Differential Geometry on Generalizations of Grassmannians and Flag Manifolds in C^*-Algebras

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1 Introduction

Within the context of C^*-algebras there exist objects which are natural generalizations of the well known Grassmann and flag manifolds which occur in the framework of the finite-dimensional spaces $\mathbb{C}^n$ (or $\mathbb{R}^n$). These objects which occur in C^*-algebras have interesting differential geometric properties which have been studied in several papers, including [1]-[3],[5]-[7]. The differential geometric properties of these manifolds have generally been studied using more or less ad hoc methods, but recently, in [5], these manifolds have been studied in a setting which allows methods from differential geometry to be used more fully. In this paper we will provide an introduction to generalizations of the usual Grassmann and flag manifolds, and we will give a survey of some of the results which are known about the differential geometric structures of these manifolds.

2 Generalizations of Grassmann and Flag Manifolds

To motivate the construction of the Grassmann and flag manifolds in a C^*-algebra, we will begin with a few different constructions of the Grassmann and flag manifolds, which can be found in [5]. For positive integers n and N, with $n \leq N$ we define an n-flag in $\mathbb{C}^N$ to be a filtration (by vector spaces) of $\mathbb{C}^n$ with length n. That is, a collection of vector spaces V_j, for $0 \leq j \leq n$, with

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$$

The set of all $n-$flags in $\mathbb{C}^n$ will be denoted by $\mathcal{F}_n(\mathbb{C}^N)$. The simplest nontrivial case, $n = 2$, corresponds to the usual Grassmann manifold of $\mathbb{C}^N$.

This is, of course, not the only way to describe flag or Grassmann manifolds. If we let $A = \mathcal{L}(\mathbb{C}^N)$ denote the C^*- algebra of linear operators on $\mathbb{C}^N$, then the set $\mathcal{F}_n(\mathbb{C}^N)$ is in 1-1 correspondence with the set of all $n-$ tuples $(e_1, \ldots, e_n)$ of mutually orthogonal self-adjoint projections in A, such that

$$e_1 + \cdots + e_n = 1,$$

via the map which sends an n-flag

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$$

to the $n-$tuple $(e_1, \ldots, e_n)$, where e_i is the orthogonal self-adjoint projection onto the subspace $V_i \ominus V_{i-1}$. Thus, the manifold $\mathcal{F}_n(\mathbb{C}^N)$ can be viewed as systems of orthogonal self-adjoint projections which sum to the identity.

We will now use the above characterization of flag and Grassmann manifolds to extend these notions to C^*-algebras, as done in [1]. The latter description of $\mathcal{F}_n(\mathbb{C}^N)$ leads one to the following definition.

Definition 1 *Let A be any C^*-algebra, and let n be a positive integer. Define the space*

$$Q_n(A) = \left\{ (e_1, \ldots, e_n) \in A^n : e_i e_j = \delta_{ij} e_i, \sum_{i=1}^{n} e_i = 1 \right\},$$

If A has an involution $(\cdot)^* : A \to A$, we define

$$P_n(A) = \{(e_1, \ldots, e_n) \in Q_n(A) : e_i^* = e_i, \textit{ for } 1 \leq i \leq n\}$$

where δ_{ij} is the Kronecker-delta function.

We see that, if $A = \mathcal{L}(\mathbb{C}^N)$, then $P_n(A) = \mathcal{F}_n(\mathbb{C}^N)$ (provided $n \leq N$). The spaces $Q_n(A)$ and $P_n(A)$, which can be thought of as systems of projections ($P_n(A)$ consists of systems of orthogonal self-adjoint projections), are extensions of the notion of Grassmann and flag manifolds for the spaces $\mathbb{C}^N$ to any C^*-algebra. The way that $Q_n(A)$ is fibered over $P_n(A)$ (especially n=1) is studied in [2], and the geometry of $P_n(A)$ plays a role in the study of geodesics in $Q_n(A)$.

The usual flag manifolds can be realized as a disjoint union of reductive homogeneous spaces. The same is true about the generalization of Grassmann and flag manifolds which occurs in the framework of environments. The setting of an environment, which was introduced in [5] and will be defined below, also allows methods from differential geometry to be used more fully.

Definition 2 *Let B be a unital complex algebra. An environment over B is a pair $\mathcal{E} = (E, \Pi)$, where*

1. *E is a complex algebra equipped with a compatible $B-$bimodule structure*
2. *$\Pi : E \to B$ is a left and right $B-$linear map.*

One can define the Grassmannian $\mathcal{G}(\mathcal{E})$ of the environment $\mathcal{E}$ as follows:

Definition 3 *Let $\mathcal{G}(\mathcal{E})$ be the set of all $\alpha \in \mathcal{E}$ such that*

1. *$\Pi(\alpha \times \alpha) = 1$,*
2. *$\alpha \cdot \Pi(\alpha \times \phi) = \alpha \times \phi$, and*
3. *$\Pi(\phi \times \alpha) \cdot \alpha = \phi \times \alpha$,*

for all $\phi \in E$.

Similarly to the construction of $P_n(A)$ from $Q_n(A)$, when B is an involutive algebra, we may define the self-adjoint Grassmannian of $\mathcal{E}$.

Definition 4 *An environment $\mathcal{E} = (E, \Pi)$ over an involutive algebra A is called an involutive environment if E is an involutive algebra and Π satisfies*

$$\Pi(\phi^\sharp) = \Pi(\phi)^*, \text{ for all } \phi \in E,$$

where $*$ and $\sharp$ denote the involution on B and E, respectively. If $\mathcal{E}$ is an involutive environment, define $\mathcal{U}(\mathcal{E}) = \{\alpha \in \mathcal{G}(\mathcal{E}) : \alpha^\sharp = \alpha\}$, and refer to it as the self-adjoint Grassmannian of $\mathcal{E}$.

As an example of an environment and its associated Grassmannian, we will show that spaces of group representations can be realized as Grassmannians of environments. Let G be a finite group, B a unital complex algebra. Let $B[G]$ be the collection of all functions from G to B, with addition and scalar multiplication in $B[G]$ defined pointwise. We define a convolution product $\times : B[G] \times B[G] \to B[G]$, by setting, for $\phi, \rho \in B[G]$,

$$(\phi \times \rho)(g) = \frac{1}{|G|} \sum_{h \in G} \phi(h)\rho(h^{-1}g), \ g \in G,$$

where $|G|$ denotes the order of the finite group G. With this operation as multiplication, $B[G]$ is endowed with an algebra structure. We define a map $\Pi : B[G] \to B$ by

$$\Pi(\phi) = \phi(1_G), \text{ for } \phi \in B[G],$$

where 1_G denotes the identity element of G. The map Π is left and right B-linear, and one can check easily that the pair $\mathcal{E}$=$(B[G], \Pi)$ is an environment over B. Using the definition of the Grassmannian of an environment, one can show, by changing variables in the convolution product, that $\mathcal{G}(\mathcal{E})$ is just the set of all group homomorphisms from G into the group of invertible elements of B. That is, $\mathcal{G}(\mathcal{E})$ is the set of all $\phi \in B[G]$ that satisfy $\phi(1_G)$=1, and,

$$\phi(gh) = \phi(g)\phi(h), \text{ for all } g, h \in G.$$

In the situation where B is an involutive algebra, with involution *, a standard construction endows $B[G]$ with an involution, $\sharp$, given, for $\phi \in B[G]$, by

$$\phi^\sharp(g) = \phi(g^{-1})^*, \text{ for all } g \in G.$$

The self-adjoint Grassmannian $\mathcal{U}(\mathcal{E})$ is then, using similar techniques as above, the set of group homomorphisms from G into the group of unitary elements of B.

In this paper we are primarily concerned with C^*-algebras, and so the environments we are interested in are the Banach environments and their associated Grassmannians.

Definition 5 *$\mathcal{E} = (E, \Pi)$ is a Banach environment over B if*

1. *B and E are both Banach algebras*
2. *E is a Banach B-bimodule, and*
3. *Π is a continuous map.*

It can be shown that, for a Banach environment $\mathcal{E} = (E, \Pi)$ over an algebra B, both $\mathcal{G}(\mathcal{E})$ and $\mathcal{U}(\mathcal{E})$ (whenever B is involutive) are closed subspaces of E.

The spaces $Q_n(B)$ and $P_n(B)$, where B is a C^*-algebra can be realized as the Grassmannian $\mathcal{G}(\mathcal{E})$ (resp. self-adjoint Grassmannian $\mathcal{U}(\mathcal{E})$) of an environment over B. This can be done by considering the environment (B^n, Π) over B, with componentwise addition, multiplication, and scalar multiplication (on the right and the left) in B^n, with the map Π defined for $x = (x_1, \ldots, x_n) \in B^n$ by

$$\Pi(x) = \sum_{i=1}^{n} x_i$$

For more details, see [5]. It should be remarked that there are a variety of natural environments over algebras, and in fact there is a general method of producing environments, for which we refer the reader to [5, pages 211-212].

Note that in a Banach environment we have the following result:

Theorem 1 *If $\mathcal{E} = (E, \Pi)$ is a Banach environment over the algebra B, then $\mathcal{G}(\mathcal{E})$ is a complex analytic submanifold of E. Moreover, if B is involutive, $\mathcal{U}(\mathcal{E})$ is a real analytic submanifold of the real Banach space E_h, the set of Hermitian elements of E.*

This general result for Banach environments was proved in [5], but certain cases of this theorem were already known. For instance, the fact that $Q_n(B)$ is a complex analytic submanifold of B^n, when B is a C^*-algebra, was proved in [1].

3 Linear Connections and the Standard Lift

In order to do some differential geometry, and in particular, study geodesics, we would like to obtain linear connections on our generalized Grassmann manifolds. In particular, we would like to obtain a torsion-free connection. In the context of a Banach environment $\mathcal{E} = (E, \Pi)$, one can define a canonical linear connection on the space $\mathcal{G}(\mathcal{E})$ (similarly on $\mathcal{U}(\mathcal{E})$ in the involutive case), as follows. Let $\chi(\mathcal{G}(\mathcal{E}))$ denote the collection of tangent vector fields on the space $\mathcal{G}(\mathcal{E})$. For any point $\alpha \in \mathcal{G}(\mathcal{E})$, one can show (see [5], p. 215-216) that there is a projection map π_α which maps E onto $T_\alpha \mathcal{G}(\mathcal{E})$, which induces a linear connection ∇ on $\mathcal{G}(\mathcal{E})$. That is:

$$\nabla : \chi(\mathcal{G}(\mathcal{E})) \times \chi(\mathcal{G}(\mathcal{E})) \to \chi(\mathcal{G}(\mathcal{E}))$$

given by

$$(\nabla_X Y)_\alpha = \pi_\alpha((\partial_X Y)_\alpha)$$

where, as usual, $\partial_X Y$ denotes the directional derivative of Y along the vector field X. The connection ∇ thus defined will be called the canonical linear connection on $\mathcal{G}(\mathcal{E})$. This construction was done for $Q_n(B)$ in [1], and the general case for environments was done in [5]. One has:

Theorem 2 *The canonical linear connection ∇ is torsion-free, and it is invariant with respect to the action of the complex Lie group of invertible elements of B on $\mathcal{G}(\mathcal{E})$.*

One can also introduce other invariant linear connections on $\mathcal{G}(\mathcal{E})$. One way to obtain such a connection on $\mathcal{G}(\mathcal{E})$ is to use parallel transport on $T\mathcal{G}(\mathcal{E})$. One can define parallel transport along curves (which are suitably smooth) by using the so-called standard lift of such curves in $\mathcal{G}(\mathcal{E})$. In [1], this is done for $\mathcal{C}^1$ curves on $Q_n(B)$, and in [5] this is done for curves of locally bounded variation (as a B-valued map) in $\mathcal{G}(\mathcal{E})$ in the context of environments. For $\mathcal{C}^\infty$ curves in an environment $\mathcal{E} = (E, \Pi)$ over B, we have the following result:

Theorem 3 *Let I be an interval in $\mathbb{R}$, with $0 \in I$, and set $N(B)$ equal to the set of invertible elements of B. Let $\gamma : I \to \mathcal{G}(\mathcal{E})$ be a curve with origin α, such that γ, as an E-valued function, is smooth. Then the standard lift $\Gamma : I \to N(B)$ of γ is a smooth map from I to B. Furthermore, it is the solution of the first-order initial value problem*

$$\dot{\Gamma}(t) = \Pi(\dot{\gamma}(t) \times \gamma(t))\Gamma(t)$$
$$\Gamma(0) = 1,$$

where $\dot{\Gamma}(t)$ and $\dot{\gamma}(t)$ denote the derivatives with respect to t of Γ and γ, respectively. Moreover, $\gamma(t) = \Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1}$ for all $t \in I$.

This result is obtained also for the spaces $Q_n(A)$ in [1]. It is interesting to note, however, that the standard lift in [1] is obtained by showing that an initial value problem similar to the one above has a solution, and by general techniques the solution maps into the set of invertible elements of the C^*-algebra A. In contrast, the standard lift in [5], in the setting of an environment, is found using different techniques, for curves of locally bounded variation, and is later proven, in the theorem given above, to be the solution of an initial value problem when the curve is smooth. We also note that if $\mathcal{E}$ is an involutive environment and γ is a smooth curve in the self-adjoint Grassmann $\mathcal{U}(\mathcal{E})$, then one can define similarly a standard lift $\Gamma : I \to U(B)$, where $U(B)$ is the set of all unitary elements of B.

We are now in position to define the parallel translation along a $\mathcal{C}^\infty$ curve:

Definition 6 *Let $\gamma : (-\epsilon, \epsilon) \to \mathcal{G}(\mathcal{E})$ be a $\mathcal{C}^\infty$ curve with origin α, and let $\Gamma : (-\epsilon, \epsilon) \to N(B)$ be the standard lift of γ. The parallel translation along γ of tangent vectors from $\gamma(0) = \alpha$ to $\gamma(t)$, for $t \in (-\epsilon, \epsilon)$, is the map*

$$\tau_t : T_\alpha \mathcal{G}(\mathcal{E}) \to T_{\gamma(t)} \mathcal{G}(\mathcal{E}), \text{ given by } \tau_t(\theta) = \Gamma(t) \cdot \theta \cdot \Gamma(t)^{-1}.$$

Using this, we can introduce an invariant linear connection

$$\nabla^+ : \chi(\mathcal{G}(\mathcal{E})) \times \chi(\mathcal{G}(\mathcal{E})) \to \chi(\mathcal{G}(\mathcal{E}))$$

given by

$$(\nabla^+_X Y)_\alpha = \lim_{t \to 0} \frac{1}{t} \left(\tau_t^{-1} \left(Y_{\gamma}(t) \right) - Y_\alpha \right),$$

for any $X, Y \in \chi(\mathcal{G}(\mathcal{E}))$, where γ is a curve which satisfies the conditions given in the definition of parallel transport, with $\gamma(0) = \alpha$ and $\dot{\gamma}(0) = X_\alpha$. In [5], an explicit description of ∇^+ is also obtained.

4 Geodesics

Since we have a connection on our Grassmann manifold, we would now like to study geodesic curves. We begin with the usual definition:

Definition 7 *Let γ be a smooth curve ($\mathcal{C}^1$, $\mathcal{C}^\infty$) defined on [-1,1]. Then γ is geodesic (with respect to a connection ∇) provided*

$$\left(\nabla_{\dot\gamma(t)}\dot\gamma(t)\right)_{\gamma(t)} = 0 \textit{ for all } t \in [-1,1]$$

In the setting of a Grassmannian (or self-adjoint Grassmannian) of an environment, geodesics with respect to ∇ the canonical linear connection can be described alternately using the definition of ∇. With respect to the canonical linear connection, γ is a geodesic if and only if

$$\pi_{\gamma(t)}\left(\ddot\gamma(t)\right) = 0 \text{ for all } t \in [-1,1]$$

Note that the above statement holds also for the spaces $Q_n(A)$ and $P_n(B)$, since they can can be realized as Grassmannians of environments, as shown above. We now have equations characterizing geodesic curves, but one needs to know under what conditions they exist. One has the following nice result:

Theorem 4 *Let $\mathcal{E} = (E, \Pi)$ be a Banach environment over an algebra B. Then for each $\alpha \in \mathcal{G}(\mathcal{E})$ and $\beta \in \mathcal{G}(\mathcal{E})$, there exists a unique geodesic $\gamma : (-\infty, \infty) \to \mathcal{G}(\mathcal{E})$ such that $\gamma(0) = \alpha$ and $\dot\gamma(0) = \beta$. In fact, γ is given by*

$$\gamma(t) = exp\ \epsilon_\alpha(t\beta) \cdot \alpha \cdot exp\ \epsilon_\alpha(-t\beta) \textit{ for all } t \in (-\infty, \infty),$$

where $\epsilon_\alpha : E \to B$ is given by $\epsilon_\alpha(\phi) = \frac{1}{2}\Pi(\phi \times \alpha - \alpha \times \phi)$ for $\phi \in E$. Moreover, $\mathcal{G}(\mathcal{E})$ is geodesically complete.

This theorem is proven by showing that a curve $\gamma : (-\epsilon, \epsilon) \to \mathcal{G}(\mathcal{E})$ is geodesic if and only if $\Gamma(t) = exp(t \cdot \epsilon_\alpha(\beta))$ for any $t \in (-\epsilon, \epsilon)$, where Γ is the standard lift of γ. We obtain the characterization of γ, using the fact that $\epsilon_\alpha : E \to B$ is linear, since

$$\gamma(t) = \Gamma(t) \cdot \alpha \cdot \Gamma(t)^{-1} = exp(\epsilon_\alpha(t\beta)) \cdot \alpha \cdot exp(\epsilon_\alpha(-t\beta))$$

This theorem was proven first for certain spaces related to $Q_n(B)$ in [7], and the generalized version was proved for environments $\mathcal{G}(\mathcal{E})$ in [5].

In addition to being globally defined, geodesics in the self-adjoint generalized Grassmannians $\mathcal{U}(\mathcal{E})$ (or $P_n(B)$) satisfy certain minimality theorems. In order to make sense of these theorems, we must define the notion of length for smooth curves; to do so we must define a norm in each tangent space to $\mathcal{U}(\mathcal{E})$. Let $\alpha \in \mathcal{U}(\mathcal{E})$. We define a norm on $T_\alpha \mathcal{U}(\mathcal{E})$ by setting:

$$\|\beta\|_\alpha = \|\epsilon_\alpha(\beta)\|, \text{ for } \beta \in T_\alpha \mathcal{U}(\mathcal{E})$$

where the map ϵ_α was defined in the previous theorem, namely

$$\epsilon_\alpha(\phi) = \frac{1}{2}\Pi(\phi \times \alpha - \alpha \times \phi), \text{ for } \phi \in E.$$

We are now able to define the length of a smooth path.

Definition 8 *The length of a smooth curve $\gamma : [0, \tau] \to \mathcal{U}(\mathcal{E})$ is defined to be*

$$\mathit{length}(\gamma) = \int_0^\tau \|\dot{\gamma}(t)\|_{\gamma(t)} dt$$

We begin with some related results which holds for a special case (n=1) of the $Q_n(B)$, $P_n(B)$ spaces considered above, as presented in [2]. For the following theorems, let B be a Banach algebra, and let Q and P denote the sets of idempotents and self-adjoint idempotents, respectively, in B (i.e. $Q_1(B), P_1(B)$). Note that, in [2], the distance function used for the theorem below is a Finsler metric.

Theorem 5 *Let $p_0, p_1 \in P$ be at distance $d(p_0, p_1) < \pi$. Then there is a unique geodesic in P joining p_0 and p_1, it has length $d(p_0, p_1)$, and it is shorter than any other curve in Q joining p_0 and p_1.*

In [2], a map $\pi : Q \to P$ is studied. For $p \in P$, we set $Q_p = \pi^{-1}(p)$. We have the following theorem:

Theorem 6 *If $p \in P$ and $q_0, q_1 \in Q_p$, then there exists a geodesic joining q_0 and q_1. Moreover, this geodesic is unique up to reparametrization.*

For more details about π, and the fibers Q_p, see [2], where among other things, alternative characterizations of the Q_p are found.

The previous theorems lead up to the following result, which is is the main minimality result proven in [2]:

Theorem 7 *Let $p \in P$, and $q \in Q_p$, and let γ be the geodesic contained in Q_p joining p and q, which is provided by the previous theorem, since $p \in Q_p$. Then γ has minimal length among all curves in Q joining p and q.*

Summarizing the above theorems about Q and P, we see that given two points in P of distance less than π apart, there is a geodesic in P joining them, and it has minimal length with respect to all curves in Q, joining the two points. We also see that if we have a point p in P, and a point q in the fiber Q_p, then the unique geodesic in Q_p is minimal with respect to all curves in Q joining p and q.

We will conclude with a minimality theorem for geodesics in the self-adjoint Grassmann $\mathcal{U}(\mathcal{E})$ in the environment $\mathcal{E}$, as in [5].

Theorem 8 *Let $\mathcal{E} = (E, \Pi)$ be a Banach environment over the involutive C^*-algebra B. Let $U(B)$ denote the set of unitary elements in B. Let $\gamma_0 : [0, \tau_0] \to \mathcal{U}(\mathcal{E})$ and $\gamma : [0, \tau] \to \mathcal{U}(\mathcal{E})$ be two smooth paths and let $\Gamma_0 : [0, \tau_0] \to U(B)$ and $\Gamma[0, \tau] \to U(B)$ be the standard lifts (as mentioned previously) of γ_0 and γ, respectively. Assume that*

1. *$\Gamma(\tau) = u\Gamma_0(\tau_0)u^*$ for some $u \in U(B)$, and*

2. *γ_0 is a geodesic curve such that length(γ_0)$< \pi$.*

Then length(γ_0)$\leq$length(γ).

That is, within a certain class of curves, geodesics are minimizing. Specifically, among curves in $\mathcal{U}(\mathcal{E})$, a geodesic γ_0 defined on [0,τ_0] of length less than π is minimizing among all curves γ defined on some [0,τ] such that $\Gamma_0(\tau_0)$ and $\Gamma(\tau)$ are unitarily equivalent. This theorem is quite similar to the first of the minimality theorems for geodesics in Q and P above. The minimality results for geodesics in Grassmannians of environments, however, only show that short geodesics in the self-adjoint Grassmannian $\mathcal{U}(\mathcal{E})$ are minimal among a certain class of curves in $\mathcal{U}(\mathcal{E})$. The corresponding result (Theorem 5) for geodesics in the self-adjoint space P, shows that short geodesics are in fact minimal among all curves in Q, which is a stronger result. Moreover, Theorems 6 and 7 contain minimality results for certain geodesics in Q, not just in P. Thus, it seems that the strength of the minimality properties of geodesics may be the price we must pay to deal with the generality of environments.

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