. If we collapse the torus T^n to T^k, then the corresponding action is the one by T^{n-k}. S^1 acts on the surface of revolution and also on $S^3 \subset \mathbb{C}^2$, by coordinatewise complex multiplication.

Further illuminating examples can be found in [CG1]. Finally, we illustrate that the collapse in the sense of Definition 3.1. can "go wrong:"

Example 3.4. Consider the local structure of a graph manifold near two identified boundary components. Assume that the identification of the two tori was carried out by simply interchanging parallel and meridian. So we have $(-\varepsilon, 0] \times S^1 \times S^1$ on the "left" and $[0, \varepsilon) \times S^1 \times S^1$ on the "right." On the left, we shrink to $(-\varepsilon, 0] \times * \times S^1$ and on the right, to $[0, \varepsilon) \times S^1 \times *$ ($*$ denotes a point). But this means that in the intersection $0 \times S^1 \times S^1$, both circles are shrunk. Thus the result of the collapse (the limit in Hausdorff distance) is locally the union of two open 2-discs with their midpoints identified.

4 Collapsing in general

The actual definition of an F-structure is quite technical and needs several ingredients. We start with the concept of sheaf, taken here from [GH].

Definition 4.1. Given a topological space X, a *sheaf (of groups)* $\mathcal{F}$ on X associates to each open set $U \subset X$ a group $\mathcal{F}(U)$, called the *sections* of $\mathcal{F}$ over U, and to each pair $U \subset V$ of open sets a map $r_{U,V} \colon \mathcal{F}(V) \to \mathcal{F}(U)$, called the *restriction map*, satisfying the following three assertions:

1. For any triple $U \subset V \subset W$ of open sets,

$$r_{W,U} = r_{V,U} \circ r_{W,V}.$$

   By virtue of this relation, we may write $\sigma\big|_U$ for $r_{V,U}(\sigma)$ without loss of information.

2. For any pair of open sets $U, V \subset M$ and sections $\sigma \in (U)$, $\tau \in (V)$ such that $\sigma\big|_{U\cap V} = \tau\big|_{U\cap V}$, there exists a section $\rho \in \mathcal{F}(U \cup V)$ with $\rho\big|_U = \sigma$ and $\rho\big|_V = \tau$.

3. If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma\big|_U = 1$, $\sigma\big|_V = 1$ then $\sigma = 1$.

The *stalk* of the sheaf $\mathcal{F}$ at the point $x \in X$ is the group $F_x = \varinjlim_{U \text{ open},\, x \in U} \mathcal{F}(U)$, which is defined as follows. Take the disjoint union

$$\bigsqcup_{U \text{ open},\, x \in U} \mathcal{F}(U)$$

and factor it by the equivalence relation $a_1(\in \mathcal{F}(U_1)) \sim a_2(\in \mathcal{F}(U_2)) \iff \exists V \subset U_1 \cap U_2,\ x \in V$ such that $r_{U_1,V}(a_1) = r_{U_2,V}(a_2)$.

The most common examples of sheaves associate to U certain spaces of functions defined on U (e. g. $C(U)$, $C^\infty(U)$, or, more generally, the space of sections $\Gamma(\xi\big|_U)$ of a vector bundle ξ). The sheaves appearing later in our context, however, will not have such a concrete nature.

A *partial action* A of a topological group G on a Hausdorff space X is given by the following data:

- A neighborhood $\mathcal{D} \subset G \times X$ of $\{e\} \times X$, called the *domain (of definition)* of the action, where $e \in G$ is the identity element.

- A continuous map $A\colon \mathcal{D} \to X$, written $(g, x) \mapsto gx$, such that $(g_1g_2)x = g_1(g_2x)$ whenever (g_1g_2, x) and (g_1, g_2x) lie in $\mathcal{D}$ and such that $ex = x$ for all $x \in X$.

We call two partial actions $(A_1, \mathcal{D}_1)$, $(A_2, \mathcal{D}_2)$ *(locally) equivalent* if there is a domain $\mathcal{D} \subset \mathcal{D}_1 \cap \mathcal{D}_2$, containing $\{e\} \times X$, such that $A_1\big|_{\mathcal{D}} = A_2\big|_{\mathcal{D}}$. The equivalence class containing the partial action A is called a *local action*, denoted $\{A\}$. We call the local action *complete* if it contains a global action $A\colon G \times X \to X$; in this case we also use the notation $\{A\} = A_{\mathrm{loc}}$.

Remark 4.2. In case G is connected, A_{loc} determines A uniquely. This is shown by a connectedness argument, using the fact that any neighborhood of e generates the group.

If X is a smooth manifold and G is a Lie group, then the category of local actions is equivalent to the category of *infinitesimal actions*, that is, continuous homomorphisms of the Lie algebra of G to the Lie algebra of vector fields on X (in fact, all possible homomorphisms are continuous as G is finite dimensional). For example, if $G = \mathbb{R}$, a local action is given by a vector field on X and completeness amounts to the integrability of the field.

From now on, G is assumed connected.

A subset $Y \subset X$ is called *(locally)* $\{A\}$*-invariant* if for some representative $(A, \mathcal{D}) \in \{A\}$ one has $gy \in Y$ for all $(g, y) \in \mathcal{D} \cap (G \times Y)$. Since the intersection of $\{A\}$-invariant sets is $\{A\}$-invariant, it follows that each point $x \in X$ is contained in a unique minimal $\{A\}$-invariant subset called the *orbit* $\mathcal{O}_x \subset X$ and that these orbits partition X. For a complete local action A_{loc}, the orbits of A and A_{loc} coincide.

A local action $\{A\}$ can be *restricted* to an open subset $U \subset X$ by taking an open subset $\mathcal{D}' \subset G \times X$ which contains $\{e\} \times U$ and is such that $gx \in U$ for all $(g, x) \in \mathcal{D}' \cap (G \times U)$. If $f \colon \tilde{X} \to X$ is a local homeomorphism then the *pullback action* $f^*\{A\}$ on $\tilde{X}$ is defined in a similar way.

Definition 4.3. Let $\mathcal{G}$ be a sheaf of connected topological groups over X. An *action* of $\mathcal{G}$ on X is a collection of local actions of the groups $\mathcal{G}(U)$ on the corresponding connected open sets $U \subset X$ such that the restrictions $r_{V,U}$ are compatible with the restrictions of local actions from V to U.

A set $S \subset X$ is called *invariant* if for all open sets $U \subset X$, $S \cap U$ is invariant for $\mathcal{G}(U)$. X is partitioned into minimal invariant subsets called *orbits*. A (disjoint) union of orbits is called a *saturated set*.

Let G_x denote the stalk of $\mathcal{G}$ at x. For a locally homeomorphic map $f \colon \tilde{X} \to X$ let $f^*(\mathcal{G})$ denote the *pullback sheaf*. For $f(\tilde{x}) = x$ it has stalk $G_{\tilde{x}} = G_x$.

Definition 4.4. An action of a sheaf $\mathcal{G}$ of connected groups on X is *complete* if for all $x \in X$, there exists an open neighborhood $V(x)$ and a locally homeomorphic map of a Hausdorff space, $p \colon \tilde{V}(x) \to V(x)$ such that

- If $p(\tilde{x}) = x$ then for any open neighborhood $W \subset \tilde{V}(x)$ of $\tilde{x}$, the restriction $p^*(\mathcal{G})(W) \to G_{\tilde{x}}$ is an isomorphism.
- The local action of $p^*(\mathcal{G})$ on $\tilde{V}(x)$ is complete.

Orbits of $p^*(\mathcal{G})$ on $\tilde{V}(x)$ project to orbits of $\mathcal{G}$ on $V(x)$.

Recall that a covering map $p \colon \tilde{V} \to V$ is *normal* if $p_*(\pi_1(\tilde{V}))$ is a normal subgroup of $\pi_1(V)$.

Definition 4.5. A $\tilde{G}$*-structure* on X is a sheaf $\mathcal{G}$ of connected topological groups on X and a complete local action of $\mathcal{G}$ on X such that the sets $V(x)$ and $\tilde{V}(x)$ of Definition 4.4 can be chosen to satisfy the following conditions.

1. $\pi\colon \tilde{V}(x) \to V(x)$ is a normal covering.

2. $V(x)$ is saturated.

3. For all orbits $\mathcal{O} \subset X$ and $x, y \in \mathcal{O}$, $V(x) = V(y)$.

In particular, a $\tilde{G}$-structure is called an *F-structure* if for all x, G_x is isomorphic to a torus and the coverings $\pi\colon \tilde{V}(x) \to V(x)$ are finite. We say it has *positive rank* if its orbits are positive dimensional.

The existence of an F-structure puts some constraints on the topology of the space; for example, in the compact case it implies that the Euler characteristic is zero.

Now we are in a position to state the main theorem of Cheeger and Gromov's papers. The first part of it is proven in [CG1] and the (strengthened) converse in [CG2].

Theorem 4.6. *If the smooth manifold M admits an F-structure of positive rank then it also collapses (with bounded curvature). Conversely, for all n there exists a positive number $\varepsilon(n)$ such that if the n-dimensional smooth manifold M^n admits an $\varepsilon(n)$-collapsed metric with sectional curvature $|K| \leq 1$ then M^n also admits an F-structure of positive rank.*

Thus, if M has a sufficiently collapsed metric then in fact it admits a family of metrics that collapse with bounded curvature. The collapsing family of metrics, g_δ, is obtained roughly as follows. Start with a metric g which is invariant for the given F-structure of positive rank, meaning that all local actions are isometric. Then shrink g in certain directions tangent to the orbits. Sometimes it is also necessary to expand g in directions orthogonal to the orbits, in order to keep the curvature bounded. Thus the diameter $\mathrm{diam}(M, g_\delta)$ and the volume $\mathrm{vol}(M, g_\delta)$ may go to infinity as M collapses. They may also stay bounded or even go to 0.

5 Collapsing to a manifold

Now we put the contents of the previous section in the context of the Hausdorff metric, turning to the question of the Hausdorff limit of a collapsing family of metrics. Namely, we will seek conditions for the limit being a Riemannian manifold. It is clear that if there is a Hausdorff limit of a family of metrics and it is a lower dimensional manifold, then the injectivity radii have to go to 0. So in a certain sense the existence of an F-structure is a necessary condition.

In [F1] and [F3] Fukaya proved the following necessary and sufficient condition, separated in two theorems:

Theorem 5.1. *Let (M_i) be a sequence of $(n+m)$-dimensional compact Riemannian manifolds with sectional curvatures (K_i) and N an n-dimensional compact Riemannian manifold. Assume*

(i) $M_i \to N$ *in the Hausdorff metric,*

(ii) $|K_i| \leq 1$.

Then for sufficiently large i, there exists a map $\pi_i \colon M_i \to N$ such that the following hold.

1. *π_i is a fiber bundle.*
2. *The fiber is G/Γ, where G is a nilpotent Lie group and Γ is a discrete group of affine transformations of G such that the index $[\Gamma : G \cap \Gamma]$ is finite. Here affine means that the elements of Γ preserve the unique connection on G which makes all right invariant vector fields parallel. Thus G is a group of affine transformations on itself, acting by right multiplication.*
3. *The structure group of π_i is contained in the skew product*

$$(C(G)/(C(G) \cap \Gamma)) \tilde{\times} \operatorname{Aut}(\Gamma),$$

where $C(G)$ denotes the center of G.

Part 2 of the conclusion means that the fiber has a finite covering which is diffeomorphic to a quotient of a nilpotent Lie group. Such manifolds are called *infranilmanifolds*.

Theorem 5.2. *Let M be an $(n+m)$-dimensional manifold, N an n-dimensional complete Riemannian manifold with bounded sectional curvature and $\pi \colon M \to N$ a smooth map. Suppose that π satisfies the assertions 1, 2 and 3 in the previous theorem. Then there exists a family g_δ of Riemannian metrics on M such that the following hold.*

1. The sequence (M, g_δ) converges to N in the Hausdorff metric.

2. There exists a constant C, independent of δ, such that the sectional curvatures satisfy $|K_\delta| \leq C$.

There are also results available about the consequences of a collapse to certain special manifolds, e. g. flat manifolds (see [T2]) or even more special, tori ([T1], [I]).

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