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Strings in AdS_3

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Abstract

I study the geometry of Anti-de Sitter space and its constant time hypersurfaces, hyperbolic space. I discuss string propagation on this space and finally answer the question, "Do strings in Anti-de Sitter space sweep out minimal surfaces?"

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1 Introduction

For the last twenty-five years, experimental particle physicists around the world have tried in vain to disprove the standard model of elementary particle interactions and the quantum field theory framework on which it is built. Modern colliders can test the standard model as far as the 13th decimal place, further than any other scientific theory in history, and yet (excepting some recent data from solar neutrino experiments) every high energy experiment is absolutely consistent with its theoretical predictions. Despite unprecedented experimental success, the standard model is clearly incorrect. The standard model does not incorporate gravity, and naive attempts to include gravity as another gauge theory lead to nonsensical predictions (infinite probability amplitudes).

There are, at the present, several possible solutions to this problem under investigation. One solution is that, as twentieth century physics was revolutionized by the discovery that measurements of position and momentum do not commute (which led to quantum mechanics) today's revolution will be the discovery that measurements along different position directions do not even commute with each other. During the last year much progress has been made towards understanding the implications that such a noncommutative geometry would have on our world.

An older solution is to replace particles, zero-dimensional objects which trace out one-dimensional worldlines in space-time, with strings, one-dimensional objects which trace out two-dimensional *worldsheets*. This solution incorporates gravity beautifully, with Einstein's equation appearing naturally as a necessary condition for conformal invariance (or vanishing of the beta function for physics-inclined readers).

String theory has a shortcoming as well. While it may well be the correct theory of nature, it is too difficult for humans to solve even the simplest problems, such as understanding the vacuum. Perturbation theory has been, with only a handful of exceptions, the only tool available to physicists studying quantum mechanical systems. However it is believed that for the vast majority of realizations of string theory, perturbation theory does not converge. As a result, the primary focus of theoretical physicists for the last five years has been to try to reduce various problems to different problems in which perturbation theory does converge. This endeavor has led to the study of beautiful webs of mirror symmetries and S, T and U dualities relating various regions of the solution space of string theory. Two years ago it led to a more shocking discovery.

It was conjectured in November of 1997 by Juan Maldacena [1, 2] that string theory on an Anti-de Sitter space can be mapped to a conformal field theory on its boundary. This means that, at least in a specific case, string theory can be reduced to a far simpler and better understood problem. To capitalize on this connection it is necessary to understand what string theory predicts about strings propagating in Anti-de Sitter space and the 2-manifolds that they trace out.

In this paper I will investigate the simplest case of such propagation, strings in the three-dimensional Anti-de Sitter space AdS_3 in the classical (non-quantum mechanical) limit as described recently by Juan Maldacena and Berkeley's own Hiroshi Ooguri [3]. I will attempt to apply what I have learned from this class to understand and to justify several of their claims. In particular I will use two different approaches to determine whether these strings sweep out minimal surfaces, as they do in flat space.

2 Building Blocks of AdS_3

To understand string propagation in AdS_3 one must first understand the geometry of AdS_3. I will build this geometry up a piece at a time, starting with a hyperbolic space, extending it to the group manifold of $SL(2,\mathbb{R})$ and then taking its universal cover.

2.1 The Poincare Disk Model of Hyperbolic Space

A constant time slice of AdS_3 is the two-dimensional hyperbolic space $\mathbb{H}^2$. This, along with the two-sphere and two-dimensional Euclidean space are the only three isotropic and homogeneous simply-connected two-manifolds and therefore play a fundamental role in the understanding of Riemannian two-manifolds. In particular, the universal covers of Riemann surfaces are each isometric to one of these spaces. AdS_3 will enjoy a similar prominence among three-manifolds, being among the eight model geometries.

Over a half dozen pictures of $\mathbb{H}^2$ are developed by Thurston in [4], each making some subset of the characteristics of $\mathbb{H}^2$ more apparent. Among these, I will be interested in the Poincare disk and hyperboloid models. I will begin with a discussion of the Poincare disk model. In this model, $\mathbb{H}^2$ is an open disk of radius 1 in two-dimensional Euclidean space. While topologically $\mathbb{H}^2$ is just the two disk, the non-Euclidean nature of this space becomes apparent when one includes in this picture the geodesics. These are arcs of circles or straight lines that orthogonally intersect the boundary of the Poincare disk.

The isometries of $\mathbb{H}^2$ are generated by what Ref. [4] calls *inversions*, which are reflections across geodesics. More concretely, every geodesic is a restriction to $\mathbb{H}^2$ of a circle with some center p and radius r in the compactified[a] $\mathbb{R}^2 \supset \mathbb{H}^2$ and for each such geodesic there is an isometry which takes

$$q \to p + r^2(q-p)/(|q-p|^2). \tag{2.1}$$

[a] I compactify Euclidean space by adding the point at infinity, so that the diameters are restrictions of circles with infinite radius.

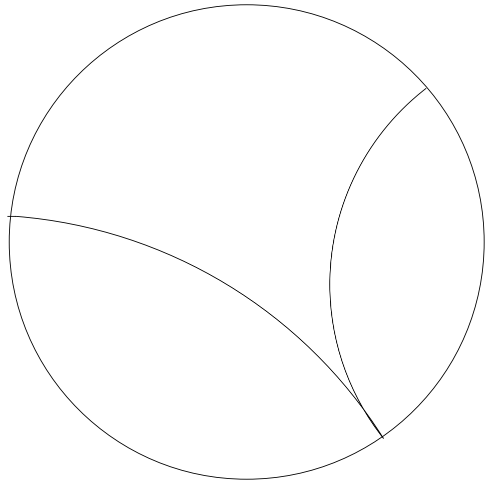

Figure 1: The Poincare disk with two geodesics

This, in particular, fixes the circle. Not only is this inversion an isometry of $\mathbb{H}^2$, but Thurston further claims that in fact all isometries of $\mathbb{H}^2$ are compositions of such isometries. Thurston also shows that these isometries preserve angles and take circles to circles (as shown in Figure 2), where as above, a line is considered to be a special case of a circle.

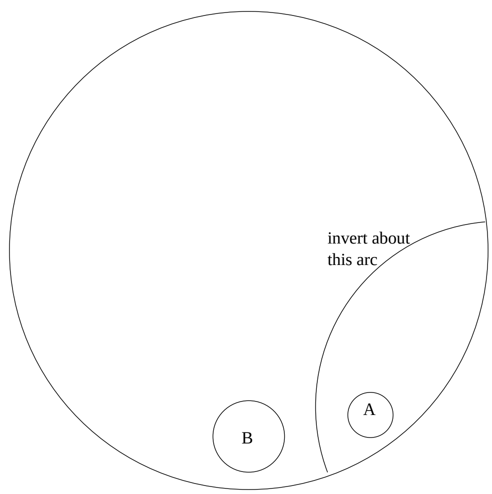

Figure 2: Isometries of $\mathbb{H}^2$ take circles (A) to circles (B)

The composition of two isometries is itself an isometry. In particular consider the composition of two isometries f_1, f_2 that are inversions about circles C_1 and C_2 which have an orthogonal (with respect to the Euclidean metric of $\mathbb{R}^2 \supset \mathbb{H}^2$) intersection at some point q. In fact, these two isometries commute and furthermore, because these isometries preserve angles, the intersection of these circles remains perpendicular after the action of either isometry. The tangent space $T_q\mathbb{H}^2$ is spanned by the tangent vectors to both circles, $v_1 \in T_qC_1$ and $v_2 \in T_qC_2$ respectively. From (2.1) we see that

$$D(f_2 \circ f_1)(v_1) = D(f_2)(D(f_1)(v_1)) = D(f_2)(v_1) = -v_1 \tag{2.2a}$$

$$D(f_2 \circ f_1)(v_2) = D(f_2)(D(f_1)(v_2)) = D(f_2)(-v_2) = -v_2 \tag{2.2b}$$

and so

$$D_q(f_2 \circ f_1) = -Id. \tag{2.3}$$

Clearly

$$q \in C_1 \cap C_2 \ \Rightarrow \ f_2 \circ f_1(q) = q \tag{2.4}$$

and so for every $q \in \mathbb{H}^2$, there is an involution $f_2 \circ f_1$ which fixes q. We have shown

Fact 1 $\mathbb{H}^2$ *is a symmetric space.*

This argument will generalize to AdS_3.

Now allow C_1 and C_2 to intersect at some arbitrary angle θ. Then $D_q(f_2 \circ f_1)$ will rotate $T_q\mathbb{H}^2$ by 2θ. In the case $\theta = \pi/2$ this is just the involution described above. However, because there exists this isometry which rotates the tangent space at q by an arbitrary angle, we learn that $\mathbb{H}^2$ is isotropic. This argument will not generalize to $SL(2,\mathbf{R})$ nor to AdS_3.

2.2 The Hyperboloid Model of Hyperbolic Space

Consider three-dimensional Minkowski (aka Lorentz) space, which is $\mathbf{R}^3$ with the following indefinite metric:

$$ds^2 = -dz^2 + dr^2 + r^2 d\theta^2. \tag{2.5}$$

The set of vectors $(z > 0, r, \theta)$ of length -1 is a subset of $T\mathbf{R}^3 \cong \mathbf{R}^3$ which forms a hyperboloid, the surface of revolution of a hyperbola about the z axis. This hypersurface, with the induced metric from Minkowski space, is also $\mathbb{H}^2$ and in particular is symmetric and so homogeneous. This means that any $p \in \mathbb{H}^2$ is mapped by some (actually, infinitely many, but I will only need one) isometry to $p = (z = 1, r = 0)$. This is the fixed point of the revolution about $(z, r = 0)$ and so the tangent plane to $\mathbb{H}^2$ at p is $(z = 1, r, \theta)$. The induced metric on this tangent plane is just the restriction of (2.5) to the last two coordinates

$$ds^2(p) = dr^2(p) + r^2 d\theta^2(p) \tag{2.6}$$

and so is positive definite.

To solve for the metric of $\mathbb{H}^2$ everywhere, define ρ by

$$r =: \sinh(\rho). \tag{2.7}$$

Then, on $\mathbb{H}^2$,

$$z = \cosh(\rho) \tag{2.8a}$$

$$\begin{aligned} ds^2 &= -dz^2 + dr^2 + r^2 d\theta^2 = ((\frac{dr}{d\rho})^2 - (\frac{dz}{d\rho})^2)d\rho^2 + \sinh^2(\rho)d\theta^2 \\ &= (\cosh^2(\rho) - \sinh^2(\rho))d\rho^2 + \sinh^2(\rho)d\theta^2 \\ &= d\rho^2 + \sinh^2(\rho)d\theta^2. \end{aligned} \tag{2.8b}$$

Now I will use this to calculate the Levi-Civita connection in the $(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\theta})$ basis (using the standard formula that Jost, in Ref. [5], calls Corollary 3.3.1)

$$g_{\rho\rho} = g^{\rho\rho} = 1, \qquad g_{\theta\theta} = \sinh^2(\rho), \qquad g^{\theta\theta} = csch^2(\rho) \tag{2.9a}$$

$$\Gamma^\rho_{\rho\rho} = \Gamma^\theta_{\rho\rho} = \Gamma^\rho_{\rho\theta} = \Gamma^\rho_{\theta\rho} = \Gamma^\theta_{\theta\theta} = 0 \tag{2.9b}$$

$$\Gamma^\rho_{\theta\theta} = \frac{1}{2}(0 + 0 - 2\sinh(\rho)\cosh(\rho)) = -\sinh(\rho)\cosh(\rho) \tag{2.9c}$$

$$\Gamma^\theta_{\theta\rho} = \Gamma^\theta_{\rho\theta} = \frac{1}{2}csch^2(\rho)(2\sinh(\rho)\cosh(\rho)) = \coth(\rho). \tag{2.9d}$$

I can solve for the Riemann curvature tensor from the connection using

$$R^k_{lij} = \Gamma^k_{jl,i} - \Gamma^k_{il,j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}. \tag{2.10}$$

The result is

$$R^k_{lii} = R^l_{lij} = 0 \tag{2.11a}$$

$$R^\theta_{\rho\theta\rho} = -R^\theta_{\rho\rho\theta} = -1 + \coth^2(\rho) - \coth^2(\rho) = -1 \tag{2.11b}$$

$$R^\rho_{\theta\rho\theta} = -R^\rho_{\theta\theta\rho} = -\sinh^2(\rho) - \cosh^2(\rho) + \cosh^2(\rho) = -\sinh^2(\rho) \tag{2.11c}$$

and in particular the Ricci, scalar and sectional curvatures are

$$Ric_{\rho\rho} = -1, \qquad Ric_{\theta\theta} = -\sinh^2(\rho), \qquad R = -2, \qquad Curv = -1. \tag{2.12}$$

I have demonstrated

Fact 2 *H^2 is homogeneous and has constant negative sectional curvature.*

The geodesic equations in $\mathbb{H}^2$ are

$$\ddot{\rho} - \sinh(\rho)\cosh(\rho)\dot{\theta}\dot{\theta} = 0, \qquad \ddot{\theta} + \coth(\rho)\dot{\rho}\dot{\theta} = 0. \tag{2.13}$$

Using the homogeneity, a geodesic through any point $q \in \mathbb{H}^2$ can be mapped, by an isometry of $\mathbb{H}^2$, to a geodesic through $\rho = 0$. The one parameter family of curves of constant θ satisfy the geodesic equation and pass through $\rho = 0$. These curves are all of the geodesics passing through $\rho = 0$ as they are the image via $exp_{\rho=0}$ of all of the rays passing through the origin of $T_{\rho=0}\mathbb{H}^2$. Then the homogeneity argument implies that all other geodesics can be obtained by the maps induced on the constant θ geodesics by the isometries of $\mathbb{H}^2$. In particular, as the constant θ curves are embedded in $\mathbb{H}^2$, no geodesics form closed loops and so $\mathbb{H}^2$ is simply connected (as the shortest loop in each nontrivial homotopy class is a geodesic). Notice also that the geodesics of constant θ diverge monotonically with ρ, and therefore all geodesics in $\mathbb{H}^2$ diverge. This agrees with the description given in class of negative curvature in terms of a repulsive force. The generalization of this statement to AdS_3 will tell us that observers at rest with respect to each other at one moment will eventually drift apart.

2.3 SL(2,R)

In terms of the Pauli sigma matrices, the fundamental representation of the Lie algebra $sl(2,\mathbb{R})$ is generated by

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad -i\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.14a}$$

$$[\sigma_1, -i\sigma_2] = \sigma_3, \qquad [-i\sigma_2, \sigma_3] = \sigma_1, \qquad [\sigma_3, \sigma_1] = i\sigma_2. \tag{2.14b}$$

Any element of the corresponding Lie group, $SL(2,\mathbb{R})$, in the fundamental representation can be written as an exponential of $sl(2,\mathbb{R})$ generators

$$\begin{aligned} g &= e^{(t+\theta)i\sigma_2/2} e^{\rho\sigma_3} e^{(t-\theta)\, i\sigma_2/2} \\ &= \begin{pmatrix} \cos(t)\cosh(\rho) + \cos(\theta)\sinh(\rho) & \sin(t)\cosh(\rho) - \sin(\theta)\sinh(\rho) \\ -sin(t)\cosh(\rho) - \sin(\theta)\sinh(\rho) & \cos(t)\cosh(\rho) - \cos(\theta)\sinh(\rho) \end{pmatrix}. \end{aligned} \tag{2.15}$$

Notice that the above parameterization of $SL(2,\mathbb{R})$ is periodic in both t and θ with period 2π. I will use the indefinite metric

$$ds^2 = -\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\theta^2 \tag{2.16}$$

which, on a line element at fixed t, is the metric of $\mathbb{H}^2$. Thus we see that

Fact 3 *The fixed time slices of the group manifold $SL(2,\mathbb{R})$ are identical copies of $\mathbb{H}^2$.*

In particular, because all time slices are identical, time translation

$$t \rightarrow t+\delta \tag{2.17}$$

is an isometry of $SL(2,\mathbb{R})$. This isometry takes any t to any t', while the isometries of $\mathbb{H}^2$ can be extended to $SL(2,\mathbb{R})$ by fixing t.

Taking the semidirect product of both kinds of isometry, one finds a transitive group of isometries and thus

Corollary 1 *The group manifold $SL(2,\mathbb{R})$ is homogeneous.*

Actually there are many more isometries than just the three-dimensional space of isometries described above. For example, time reversal is in a non-identity component of the space of all isometries. We will be interested in the six-dimensional group of isometries corresponding to left and right multiplication by arbitrary group elements. The space of isometries cannot have more than six dimensions, as each dimension of isometry is generated by Killing vector fields, which are Jacobi fields. But we know that the space of Jacobi fields is isomorphic to the space of initial conditions (vectors and derivatives) of the Jacobi equation, which is $2n=6$-dimensional. Thus six is an upper bound on the dimensionality of the space of isometries imposed by the form of the Jacobi equation.

The non-vanishing Christoffel symbols are

$$g_{\rho\rho}=g^{\rho\rho}=1, \qquad g_{\theta\theta}=\sinh^2(\rho), \qquad g^{\theta\theta}=csch^2(\rho) \tag{2.18a}$$

$$g_{tt}=-\cosh^2(\rho), \qquad g^{tt}=-sech^2(\rho) \tag{2.18b}$$

$$\Gamma^{\theta}_{\theta\rho}=\Gamma^{\theta}_{\rho\theta}=\coth(\rho) \tag{2.18c}$$

$$\Gamma^{\rho}_{\theta\theta}=-\Gamma^{\rho}_{tt}=-\sinh(\rho)\cosh(\rho) \tag{2.18d}$$

$$\Gamma^{t}_{t\rho}=\Gamma^{t}_{\rho t}=\frac{1}{2}sech^2(\rho)(2\sinh(\rho)\cosh(\rho))=\tanh(\rho). \tag{2.18e}$$

The following components of the Riemann curvature tensor are non-vanishing

$$R^{\theta}_{\rho\theta\rho}=-R^{\theta}_{\rho\rho\theta}=-1 \tag{2.19a}$$

$$R^{\rho}_{\theta\rho\theta}=-R^{\rho}_{\theta\theta\rho}=-\sinh^2(\rho) \tag{2.19b}$$

$$R^{t}_{\rho t\rho}=-R^{t}_{\rho\rho t}=-1 \tag{2.19c}$$

$$R^{\rho}_{t\rho t} = -R^{\rho}_{tt\rho} = \cosh^2(\rho) \tag{2.19d}$$

$$R^{\theta}_{t\theta t} = -R^{\theta}_{tt\theta} = \cosh^2(\rho) \tag{2.19e}$$

$$R^{t}_{\theta t\theta} = -R^{t}_{\theta\theta t} = -\sinh^2(\rho). \tag{2.19f}$$

The Ricci, scalar and all sectional curvatures are therefore

$$Ric_{\rho\rho} = -2, \qquad Ric_{\theta\theta} = -2\sinh^2(\rho), \qquad Ric_{tt} = 2\cosh^2(\rho), \qquad R = -6, \qquad Curv = -1. \tag{2.20}$$

To understand the topology and geometry of $SL(2,\mathbf{R})$ and the transition to AdS_3 it will prove useful to consider an alternate basis. Rewrite the general element of $SL(2,\mathbf{R})$ in the form

$$\begin{aligned} g &= \begin{pmatrix} \cos(t)\cosh(\rho)+\cos(\theta)\sinh(\rho) & \sin(t)\cosh(\rho)-\sin(\theta)\sinh(\rho) \\ -\sin(t)\cosh(\rho)-\sin(\theta)\sinh(\rho) & \cos(t)\cosh(\rho)-\cos(\theta)\sinh(\rho) \end{pmatrix} \\ &= \begin{pmatrix} X_{-1}+X_1 & X_0-X_2 \\ -X_0-X_2 & X_{-1}-X_1 \end{pmatrix} \end{aligned} \tag{2.21}$$

where X_i are restricted to the solutions of

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1 \tag{2.22}$$

in the Lorentzian space with

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2. \tag{2.23}$$

The parameterization (2.21) is an isometric embedding from the subset of the Lorentzian space

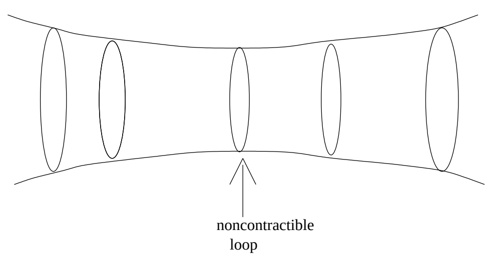

Figure 3: $X_2 = 0$ Cross-section of $SL(2,\mathbf{R})$

$E^{2,2}$ that satisfies (2.22) to the group manifold $SL(2,\mathbf{R})$. This hypersurface is easy to visualize, it is the just a product of a hyperboloid with coordinates

$$\left(\sqrt{X_{-1}^2+X_0^2}, X_1, X_2\right) \tag{2.24}$$

with a circle whose angular coordinate satisfies

$$\tan(t) = \frac{X_0}{X_{-1}}. \tag{2.25}$$

Claim 1 *The above metric is the Killing metric of $sl(2,\mathbf{R})$.*

Proof 1 *The Killing metric of $sl(2,\mathbf{R})$ is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.26}$$

Consider the short arc swept by acting $exp_g(\varepsilon)$ on an arbitrary $g \in SL(2,\mathbf{R})$. The image of g is (dropping all terms of order ε^2)

$$\begin{aligned} g' &= exp_g(\varepsilon)g = \begin{pmatrix} 1+\varepsilon_3 & \varepsilon_1-\varepsilon_2 \\ \varepsilon_1+\varepsilon_2 & 1-\varepsilon_3 \end{pmatrix}\begin{pmatrix} X_{-1}+X_1 & X_0-X_2 \\ -X_0-X_2 & X_{-1}-X_1 \end{pmatrix} \\ &= \begin{pmatrix} (1+\varepsilon_3)(X_{-1}+X_1)+(\varepsilon_2-\varepsilon_1)(X_0+X_2) & (1+\varepsilon_3)(X_0-X_2)+(\varepsilon_1-\varepsilon_2)(X_{-1}+X_1) \\ (\varepsilon_1+\varepsilon_2)(X_{-1}+X_1)+(\varepsilon_3-1)(X_0+X_2) & (\varepsilon_1+\varepsilon_2)(X_0-X_2)+(1-\varepsilon_3)(X_{-1}-X_1) \end{pmatrix} \\ &= \begin{pmatrix} X'_{-1}+X'_1 & X'_0-X'_2 \\ -X'_0-X'_2 & X'_{-1}-X'_1 \end{pmatrix} \end{aligned} \tag{2.27}$$

Therefore

$$\begin{aligned} X'_{-1} &= X_{-1}+\varepsilon_2X_0+\varepsilon_3X_1+\varepsilon_1X_2 \\ X'_0 &= -\varepsilon_2X_{-1}+X_0-\varepsilon_1X_1+\varepsilon_3X_2 \\ X'_1 &= \varepsilon_3X_{-1}-\varepsilon_1X_0+X_1+\varepsilon_2X_2 \\ X'_2 &= -\varepsilon_1X_{-1}-\varepsilon_3X_0-\varepsilon_2X_1+X_2. \end{aligned} \tag{2.28}$$

Using the metric (2.23), the small arc traced out by this exponential map has a length squared (to leading order in ε)

$$\begin{aligned} ds^2 &= -dX_{-1}^2-dX_0^2+dX_1^2+dX_2^2 \\ &= -(\varepsilon_2X_0+\varepsilon_3X_1-\varepsilon_1X_2)^2-(-\varepsilon_2X_{-1}-\varepsilon_1X_1+\varepsilon_3X_2)^2 \\ &\quad +(\varepsilon_3X_{-1}-\varepsilon_1X_0+\varepsilon_2X_2)^2+(-\varepsilon_1X_{-1}-\varepsilon_3X_0-\varepsilon_2X_1)^2 \\ &= \varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2 \end{aligned} \tag{2.29}$$

which agrees with the norm squared of $\varepsilon \in sl(2,\mathbf{R})$ using the Killing metric.

Corollary 2 *The metric is bi-invariant.*

Claim 2 *The fundamental group of $SL(2,\mathbf{R})$ is $\pi_1(SL(2,\mathbf{R})) = \mathbb{Z}$.*

Proof 2 *Deformation retract each hyperboloid by*

$$\begin{aligned} F_t: \quad & SL(2,\mathbf{R}) \to SL(2,\mathbf{R}) : (X_{-1}, X_0, X_1, X_2) \mapsto \\ & \left(\frac{\sqrt{1+(1-t)^2(X_1^2+X_2^2)}}{\sqrt{1+X_1^2+X_2^2}} X_{-1}, \frac{\sqrt{1+(1-t)^2(X_1^2+X_2^2)}}{\sqrt{1+X_1^2+X_2^2}} X_0, (1-t)X_1, (1-t)X_2\right) \end{aligned} \tag{2.30}$$

and so we see that $SL(2,\mathbf{R})$ is homotopy equivalent to the circle.

The hyperboloids that are fixed time cross-sections of $SL(2,\mathbf{R})$ are simply copies of $\mathbb{H}^2$. This means that, if we are willing to sacrifice our embedding into $E^{2,2}$, we could replace the fixed time slices with Poincare disks, as these also realize $\mathbb{H}^2$. In this case our space would appear to be a disk cross a circle or equivalently a solid torus.

3 AdS_3

3.1 The Space

At last we are ready to construct AdS_3. Simply take the universal cover of $SL(2,\mathbf{R})$. In the model where $SL(2,\mathbf{R})$ is a disk crossed with a circle, this gives a solid cylinder of infinite height (time), as seen in Figure 4. Due to its extraordinary amount of symmetry, this space plays a critical role in 3D geometry as one of the eight three-dimensional model geometries [4] ($SL(2,\mathbf{R})$ would have been a model geometry, but model geometries have to be simply connected.) AdS_3 is homogeneous, as time translation is still an isometry after taking the universal cover and the isometries of the fixed time slices $\mathbb{H}^2$ are unchanged. Not only is AdS_3 homogeneous, but in fact it is itself a Lie group [6], realized by the following transformations of $\mathbf{R}$

$$x \mapsto x + 2\pi a - i \ln \frac{1 - ze^{-ix}}{1 - \bar{z}e^{ix}}. \tag{3.1}$$

We will be interested in the surfaces that moving strings trace out in AdS_3. The simplest such surfaces are geodesics, corresponding to strings contracted to a point. Thus, our first goal will be to find the geodesics of AdS_3. To do this it will suffice to begin with only a few geodesics passing through the origin, as isometries will allow us to construct the other geodesics from these [3].

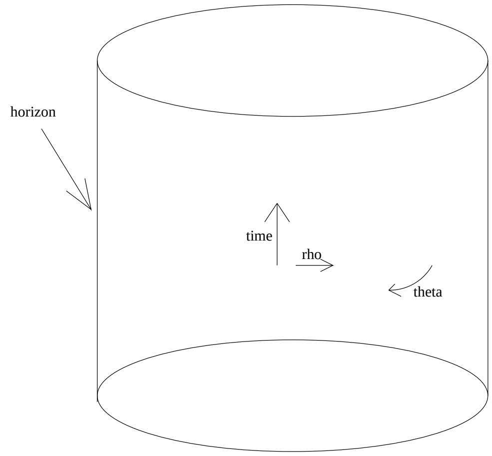

Figure 4: The Cylindrical Model of AdS_3

3.2 The Geodesics

The geodesic equations are

$$\ddot{\rho}+\sinh(\rho)\cosh(\rho)(\dot{t}\dot{t}-\dot{\theta}\dot{\theta})=0 \tag{3.2a}$$

$$\ddot{\theta}+\coth(\rho)\dot{\rho}\dot{\theta}=0, \qquad \ddot{t}+\tanh(\rho)\dot{\rho}\dot{t}=0. \tag{3.2b}$$

These three coupled second order differential equations are difficult to solve, and so, following [3], I will find some simple geodesics in $SL(2,\mathbf{R})$, act on them with the isometries given by the right and left group multiplication and then lift these geodesics onto the universal cover.

The metric is bi-invariant and so the geodesics through the origin are precisely the one parameter subgroups. Perhaps the simplest of these is

$$g=\begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \tag{3.3}$$

or alternately, in the hyperboloid picture (or AdS_3),

$$t=s, \qquad \rho=0. \tag{3.4}$$

This describes a particle at rest, at the origin, traveling through time. Ref. [3] claims that a general timelike geodesic is just (3.3) left and right multiplied by arbitrary constant matrices in $SL(2,\mathbf{R})$ (this gives six dimensions of geodesics, which is plausible as there are six dimensions of solutions to the Jacobi equation). To lift these geodesics into AdS_3, one may write the desired geodesic in terms of ρ, θ and t as in (3.4) and then allow t to run over all reals.

I will give an example of a nontrivial timelike geodesic using the isometry given by left multiplication (fix α)

$$\begin{aligned} g &= \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\alpha)\cos(s) - \sinh(\alpha)\sin(s) & \cosh(\alpha)\sin(s) + \sinh(\alpha)\cos(s) \\ \sinh(\alpha)\cos(s) - \cosh(\alpha)\sin(s) & \sinh(\alpha)\sin(s) + \cosh(\alpha)\cos(s) \end{pmatrix} \end{aligned} \tag{3.5}$$

which lifts to the geodesic

$$\rho = \alpha, \qquad t = s, \qquad \theta = -s - \frac{\pi}{2}. \tag{3.6}$$

Notice that this solves the geodesic equations (3.2). Both this geodesic and the simple timelike geodesic presented earlier can be found sketched freehand in Figure 5.

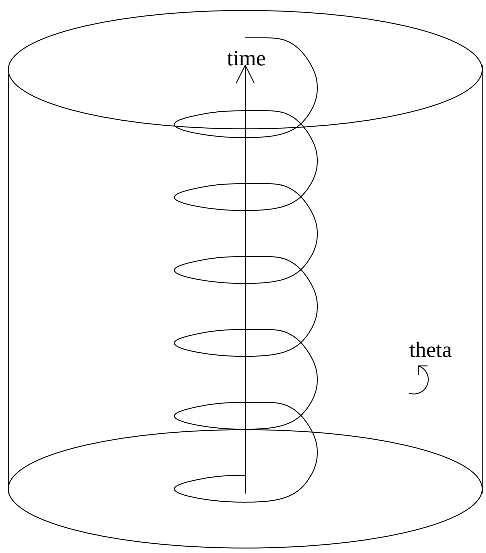

Figure 5: Two Timelike Geodesics in AdS_3

Similarly, the simplest spacelike geodesic is

$$g = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \tag{3.7}$$

or alternately

$$t = 0, \qquad \rho = s, \qquad \theta = 0. \tag{3.8}$$

For brevity I have allowed ρ to be negative, negative ρ just means add π to θ. Again, Ref. [3] claims that all spacelike geodesics can be attained by left and right multiplication of (3.7) by constant elements of $SL(2,\mathbb{R})$. For example,

$$\begin{aligned} g &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)e^{s} & \sin(\alpha)e^{-s} \\ -\sin(\alpha)e^{s} & \cos(\alpha)e^{-s} \end{pmatrix} \end{aligned} \tag{3.9}$$

which lifts to the geodesic

$$\rho = s, \qquad t = \theta = \alpha. \tag{3.10}$$

Both of these geodesics correspond to lines at constant time and angle extending to the horizon (these are both quite trivial examples of spacelike geodesics.) These geodesics are illustrated in Figure 6.

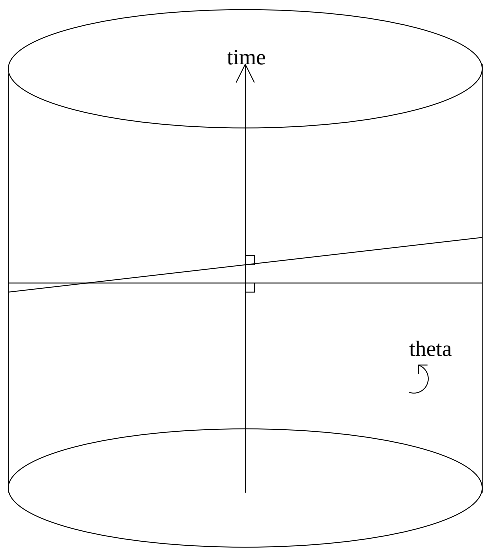

Figure 6: Two Spacelike Geodesics in AdS_3

It took me a little more work to find a lightlike geodesic. I began by considering geodesics at $\theta = 0$ and so the null condition on a line segment gave

$$0 = ds^2 = -\cosh^2(\rho)dt^2 + d\rho^2 \tag{3.11}$$

which describes a first order differential equation solved by integrating

$$dt = \mathrm{sech}(\rho)d\rho. \tag{3.12}$$

After picking my favorite boundary conditions I found

$$s = \tan(\frac{t}{2}) = \tanh(\frac{\rho}{2}) \tag{3.13}$$

where s is the variable that I will use to parametrized the null geodesic. After the application of some trigonometric identities, the inclusion of the worldsheet is seen to be

$$g = \frac{1}{1-s^2}\begin{pmatrix} 1+2s-s^2 & 2s \\ -2s & 1-2s-s^2 \end{pmatrix} \tag{3.14}$$

or alternately

$$t = \sin^{-1}(\frac{2s}{1+s^2}), \qquad \rho = \sinh^{-1}(\frac{2s}{1-s^2}), \qquad \theta = 0. \tag{3.15}$$

Notice that at $s = 0$ the geodesic is at the origin while at $s = 1$ it is at the horizon. These correspond to $t = 0$ and $t = \pi/2$ respectively and so light can reach the horizon in finite time.

As another example of a lightlike geodesic consider

$$g = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \tag{3.16}$$

or equivalently

$$\theta = -\frac{\pi}{2}, \qquad \sinh\rho = \frac{s}{2}, \qquad \sin(t) = \frac{s/2}{\sqrt{1+s^2/4}}. \tag{3.17}$$

When $s = 0$ the light leaves the origin and at $s = \infty$ it reaches the horizon. These two points correspond again to times $t = 0$ and $t = \pi/2$ and so again we see that light can reach the horizon in time $\pi/2$. In fact, by homogeneity, all observers will believe that light from them reaches the horizon in time $\pi/2$, although they will not agree on the direction of the time axis.

This may seem like a feature of a bad choice of coordinates, but actually it has consequences that are frightfully apparent to any resident of a small AdS^3. Light leaving an observer will reach the horizon in every direction and reflect back from the horizon[b] in time π. This means that, if the universe is relatively empty, whichever way an observer looks, he will see himself as he was π moments before. The trajectory of some of these light rays is illustrated in figure 7.

[b]Light does not necessarily bounce back from the horizon. As described in Ref. [7], to define a field theory in AdS_3 one needs to define Cauchy surfaces (surfaces such that, if a field and its derivative are unknown on the surfaces, they are known everywhere). There is no unique way to do this, but if one imposes energy and momentum conservation on AdS^3, one finds *reflective boundary conditions* which force light approaching the boundary to reflect.

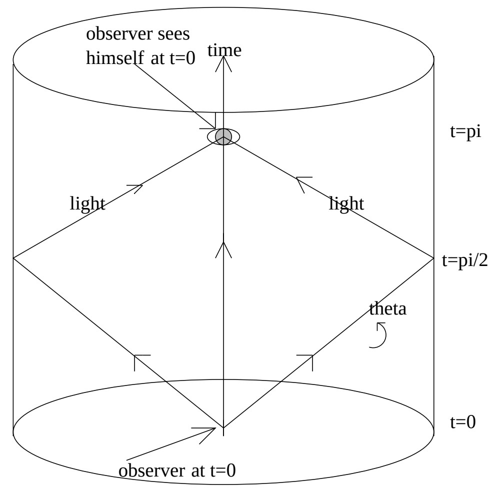

Figure 7: An Introspective Observer in AdS_3

4 Strings in AdS_3

4.1 The Wess Zumino Witten Model

Following Ref. [3] I will consider strings propagating on the Cartesian product of AdS_3 and an arbitrary compact manifold N. The choice of N will be unimportant in the following discussion as the degrees of freedom of the inclusion into N will decouple from the degrees of freedom of the inclusion into AdS_3 (with one exception to be described below).

I will use the $SL(2,\mathbf{R})$ Wess Zumino Witten (WZW) model of string propagation [8] in AdS_3. In this model, a string trajectory is described by an AdS_3-valued function $\tilde{g}$ on a surface called *the worldsheet* which describes the inclusion of the worldsheet into AdS_3 and whose image is interpreted as the surface in space-time swept out by the string. Local coordinates on the worldsheet are called σ and τ, and there is a complex structure with complex coordinates

$$x^{\pm} = \sigma \pm i\tau \tag{4.1}$$

making the worldsheet a Riemann surface.

Associated to every worldsheet is the WZW action, which is the integral over the worldsheet of a kinetic term plus the integral of a total derivative term over a three-manifold M whose boundary is the worldsheet:

$$S_{WZW} = \frac{k}{8\pi}\int d\sigma d\tau h^{ab}\mathrm{Tr}((g^{-1}\partial_a g)(g^{-1}\partial_b g)) + \frac{k}{24\pi}\int_M d^3x\varepsilon^{ijk}Tr((g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)). \tag{4.2}$$

Here k is a real number, h^{ab} is the inverse of the worldsheet metric, $\{a,b\}$ run over σ and τ and $\{i,j,k\}$ index a basis of a tangent space to any manifold bounded by the worldsheet which is mapped to M by an extension of the inclusion of the worldsheet into AdS_3. g is the projection of $\tilde{g}$ onto $SL(2,\mathbf{R})$. The choice of M is not unique; however, because $H^3(AdS_3) = 0$ and the second term is a total derivative, the choice of M will not affect the action. For compact Lie groups $H^3 \neq 0$ and so in that case one forces e^{iS} to be uniquely defined by imposing a quantization condition on k.

The general solution of the equations of motion obtained by varying g and extremizing S_{WZW} is

$$g(x) = g_+(x^+)g_-(x^-) \tag{4.3}$$

where g_+ and g_- are arbitrary $SL(2,\mathbf{R})$ valued functions. I will consider the case where the worldsheet is the Riemann sphere and σ and τ are polar coordinates in the coordinate patch not including the north pole. (In particular, σ is the longitude.) And so a necessary condition for g to be globally defined is

$$g(\sigma,\tau) = g(\sigma+2\pi,\tau). \tag{4.4}$$

The equations of motion from varying h_{ab} are that the stress tensor (defined to be the variation of the action with respect to the worldsheet metric) vanishes. However the full stress tensor also contains a constant contribution from the stress tensor of the compact manifold N mentioned above. Thus the stress tensor on AdS_3 is merely restricted to be an arbitrary constant. Notice that the geodesics are the solutions to the equations of motion that correspond to strings collapsed to a point.

Given any solution g to the equations of motion obtained from the action (4.2) and the periodicity (4.4), another solution g' is given by an operation called (for reasons that I will not explain here) spectral flow

$$g' = g'_+(x^+)g'_-(x^-) \tag{4.5a}$$

$$g'_+(x^+) = e^{\frac{1}{2}\omega x^+ i\sigma_2} g_+(x^+), \qquad g'_-(x^-) = g_-(x^-)e^{\frac{1}{2}\omega x^- i\sigma_2} \tag{4.5b}$$

or equivalently

$$t \;\to\; t+\omega\tau, \qquad \theta \;\to\; \theta+\omega\sigma \tag{4.6}$$

where ω is an arbitrary integer.

4.2 An Example

If we apply the spectral flow (4.6) to the timelike geodesics (3.6), with $\omega = 1$, then, after a minor reparametrization, I obtain the cylinders

$$\rho = \alpha, \qquad t = \sigma, \qquad \theta = \tau. \tag{4.7}$$

Therefore these cylinders are examples of worldsheets embedded in AdS_3.

Claim 3 *These cylinders are not minimal surfaces.*

I will show this using two distinct methods. First, I will calculate the second fundamental form and use it to show that the mean curvature is not zero. Second, I will show that the coordinate functions are not harmonic.

The principal curvatures are the eigenvalues of the second fundamental form

$$S_{\frac{\partial}{\partial\rho}}(\frac{\partial}{\partial\theta}) = (\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\rho})^{\perp} = \Gamma^{\theta}_{\theta\rho}\frac{\partial}{\partial\theta} = \frac{\cosh(\rho)}{\sinh(\rho)}\frac{\partial}{\partial\theta} \tag{4.8a}$$

$$S_{\frac{\partial}{\partial\rho}}(\frac{\partial}{\partial t}) = (\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\rho})^{\perp} = \Gamma^{\theta}_{t\rho}\frac{\partial}{\partial t} = \frac{\sinh(\rho)}{\cosh(\rho)}\frac{\partial}{\partial t}. \tag{4.8b}$$

Thus the principal curvatures are

$$\kappa_1 = \frac{\cosh(\rho)}{\sinh(\rho)}, \qquad \kappa_2 = \frac{\sinh(\rho)}{\cosh(\rho)} \tag{4.9}$$

which do not sum to zero and so the mean curvature is non-vanishing. Thus, these strings do not sweep out minimal surfaces, as do strings in flat space.

Next, I will show that the above coordinate functions (4.7) for these worldsheets are not harmonic. The coordinate functions are

$$(\rho(\sigma,\tau), t(\sigma,\tau), \theta(\sigma,\tau)) = (\alpha, \tau, \sigma) \tag{4.10}$$

whose derivatives are

$$Df = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.11}$$

Thus the induced metric on the worldsheet is

$$g_{\sigma\sigma} = \langle\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\sigma}\rangle = \langle\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\rangle = \sinh^2(\alpha) \tag{4.12a}$$

$$g_{\sigma\tau} = g_{t\theta} = g_{\tau\sigma} = 0 \tag{4.12b}$$

$$g_{\tau\tau} = \langle \frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau} \rangle = \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = -\cosh^2(\alpha) \tag{4.12c}$$

$$\sqrt{-\det(g)} = \sinh(\alpha)\cosh(\alpha). \tag{4.12d}$$

The metric is constant on each cylindrical worldsheet M (they are homogeneous submanifolds) and so the Laplace-Beltrami operator on the coordinate functions is a linear combination of second derivatives of the coordinate functions. However the coordinate functions are linear in σ and τ and so they are annihilated by the Laplace-Beltrami operator:

$$\triangle_M \rho = \triangle_M t = \triangle_M \theta = 0. \tag{4.13}$$

Thus Jost's condition (3.6.35) from Ref. [5] for the coordinate functions to be harmonic and therefore for the cylinder M to be minimal is that the following expression must vanish:

$$g^{\alpha\beta}\Gamma^j_{ik}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} = \sinh^2(\alpha)\Gamma^j_{\theta\theta} + \cosh^2(\alpha)\Gamma^j_{tt} = \sinh(\alpha)\cosh(\alpha) \neq 0. \tag{4.14}$$

Again we see that these strings do not sweep out minimal surfaces.

5 Conclusion

I have always believed, based on intuition from strings included in flat space-time [9], that strings sweep out minimal surfaces. String theory, as I have learned it, is always motivated by a generalization of general relativity, where *Action=Arc Length* is replaced by *Action=Surface Area*. When arriving at my above conclusion I was certain that I was incorrect. However I have found no flaw in my calculations and have now verified my result in two very different ways. Thus my intuition for the propagation of strings in curved space is forever changed. Next, I will need to continue to study these worldsheets in Anti-de Sitter space (and maybe WZW on $S^3 \cong SU(2)$) until I have developed a new, superior geometric description of the WZW action.

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