

THE COMPETITION OF ROUGHNESS AND CURVATURE IN AREA-CONSTRAINED POLYMER MODELS

RIDHIPRATIM BASU, SHIRSHENDU GANGULY, AND ALAN HAMMOND

ABSTRACT. The competition between local Brownian roughness and global parabolic curvature experienced in many random interface models reflects an important aspect of the KPZ universality class. It may be summarized by an exponent triple $(1/2, 1/3, 2/3)$ representing local interface fluctuation, local roughness (or inward deviation) and convex hull facet length. The three effects arise, for example, in droplets in planar Ising models [20, 21, 19, 2]. In this article, we offer a new perspective on this phenomenon. We consider the model of directed last passage percolation in the plane, a paradigmatic example in the KPZ universality class, and constrain the maximizing path under the additional requirement of enclosing an atypically large area. The interface suffers a constraint of parabolic curvature, as the Ising droplets do, but now its local interface fluctuation exponent is governed by KPZ relations, and is thus two-thirds rather than one-half. We prove that the facet lengths of the constrained path's convex hull are governed by an exponent of $3/4$, and inward deviation by an exponent of $1/2$. That is, the exponent triple is now $(2/3, 1/2, 3/4)$ in place of $(1/2, 1/3, 2/3)$. This phenomenon appears to be shared among various isoperimetrically extremal circuits in local randomness. Indeed, we formulate a conjecture to this effect, concerning such circuits in supercritical percolation, whose Wulff-like first-order behaviour was recently established by Biskup, Louidor, Procaccia and Rosenthal in [9], settling a conjecture of Benjamini.

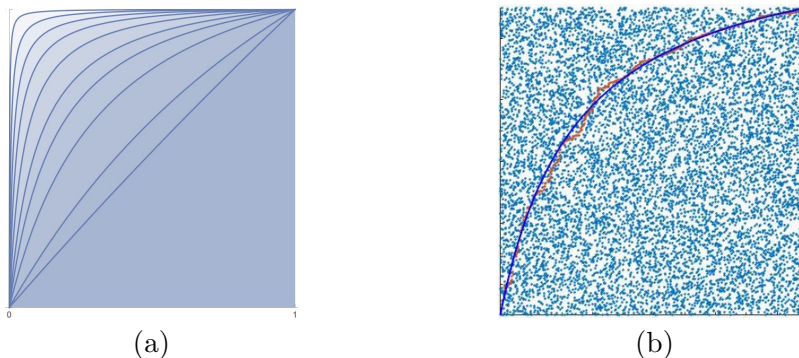


FIGURE 1. (a) Limiting curves for the constrained geodesics for various trapped area values. (b) A typical realization for the area trapping polymer model.

CONTENTS

1.	Introduction and main results	2
2.	Geometric and probabilistic inputs	9
3.	A Variational Approach to the Constrained Geodesic	13
4.	Law of large numbers	17
5.	Lower Bound for Scaling Exponents	22
6.	Upper Bound for Scaling Exponents	28
7.	Proofs of some of the earlier statements	31
	References	36

1. INTRODUCTION AND MAIN RESULTS

The geometric properties of random interfaces are a vast arena of study in rigorous statistical mechanics. Two important classes of interface models are *phase separation* models that idealize the boundary between a droplet of one substance suspended in another, and *last passage percolation* models, where a directed path in independent local randomness maximizes a random weight determined by the environment.

One of the best known mathematical examples of phase separation is the two dimensional supercritical Ising model in a large box with negative boundary condition. Conditioned to have an atypically high number of positive spins in the box, the vertices in the plus phase tend to form a droplet surrounded by a sea of negative spins. The random phase boundary of such a droplet has been the object of intense study. Wulff proposed that the profile of such constrained circuits would macroscopically resemble a dilation of an isoperimetrically optimal curve. This was established rigorously in [15] and [23]. Via the FK representation of the Ising model, one observes a similar situation in the setting of subcritical two dimensional percolation, where the analogous object is the boundary of the cluster containing the origin after this cluster is conditioned to be atypically large. The Wulff shape captures the macroscopic profile of the circuit in such models, but what of fluctuations? Several definitions may be considered that seek to capture fluctuation behaviour on the part of the circuit, including the deviation in the Hausdorff metric of the convex hull of the circuit from an appropriately scaled Wulff shape. Alternative definitions, more local in nature, serve better to capture the transition in circuit geometry from local Brownian randomness to a smoother profile dictated by the constraints of global curvature. In fact, a pair of definitions is natural, one to capture the longitudinal distance at which the transition takes place, and the second to treat the orthogonal inward deviation of the interface at this transition-scale. To specify the characteristic longitudinal distance, we may note that the convex hull of the circuit is a polygonal path that is composed of planar line segments or *facets*; we may treat the typical or maximum facet length as a barometer of the transition from the shorter scale of local randomness to the longer scale of curvature. Latitudinally, we may note that any point in the circuit has a *local roughness*, given by its distance from the convex hull boundary. The typical or maximum local roughness along the circuit is a latitudinal counterpart to facet length. Alexander [2], and Hammond [20, 21, 19] analysed such conditioned circuit models and determined that when the area contained in the circuit is of order n^2 , so that the circuit has diameter of order n , facet length and local roughness scale as $n^{2/3}$ and $n^{1/3}$. A similar situation is witnessed when a parabola $x \rightarrow t^{-1}x^2$ is subtracted from a two-sided Brownian motion $B : \mathbb{R} \rightarrow \mathbb{R}$. When $t > 0$ is large, facets of the motion's convex hull have length $\Theta(t^{2/3})$ and inward deviation $\Theta(t^{1/3})$. This phenomenon is expected to embrace many models in which the local structure of the interface is Brownian: another example, which was analysed by Ferrari and Spohn [17], is Brownian bridge pinned to zero at $-t$ and t and conditioned to remain above the semi-circle of radius t centred at the origin.

The second class of random geometric paths we have mentioned are last passage percolation models. These models form part of a huge, Kardar-Parisi-Zhang, class of statistical mechanical models in which a path through randomness is selected to be extremal for a natural weight determined by that randomness. The maximizing paths are often called *polymers*. The fluctuation behaviour of a length n polymer with given endpoints may be gauged either in terms of the scale of deviation of its weight from the mean value, or by the scale of deviation of say the polymer's midpoint from the planar line segment that interpolates the polymer's endpoints. The two deviations are given by scales $n^{1/3}$ and $n^{2/3}$. The scaling has been predicted in the statistical physics literature by Kardar, Parisi, and Zhang [25]. The length fluctuation exponent of $1/3$ was first proved for planar Poissonian directed last passage percolation, in the seminal work of Baik, Deift and Johansson [4] who also identified the GUE Tracy-Widom scaling limit; the scaling limit of $2/3$ for transversal deviation was proved by Johansson [24]. Since then, this and other integrable models in the same

KPZ universality class have been extensively analysed and detailed information about this model has been obtained [27, 28].

It is of much interest to study models that combine phase separation and path minimization (or maximization) in random environment. Consider an environment with local independent randomness, and the circuit through that randomness whose weight determined by that randomness is extremal among those circuits that trap a given area. Studying the random geometry of such a circuit is a problem of extremal isoperimetry. Taking the randomness to be supercritical percolation, Itai Benjamini conjectured the first-order, Wulff-like, behaviour of the boundary of the set in the supercritical percolation cluster attaining the so-called anchored expansion constant. This was proved by [9] by showing that the curve in the limit solves a natural isoperimetric variational problem; this has recently been extended to higher dimensions by Gold [18]. Given the role of facet length and local roughness in capturing the local-random-to-global-curvature transition in circuit geometry, a natural next task is to seek to understand the scaling exponents of such objects. This is our pursuit in this paper. Our choice of model preserves the qualitative features of the problem of interface fluctuation in an isoperimetrically extremal droplet in supercritical percolation and at the same time ensures that key algebraic aspects of KPZ theory can be harnessed to yield sharp fluctuation estimates.

The particular setting we consider in this paper is planar Poissonian directed last passage percolation. To impose the required properties, we study this model under a quadratic curvature constraint: that is, we force the best path to move away from the straight line and have a quadratic curvature on the average. This is done naturally by considering the longest upright path in a Poissonian environment joining $(0, 0)$ and (n, n) which has the additional *area trap* property that the area under the curve is at least $(\frac{1}{2} + \alpha)n^2$ for some $\alpha \in (0, \frac{1}{2})$. We call this random system the *Area Trapping Polymer* model (though we defer precise definitions to the next section). Note that from the discussion above it follows that the unconstrained longest path stays close to the diagonal and hence encloses area $(\frac{1}{2} + o(1))n^2$. Also postponing precise statements (to Section 1.2), our main results quantify the competition along the contour between a global shape, dictated by a parabola, and local behaviour, guided by KPZ relations. These assertions show that the maximum facet length of the contour's least concave majorant scales as $n^{3/4+o(1)}$, while the local roughness, namely inward deviation from the concave majorant, scales as $n^{1/2+o(1)}$.

We also establish a law of large numbers for the length of the optimal path. The proof proceeds by setting up a variational problem, as is natural in such contexts (see [12] and [9]). Perhaps surprisingly, however, the problem turns out to have a very explicit solution. The geometric information about the limit shape of the constrained polymer is also used as an input in some of the arguments about fluctuations. The question of fluctuation in the context of isoperimetrically extremal circuits in supercritical percolation can be formulated as a first passage percolation analogue of our setting of last passage percolation. Although the first passage percolation model is not exactly solvable, it, too, is believed to be in the KPZ universality class, and the geodesics there are believed to have the same $n^{2/3}$ scaling of transversal fluctuation as in our case (see [3] and the references therein). Thus our results suggest that a certain universality is being witnessed and, according to this belief, we formulate a conjecture concerning percolation. We elaborate on this and a number of other interesting questions in Section 1.4.

Finally, we discuss briefly the key inputs used in our paper and how it contrasts with the other examples of fluctuation results already mentioned. The results in [20, 21, 19] crucially used refined understanding and geometric estimates concerning percolation clusters, while a study of area trapping planar Brownian loop [22] used well known estimates for Brownian motion. Exact expressions involving Brownian motion conditioned to stay above a parabola were the key ingredient in the proofs in [17]. However, in our setting, even though the unconstrained model has integrable properties, the lack of general geometric understanding as an integrable model is perturbed causes

a big challenge. Nonetheless, we are able to utilize as blackbox estimates certain known facts about the unconstrained model and their robust variants established recently in [8]. By so doing, we rigorously establish local roughness exponents for the Area Trapping Polymer model. In this way, this work is an example of the use of integrable probabilistic inputs in obtaining geometric conclusions about settings which lie outside the realm of exact solvability (see [8] for another example of this). We believe a general program of developing such geometric arguments would lead to more robust proofs which could work for settings that are non-integrable but are variants of solvable models.

1.1. Model Definitions. We now recall the planar Poissonian directed last passage percolation model.

Let Π be a homogeneous rate one Poisson Point Process (PPP) on the plane. A partial order on \mathbb{R}^2 is given by $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. For $u \preceq v$, a directed path γ from u to v is a piecewise linear path that joins points $u = \gamma_0 \preceq \gamma_1 \preceq \cdots \preceq \gamma_k = v$ where each γ_i for $i \in [k-1]$ (throughout the article we will adopt the standard notation $[n] = \{1, 2, \dots, n\}$) is a point of Π . Define the length of γ , denoted $|\gamma|$, to be the number of Π -points on γ .

Definition 1.1. *Define the last passage time from u to v , denoted by $L(u, v)$, to be the maximum of $|\gamma|$ as γ varies over all directed paths from u to v . There may be several maximizing paths between u and v , and throughout the paper we will refer to the top most path (it is easy to see that the top most path is well defined here) among those, as the geodesic between u and v and denote it by $\gamma(u, v)$.*

We will often refer to $\gamma(u, v)$ as the polymer between u and v . This polymer's length, or weight, is then $|\gamma(u, v)|$.

Next we introduce the area constraint to this classical model.

1.1.1. Area trapped by a path. Consider a path γ between the origin $(0, 0)$ and a point (x, y) in the positive quadrant. The area trapped by the path γ , denoted by $A(\gamma)$, is defined to be the area of the closed polygon determined by the x -axis, the vertical line segment joining $(x, 0)$ to (x, y) , and the line segments of the path γ . Let γ_n denote the geodesic between $(0, 0)$ and (n, n) . Well-known facts about Poissonian LPP readily imply that $A(\gamma_n) = (\frac{1}{2} + o(1))n^2$ asymptotically almost surely. We constrain the model and consider maximizing paths subject to trapping a much larger area. To this end fix $\alpha \in (0, \frac{1}{2})$, and let

$$L_\alpha(n) := \max \{ |\gamma| : \gamma \text{ path from } (0, 0) \text{ to } (n, n) \text{ and } A(\gamma) \geq (\frac{1}{2} + \alpha)n^2 \}. \quad (1)$$

It is easily seen that, among the paths that attain this maximum, there is almost surely exactly one that traps the least area. This path will be called $\Gamma_{\alpha, n}$. We will write Γ_n provided that the context clarifies the value of α in question. We shall call Γ_n the constrained (or α -constrained) geodesic. In analogy with the phase separation example in percolation mentioned in the introduction, it might seem more natural to consider the family of down right paths joining $(n, 0)$ and $(0, n)$, since all of these curves enclose the origin. However, because of the obvious underlying symmetry, and to take advantage of standard notational conventions, throughout the sequel we will consider the contour joining the origin to the point (n, n) .

Our main objects of interest are two quantities that measure local regularity of the constrained geodesic Γ_n . The following definitions are illustrated by Figure 2.

Definition 1.2. *Let $\text{conv}(\Gamma_{\alpha, n})$ denote the convex hull of the polygon determined by the path $\Gamma_{\alpha, n}$ and the lines $y = 0$ and $x = n$. Let $\Gamma_{\alpha, n}^*$ denote the closure of the polygonal part of the boundary of $\text{conv}(\Gamma_n)$ between $(0, 0)$ and (n, n) above the x -axis. Thus $\Gamma_{\alpha, n}^*$ is the least concave majorant of $\Gamma_{\alpha, n}$ and is a union of finitely many line segments. These segments will be called facets. Define*

maximum facet length of Γ_n , denoted $\text{MFL}(\Gamma_{\alpha,n})$, to be the maximum Euclidean length of the facets. For $x \in \Gamma_{\alpha,n}$, let $d(x, \Gamma_{\alpha,n}^*)$ denote the distance from x to $\Gamma_{\alpha,n}^*$. This is a natural notion of the local roughness at x . Define the maximum local roughness of Γ_n , denoted $\text{MLR}(\Gamma_n)$, by

$$\text{MLR}(\Gamma_{\alpha,n}) = \sup\{x \in \Gamma_{\alpha,n} : d(x, \Gamma_{\alpha,n}^*)\}. \quad (2)$$

In the following subsections we present our main results. We first state our results regarding the fluctuation exponents of $\Gamma_{\alpha,n}$.

1.2. Scaling exponents for geodesic geometry. The sense in which we capture the exponents is stronger and easier to state for the lower bounds, and so we begin with them.

Theorem 1.3. *Fix $\alpha \in (0, \frac{1}{2})$ and $\varepsilon > 0$. Then there exists $c = c(\alpha, \varepsilon) > 0$ such that, for all large enough n ,*

$$\mathbb{P}(\text{MFL}(\Gamma_n) \geq n^{3/4-\varepsilon}) \geq 1 - e^{-n^c}.$$

As we have mentioned, the scaling exponent for transversal fluctuation of point-to-point geodesic in unconstrained Poissonian last passage percolation is known to be $2/3$. This fact, alongside the last theorem, yields the following lower bound on maximum local roughness.

Theorem 1.4. *Fix $\alpha \in (0, \frac{1}{2})$ and $\varepsilon > 0$. Then there exists $c = c(\alpha, \varepsilon) > 0$ such that for all large enough n ,*

$$\mathbb{P}(\text{MLR}(\Gamma_n) \geq n^{1/2-\varepsilon}) \geq 1 - e^{-n^c}.$$

Regarding the matching upper bound, we prove that, with high probability, there exists a dense set of $\alpha \in (0, 2^{-1})$ for which the maximum length of the facets away from the boundary is bounded above by $n^{3/4+o(1)}$. To make this precise, fix $\delta \in (0, \pi/4)$, and consider a facet in $\Gamma_{\alpha,n}$ with endpoints A and B recorded in clockwise order. Setting $O = (n, 0)$, let θ_A denote the acute angle that OA makes with the y -axis and θ_B , the acute angle that OB makes with the x -axis; see Figure 2.

Definition 1.5. *The facet AB is called δ -interior if $\min(\theta_A, \theta_B) \geq \delta$.*

Note that the union of the δ -interior facets forms a polygonal path. (The union could be empty, but we will infer later from Theorem 1.10 that this event has an exponentially small probability.) Let A_0 and B_0 denote the extremities of this union path, and let $\Gamma_{\delta,\alpha,n}$ denote the subpath of $\Gamma_{\alpha,n}$ between A_0 and B_0 . Define the maximum δ -interior facet length of $\Gamma_{\alpha,n}$, denoted by $\text{MFL}(\Gamma_{\delta,\alpha,n})$, to be the maximum length of the δ -interior facets. Define the δ -interior maximum local roughness, denoted by $\text{MLR}(\Gamma_{\delta,\alpha,n})$, by altering (2) in Definition 1.2 so that now the supremum is taken over all $x \in \Gamma_{\delta,\alpha,n}$ and the distance is measured from the corresponding restricted part of $\Gamma_{\alpha,n}^*$.

Definition 1.6. *Fix $\varepsilon > 0$. We say $\alpha \in (0, \frac{1}{2})$ is (n, ε, δ) -good if $\text{MFL}(\Gamma_{\delta,\alpha,n}) \leq n^{3/4+\varepsilon}$.*

Here is our upper bound concerning the maximum length of facets.

Theorem 1.7. *Fix $\varepsilon > 0$, $\delta \in (0, \pi/4)$ and an interval $[\alpha_1, \alpha_2] \subset (0, \frac{1}{2})$. Let I_{n,α_1,α_2} denote the set of (n, ε, δ) -good $\alpha \in [\alpha_1, \alpha_2]$. Then there exists $c = c(\varepsilon, \delta, \alpha_1, \alpha_2) > 0$ such that the probability that $I_{n,\alpha_1,\alpha_2} \neq \emptyset$ is at least $1 - e^{-n^c}$ for all large enough n .*

Our final principal result concerning exponents asserts that, for $\alpha \in I_{n,\alpha_1,\alpha_2}$, it is highly likely that $\text{MLR}(\Gamma_{\delta,\alpha,n}) \geq n^{1/2+\varepsilon}$. Thus we obtain, with high probability, an $n^{1/2+\varepsilon}$ upper bound for interior maximum local roughness for a dense, though possibly n -dependent and random, set of α .

Theorem 1.8. *In the setting of Theorem 1.7, there exists $c > 0$, such that the event that there exists $\alpha \in I_{n,\alpha_1,\alpha_2}$ such that $\text{MLR}(\Gamma_{\delta,\alpha,n}) > n^{1/2+\varepsilon}$, occurs with probability at most e^{-n^c} for all large enough n .*

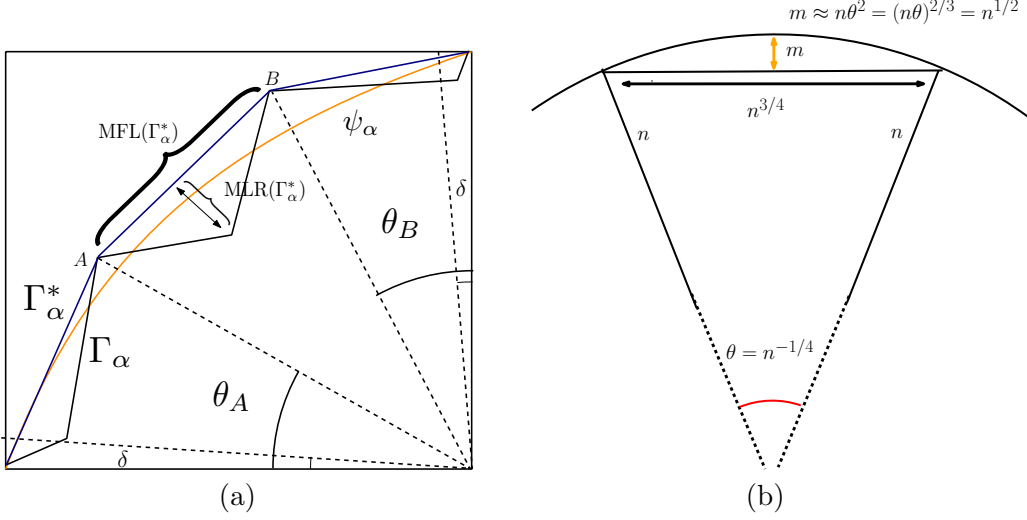


FIGURE 2. (a) The three different curves correspond to $\psi_{\alpha,n}, \Gamma_{\alpha,n}, \Gamma_{\alpha,n}^*$. The dashed lines indicate the δ -interior and hence the facets inside the sector bounded by them, are the δ -interior facets. (b) The scale at which the competition from two sources are equal to each other.

We now state a law of large numbers for $L_\alpha(n)$ and the constrained geodesic. Although the route taken here of setting up an appropriate variational problem is by now classical [12], we point out that it was not at all obvious that the solution can be explicitly described. Moreover some of the consequences of the results in the following section are used as geometric inputs for the fluctuation results stated before.

1.3. Law of Large Numbers. For the unconstrained model, a straightforward super-additivity argument yields that $n^{-1}\mathbb{E}L_n$ converges to a limit. The evaluation of the limiting constant is classical: [30, 26] showed that the limit equals 2 by using Young tableaux combinatorics and the RSK correspondence (see also [1]). However, for the constrained model, the super-additive structure is lost and it is not clear a priori that a law of large numbers for $L_\alpha(n)$ exists. Our first result here is to establish the law of large numbers for the area trapping polymer model; we are also able to evaluate the limiting constant implicitly as a function of α .

Given $\alpha \in (0, 1/2)$, let c_α be given implicitly by the following equation:

$$\frac{1 + c_\alpha}{c_\alpha} \left(1 - \frac{\log(1 + c_\alpha)}{c_\alpha} \right) = \frac{1}{2} + \alpha. \quad (3)$$

One can see that the function $f(c) = \frac{1+c}{c} [1 - \frac{\log(1+c)}{c}]$ is strictly increasing¹ in c , and converges to $1/2$ and 1 at 0 and ∞ respectively. Let $w_\alpha = \sqrt{1 + c_\alpha} \frac{\log(1+c_\alpha)}{c_\alpha}$.

Theorem 1.9. *For any $\alpha \in (0, 1/2)$,*

$$\mathbb{E}L_\alpha(n) = 2w_\alpha n + o(n)$$

as $n \rightarrow \infty$.

Notice that $w_\alpha \rightarrow 1$ as $\alpha \rightarrow 0$; hence the above theorem is consistent with the result in the classical unconstrained case. In the unconstrained model, one also has a law of large numbers for the geodesic, i.e., it is known that the geodesic is concentrated around the straight line joining

¹ $f'(c) = \frac{(2+c)\log(1+c)-2c}{c^3}$ and $\frac{d}{dc}[(2+c)\log(1+c)-2c] = \log(1+c) - \frac{c}{1+c} > 0$.

$(0, 0)$ and (n, n) . More precisely, under the rescaling that takes the $n \times n$ square to a unit square, the geodesic converges almost surely in Hausdorff distance to the diagonal of the unit square [12]; (as mentioned before, the precise order of the transversal fluctuations are known to be $n^{2/3}$ as a consequence of integrability). Even though we do not have any exactly solvable structure in the constrained model, we can establish a similar law of large numbers here too asserting that the constrained geodesic concentrates around a deterministic curve. Moreover, we can identify the limiting curve in a fairly explicit manner.

Let $\psi_\alpha : [0, 1] \rightarrow [0, 1]$ be defined as $\psi_\alpha(x) = \frac{(1+c_\alpha)x}{1+c_\alpha x}$ where c_α is as above. Also let $\psi_{\alpha,n} : [0, n] \rightarrow [0, n]$ be the n blow-up of ψ_α i.e. $\psi_{\alpha,n}(x) = n\psi_\alpha(x/n)$. We denote by $\text{dist}(\cdot, \cdot)$ the Hausdorff distance. The following theorem is our law of large numbers for the constrained geodesic.

Theorem 1.10. *For any $\alpha \in (0, 1/2)$ and $\Delta > 0$, there exists $c = c(\alpha, \Delta)$ such that, for all large enough n , it is with probability at least $1 - e^{-cn}$ that*

$$\text{dist}(\Gamma_{\alpha,n}, \psi_{\alpha,n}) \leq \Delta n.$$

In particular, this theorem states that to first order, the constrained geodesics behave like a given smooth curve. This result will be useful to us when we study the scaling exponents for local roughness of the constrained geodesic, in particular, when we seek to rule out long and flat facets (see Theorem 2.2).

1.4. Open questions and future directions. Below we list below several interesting questions for future research:

- (1) The upper bound Theorem 1.7 is weaker than the lower bound Theorem 1.3. Strengthening the former is a natural open problem.
- (2) By definition, $A(\Gamma_{\alpha,n}) \geq (\frac{1}{2} + \alpha)n^2$. The typical order of $A(\Gamma_{\alpha,n}) - (\frac{1}{2} + \alpha)n^2$ remains unknown.
- (3) For $\alpha \in (0, 1/2)$, what is the typical deviation of $\Gamma_{\alpha,n}$ from the curve $\psi_{\alpha,n}$? What is the order of fluctuations of $L_\alpha(n)$?
- (4) In a phase separation problem, [20, 21, 19] determines the polylogarithmic corrections to both elements of the $(2/3, 1/3)$ (facet length, local roughness) exponent pair. The powers of the logarithm are $(1/3, 2/3)$. Finding such corrections for the *Area Trapping Polymer* model would refine the identification of the exponent pair $(3/4, 1/2)$ made in this article and suggested by the first point.

The first two points will be discussed further in Section 6.2.

Supercritical percolation: We end this section with a conjecture regarding fluctuation exponents in the context of supercritical percolation on the nearest neighbour graph on \mathbb{Z}^2 . As we have mentioned already, [9] settles a conjecture of Benjamini regarding the limit shape of isoperimetrically extremal sets. Formally, for supercritical percolation, with the origin $\mathbf{0}$ conditioned to be in the infinite cluster $\mathcal{C}_\infty(\mathbf{0})$, the authors in [9] consider the ‘anchored isoperimetric profile’, i.e., for any $r > 0$ they discuss the set \mathcal{B}_r which solves the following isoperimetric problem:

$$\inf \left\{ \frac{|\partial B|}{|B|} : \mathbf{0} \in B, B \subset \mathcal{C}_\infty(\mathbf{0}) \text{ is connected, } |B| \leq r \right\}$$

where ∂B denotes the edge boundary of B , restricted to $\mathcal{C}_\infty(\mathbf{0})$ (for more details see [9]). The main result in [9] shows the convergence of the set \mathcal{B}_r as $r \rightarrow \infty$, after suitable renormalization, to a deterministic Wulff crystal, in the Hausdorff sense. To go beyond first order behaviour one has to understand local geometric properties of the boundary of \mathcal{B}_r . The extremal circuit broadly has to satisfy a:

- (1) volume condition,

(2) and an extremal isoperimetry condition.

The heuristic now is that the former is a global constraint while the latter is only local. Namely, each local part of the boundary does not feel the volume constraint $|\mathcal{B}_r| \leq r$, and thereby just tries to move through the open edges while trying to minimize the number of open edges that it cuts across, since these are precisely the edges that contribute to $|\partial\mathcal{B}_r|$. This brings us within the realm of first passage percolation, which is also predicted to lie in the KPZ universality class; (see [3] for more on first passage percolation). Thus, Theorems 1.3, 1.4, 1.7 and 1.8 can be thought of as rigorous counterparts to the above discussion in the integrable last passage percolation setting. In light of this discussion, these results lead us to conjecture fluctuation exponents for the above model. Next we formulate a precise statement in a slightly simpler setting.

Recall the standard definition of the dual graph of the nearest neighbour lattice on \mathbb{Z}^2 . Given a supercritical bond percolation environment on the edge set of \mathbb{Z}^2 with density $p > p_c(\mathbb{Z}^2) = 1/2$, for every positive integer n , consider the set \mathcal{D}_{n^2} of dual circuits (namely simple loops consisting of dual edges) enclosing a connected subset of \mathbb{Z}^2 of size n^2 and containing the origin such that the number of primal open edges in the percolation environment that the circuit cuts through is minimized. Thus the above model is an exact analogue of the model considered in this paper, in the context of first passage percolation in a bond percolation environment. Note, however, that such a circuit is not necessarily unique due to the discrete nature of the problem. We now state precisely our conjecture.

Conjecture: Consider bond percolation on the nearest neighbour lattice on \mathbb{Z}^2 with any supercritical parameter value $p > p_c(\mathbb{Z}^2)$. The random variables

$$\max_{D \in \mathcal{D}_{n^2}} \left| \frac{\log(\text{MFL}(D))}{\log n} - 3/4 \right| \quad \text{and} \quad \max_{D \in \mathcal{D}_{n^2}} \left| \frac{\log(\text{MLR}(D))}{\log n} - 1/2 \right|$$

converge to zero in probability as n grows to infinity where for any $D \in \mathcal{D}_{n^2}$, the maximum facet length $\text{MFL}(D)$ and maximum local roughness $\text{MLR}(D)$ are defined in the same way as in this paper by considering the convex hull of the points in D .

We end with the remark that the problem considered in [9] corresponds to a similar first passage percolation problem where the environment has bounded dependence range and hence should have the same fluctuation behaviour as the simpler model just described.

1.5. Organization of the rest of the article. In Section 2 we collect some preliminary probabilistic results: some for the constrained model and a few for the unconstrained one. The results for the unconstrained model follow from the sharp moderate deviation estimates in [27, 28] and the consequences established in [8]. In Section 3, we set up and solve the variational problem for the law of large numbers and the following Section 4 is devoted to the proofs of Theorem 1.9 and Theorem 1.10: the law of large numbers for the length of constrained geodesic and the path itself. We next turn to the proofs of the results on the scaling exponents. In Section 5 we provide proofs of Theorems 1.3 and Theorem 1.4. The proofs of Theorem 1.7 and Theorem 1.8 are completed in Section 6. Proofs of some of the auxiliary results stated and used throughout the article are postponed to Section 7. Throughout the article we will use various letters to denote universal constants whose value may change from line to line.

Acknowledgments. The authors thank Marek Biskup, Craig Evans and Ofer Zeitouni for useful discussions, and an anonymous referee for comments on the manuscript. R.B. was partially supported by an AMS-Simons Travel Grant during the completion of this work. S.G.'s research is supported by a Miller Research Fellowship at UC Berkeley. A.H. is supported by NSF grant DMS-1512908.

2. GEOMETRIC AND PROBABILISTIC INPUTS

In this section we gather the probabilistic inputs needed for our proof. First we shall record a few useful facts about the length and geometry of the constrained geodesics that will be used repeatedly. The bulk of this section will then recall results about the unconstrained model. These results are all consequences of the basic integrable ingredients obtained by Lowe, Merkl and Rolles [27, 28]. Some of the arguments are fairly standard in the exactly solvable literature, in which case we provide a sketch the proof or point to the relevant reference. A few of these consequences were established and used recently in [8] and we quote the relevant results here.

We start with some basic facts about the constrained model.

2.1. Some basic results for the Area Trapping Polymer model. We first start with concentration for $L_\alpha(n)$. In the absence of integrability, we use standard Poincaré inequality techniques and obtain a concentration at $n^{1/2+o(1)}$ scale.

Theorem 2.1. *Fix any $\alpha \in (0, 1/2)$. Then there exists a constant $C > 0$ such that for all n and all $t > 0$,*

$$\mathbb{P}(|L_\alpha(n) - \mathbb{E}(L_\alpha(n))| \geq t) \leq C e^{-t^2/Cn \log^2(n)}.$$

The proof is standard and is postponed until the end of the paper in Section 7.

Our next result rules out flat facets. This is a consequence of Theorem 1.10, which asserts that it is extremely unlikely that any interior facets (which are those in the bulk) have very shallow or steep gradient. Even though, we will need a stronger version later (see Theorem 4.6) it is instructive to state a weaker form of the result now, and give its simple proof.

Theorem 2.2. *For any small enough $\delta > 0$ there exists $\omega > 0$ such that, with probability $1 - e^{-cn}$, all δ -interior facets make an angle with the x -axis which lies in the interval $(\omega, \pi/2 - \omega)$.*

Proof. We start by recalling a simple fact: for two facets of $\Gamma_{\alpha,n}^*$ with starting points u_1, u_3 and ending points u_2, u_4 respectively where $u_1 \preceq u_2 \preceq u_3 \preceq u_4$, by convexity of $\Gamma_{\alpha,n}^*$, the angle made with the x -axis by the facet (u_1, u_2) is larger than the angle made by (u_3, u_4) .

Consider any δ -interior facet (u_1, u_2) , and let $u_1 = (x, y)$. Also let L_1, L_2 be the straight lines joining the origin to u_1 and (n, n) to u_2 respectively. Let v_1 and v_2 be the points of intersection of L_1 and L_2 with $\psi_{\alpha,n}$. Theorem 1.10 implies that, for any $\varepsilon > 0$, with probability at least $1 - e^{-cn}$, the bound

$$\max(|u_1 - v_1|, |u_2 - v_2|) \leq \varepsilon n$$

holds for all δ -interior facets, simultaneously. Now, by the convexity of $\Gamma_{\alpha,n}^*$, the gradient of the facet (u_1, u_2) is between the gradient of the lines joining $(0, 0)$ to u_1 and (n, n) to u_2 . Choosing ε to be much smaller than δ , we see that the gradient of the facet (u_1, u_2) plus an error of $O(\varepsilon)$ lies between the gradients of the line joining the origin and v_1 , and that joining (n, n) and v_2 . Thus we are done by choosing ε to be small enough and using the strict convexity of $\psi_{\alpha,n}$; (see Figure 2(a) for illustration). \square

We will later need Theorem 4.6, a strengthening of the above result, which is uniform in α . We turn to state useful results that concern the unconstrained, exactly solvable, model. All of these are corollaries of the following moderate deviation estimates.

2.2. Moderate deviation estimates. Recall the polymer length $L(\cdot, \cdot)$ from Definition 1.1. Let $u = (u_1, u_2) \preceq v = (v_1, v_2)$ be such that $|u_1 - v_1||u_2 - v_2| = t$; i.e., t is the area of the rectangle whose opposite corners are u and v . Notice that, by scale invariance of the Poisson point process, the distribution of $L(u, v)$ is a function merely of t .

Theorem 2.3 ([27, 28]). *Fix $\kappa > 1$. Let u, v as above be points such that the straight line joining u and v has gradient m , where $m \in (\kappa^{-1}, \kappa)$. There exist positive constants s_0, t_0, C and c depending only on κ such that, for all $t > t_0$ and $s > s_0$,*

$$\mathbb{P}\left(|L(u, v) - 2\sqrt{t}| \geq st^{1/6}\right) \leq Ce^{-cs^{3/2}}.$$

Observe that the above theorem implies that $|\mathbb{E}L(u, v) - 2\sqrt{t}| = O(t^{1/6})$; and hence similar tail bounds are true for the quantity $|L(u, v) - \mathbb{E}L(u, v)|$. When we make use of Theorem 2.3, it will often be in order to obtain tail bounds for the quantities $|L(u, v) - \mathbb{E}L(u, v)|$. Also the results of [4] establish that $t^{-1/6}(L(u, v) - 2\sqrt{t})$ converges weakly to the GUE Tracy-Widom distribution (which is defined for example in [4]). This law has negative mean, leading to the bound

$$\mathbb{E}L(u, v) \leq 2\sqrt{t} - Ct^{1/6} \tag{4}$$

for some $C > 0$.

2.3. Transversal fluctuations of point-to-point geodesics. Let $u = (u_1, u_2) \preceq v = (v_1, v_2)$. Let $y = p_{u,v}(x)$ denote the equation of the line segment joining u and v . For any path γ from u to v , define the maximum transversal fluctuation of the path γ by

$$\text{TF}(\gamma) := \{\sup |p_{u,v}(x) - y| : (x, y) \in \gamma\}.$$

For points $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we call $|v_1 - u_1|$ the horizontal distance between u and v and similarly define the vertical distance.

Transversal fluctuations for paths between $(0, 0)$ and (n, n) were shown to be $n^{2/3+o(1)}$ with high probability in [24]. The following more precise estimate was established in [8].

Theorem 2.4 (Tails for Transversal Fluctuations, [8]). *Fix $\kappa > 1$. Let $u \preceq v$ be points such that the x -coordinate distance between u and v is r and the gradient of the line joining u and v is m , where $m \in (\kappa^{-1}, \kappa)$. Then, for the geodesic Γ from u to v ,*

$$\mathbb{P}[\text{TF}(\Gamma) > sr^{2/3}] \leq Ce^{-cs}$$

for κ -dependent constants C, c , and r, s sufficiently large.

The proof of Theorem 2.4 from [8] in fact shows something more. Any path that has transversal fluctuations much bigger than $r^{2/3}$ is not only unlikely to be a geodesic; it is typically much shorter than a geodesic. In particular, the proof of Theorem 11.1 in [8] implies the next result.

Theorem 2.5 (Paths with off-scale TF are uncompetitive). *Fix $\kappa > 1$ and $\varepsilon > 0$. Let $u \preceq v$ be points such that the x -coordinate distance between u and v is r and the gradient of the straight line joining u and v is m where $m \in (\kappa^{-1}, \kappa)$. Let $\mathcal{E} = \mathcal{E}_{u,v}$ denote the event that there exists a path γ from u to v with $\text{TF}(\gamma) \geq r^{2/3+\varepsilon}$ and $|\gamma| > \mathbb{E}[L(u, v)] - r^{1/3+\varepsilon/10}$; (we suppress the dependence of ε in \mathcal{E} for brevity). Then there exists $c = c(\varepsilon) > 0$ such that*

$$\mathbb{P}(\mathcal{E}) \leq e^{-r^c}$$

for r sufficiently large.

The stretched exponential nature of these estimates allow us to prove uniform properties of the noise space which will be convenient for some of the arguments later. This illustrates a theme in this article: points to which we seek to apply the above results may be special, so that estimates that are uniform over all points will be needed. For the next result, recall that our noise space is nothing other than a rate one Poisson point process in the box $[0, n]^2$. For $u, v \in [0, n]^2$, let $\mathbb{L}(u, v)$ denote the line joining u and v . Now, for any $u \preceq v$, the pair (u, v) is called κ -steep if the gradient of $\mathbb{L}(u, v)$ lies in the interval $[\kappa^{-1}, \kappa]$. We also set, for $\tau > 0$,

$$S(u, v) = S_\tau(u, v) := \begin{cases} |u - v|^{1/3 + \varepsilon/10} & \text{if } |u - v| > n^\tau, \\ n^{2\tau/3} & \text{otherwise.} \end{cases} \quad (5)$$

Corollary 2.6. *Fix $\varepsilon > 0$, $\kappa > 1$, and τ sufficiently small. Then there exists $c = c(\varepsilon, \tau)$ such that*

$$\mathbb{P} \left(\bigcup_{u, v: (u, v) \text{ is } \kappa\text{-steep}} |L(u, v) - \mathbb{E}(L(u, v))| \geq S(u, v) \right) \leq e^{-n^c}.$$

Proof. This follows by using Theorem 2.3 and a standard coarse graining argument. Since we are not attempting to prove optimal bounds, we simply partition the box into squares of area one. Thus the corner points are the lattice points, i.e., elements of the set $\{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$. The proof now follows by first proving the desired statement for lattice points u, v . This just follows by Theorem 2.3 and a union bound over all pairs of lattice points u, v (of which there are n^2). The proof is then completed by showing that, since any box of area one is unlikely (with stretched exponentially small failure probability) to contain more than $n^{\varepsilon/100}$ points, for any u and v , one can approximate $L(u, v)$ by $L(u_*, v_*)$, where u_*, v_* are the nearest lattice points. \square

We now specify a modification of the event $\mathcal{E}_{u, v}$ from Theorem 2.5. Let $\mathcal{E}_{u, v}^*$ denote the event $\mathcal{E}_{u, v}$ where the permitted error term $r^{1/3 + \varepsilon/10}$ is instead taken to be the quantity $S(u, v)$ defined a few moments ago. The next corollary is proved along the same lines as Theorem 2.5, in combination with the arguments in the proof of Corollary 2.6, and we omit the proof.

Corollary 2.7. *Fix $\varepsilon > 0$, $\kappa > 1$, and τ small enough. Then there exists $c = c(\varepsilon, \tau)$ such that*

$$\mathbb{P} \left(\bigcup_{u, v: (u, v) \text{ is } \kappa\text{-steep}} \mathcal{E}_{u, v}^* \right) \leq e^{-n^c}.$$

2.4. Paths between points in an on-scale parallelogram. Next we control the deviation of lengths between pairs of points within certain parallelograms. We first need some definitions. Let $U(r, m, h)$ denote a parallelogram with the properties that:

- (a) One pair of parallel sides are vertical.
- (b) The other pair of parallel sides have gradient m .
- (c) The (horizontal) distance between the vertical sides is r .
- (d) The height of the vertical sides is h .

Note that this defines a unique parallelogram up to translation. This will be enough for our purposes because the underlying noise field is translation invariant.

Theorem 2.8 ([8]). *Let $m \in (\kappa^{-1}, \kappa)$ and $h = Wr^{2/3}$ for some $W > 0$. Then there exist (κ, W) -dependent constants $C, c > 0$, such that, for all sufficiently large r and s ,*

$$\mathbb{P} \left(\sup_{u \in A, v \in B} (L(u, v) - \mathbb{E}L(u, v)) \geq sr^{1/3} \right) \leq Ce^{-cs},$$

and

$$\mathbb{P} \left(\inf_{u \in A, v \in B} (L(u, v) - \mathbb{E}L(u, v)) \leq -sr^{1/3} \right) \leq Ce^{-cs},$$

where A and B respectively denote the left third and right third of $U(r, m, h)$.

The next theorem states that, even if the paths are restricted to stay inside the parallelogram, the fluctuation remains on-scale. Let $L(u, v; U)$ denote the length of the longest path from u to v that does not exit U .

Theorem 2.9 ([8]). *Under the assumptions of Theorem 2.8, there exist (κ, W) -dependent constants $C, c > 0$ such that, for all sufficiently large r and s ,*

$$\mathbb{P} \left(\inf_{u \in A, v \in B} (L(u, v; U) - \mathbb{E}L(u, v)) \leq -sr^{1/3} \right) \leq Ce^{-cs^{1/2}}$$

where A and B respectively denote the right third and the left third of $U(r, m, h)$.

2.5. Paths constrained in a thin parallelogram. The next set of results shows that, if a path is constrained to have much smaller than typical transversal fluctuation, then it must have much smaller length than that of a typical geodesic.

Theorem 2.10 (Paths with small TF are uncompetitive). *Fix $\varepsilon > 0$. Consider the parallelogram $U = U(r, m, h)$, where $m \in (\kappa^{-1}, \kappa)$ and $h = r^{2/3-\varepsilon}$. Let u_0 and v_0 denote the midpoints of the vertical sides A_0 and B_0 of U . Then, for some $c = c(\varepsilon) > 0$,*

$$\mathbb{P} \left(L(u_0, v_0; U) \geq \mathbb{E}L(u_0, v_0) - r^{1/3+\varepsilon/3} \right) \leq e^{-r^c}$$

for r sufficiently large.

The proof of Theorem 2.10 follows a strategy, by now well known, that has been used to show Gaussian fluctuation of paths constrained to stay in a thin rectangle in [11] in the context of first passage percolation. In the context of LPP, the Gaussian fluctuation has recently been shown in [14], and tail bounds are proved in a more general context in the preprint [7]. We now provide a sketch of Theorem 2.10's proof.

Sketch of Proof. Without loss of generality let us assume $m = 1$. Also, we can replace the parallelogram U by the rectangle U' , one of whose pairs of sides is parallel to the line segment u_0v_0 , with the other pair having midpoints u_0 and v_0 and length $r^{2/3-\varepsilon}$. Divide the rectangle U' into K equal parts (each of width $\frac{r}{K}$) by parallel line segments perpendicular to u_0v_0 . Let L_i denote the left side of the i -th rectangle and let u_i be its midpoint. Now let γ_i be the best path (i.e. with maximal value of $|\gamma|$) between L_i and L_{i+1} that stays within U' . It follows from Theorem 2.8 that it is extremely likely that $|\gamma_i| - L(u_i, u_{i+1}) \ll (K^{-1}r)^{1/3}$. It follows that up to an error of order much smaller than $K^{2/3}r^{1/3}$ we can approximate $L(u_0, v_0; U')$ by $\sum_i L(u_i, u_{i+1})$. Now observe that $L(u_i, u_{i+1})$ are independent and identically distributed with mean $2r/K - C(K^{-1}r)^{1/3}$ (here we use (4)) and standard deviation of the order of $(K^{-1}r)^{1/3}$. By standard concentration inequalities, one then shows that $\sum_i L(u_i, u_{i+1})$ concentrates around $2r - K^{2/3}r^{1/3}$ at scale $K^{1/6}r^{1/3}$. By choosing $K \gg r^{\varepsilon/2}$ properly one gets the result. \square

In the same vein, we have the following corollary.

Corollary 2.11. *Fix $\varepsilon > 0$, $\kappa > 1$, and τ small enough. Then there exists $c = c(\varepsilon, \tau)$ such that, with probability at least $1 - e^{-n^c}$,*

$$\sup_{u, v: |u-v| \geq n^\tau, (u, v) \text{ is } \kappa\text{-steep}} L(u, v, U) - \mathbb{E}(L(u, v)) \leq -|u - v|^{\frac{1}{3}+\varepsilon}.$$

2.6. Estimates for One-Sided Geodesics. We end this section by considering geodesics in a different constrained model. Baik and Rains [5, 6] considered increasing paths from $(0, 0)$ to (n, n) that lie above the diagonal line joining the points. Recall that $L(u, v)$ denotes the length of the longest increasing path between points $u \preceq v$. Let $L^\square(u, v)$ denote the length of the longest path between u and v restricted to lie above the line joining u and v and, similarly to Definition 1.1, let $\gamma^\square(u, v)$ denote the corresponding uppermost one-sided geodesic between u and v . Moreover, let $L^\square(n)$ denote $L^\square(u, v)$ in the special case that $u = (0, 0)$ and $v = (n, n)$. Baik and Rains [5, 6] proved that $\mathbb{E}L^\square(n) = 2n + o(n)$ and that the fluctuations are again of order $n^{1/3}$ (although in this case the scaling limit is different; it is the GSE Tracy-Widom distribution instead of the GUE Tracy-Widom distribution). We shall need the corresponding moderate deviation estimates, consequences of Theorem 2.9.

Theorem 2.12. *Let u, v be as in the hypothesis of Theorem 2.3. There exist positive constants s_0, t_0, C and c such that, for all $t > t_0$ and $s > s_0$,*

$$\mathbb{P}[|L^\square(u, v) - 2\sqrt{t}| \geq st^{1/6}] \leq Ce^{-cs^{1/2}}.$$

Proof. The upper tail result follows from Theorem 2.3 due to $L^\square(u, v) \leq L(u, v)$. For the lower tail, we use Theorem 2.9 with the straight line joining u and v being the bottom side of the parallelogram and use the fact that $\mathbb{E}L(u, v) = 2\sqrt{t} - \Theta(t^{1/6})$. \square

We need a uniform version of this result. Recall $S(u, v) = S_\tau(u, v)$ from (5).

Corollary 2.13. *Fix $\varepsilon > 0$, $\kappa > 1$, and $\tau > 0$ small enough. Then there exists $c = c(\varepsilon, \tau)$ such that, with probability at least $1 - e^{-n^c}$,*

$$\sup_{u, v: (u, v) \text{ is } \eta\text{-steep}} L^\square(u, v) - \mathbb{E}(L(u, v)) \geq -S(u, v).$$

Proofs of Corollaries 2.7 2.11, 2.13 follow from the corresponding theorems for fixed points, in the same way as Corollary 2.6 follows from Theorem 2.3. We omit the details.

3. A VARIATIONAL APPROACH TO THE CONSTRAINED GEODESIC

We now move towards proving Theorem 1.9 and Theorem 1.10. Deuschel and Zeitouni [13, 12] studied in detail the hydrodynamic limit of the unconstrained geodesic for inhomogeneous point processes. We follow their strategy broadly and study the limit of the constrained curve by means of appropriate variational problems. We first explain the idea. Fix $\alpha \in (0, \frac{1}{2})$ for the rest of this section. First let us assume that a limiting continuous curve ϕ of the constrained geodesics exists after scaling. This curve $\phi : [0, 1] \rightarrow [0, 1]$ will be continuous, non-decreasing and surjective. By the area constraint, $\int_0^1 \phi(s) ds \geq \frac{1}{2} + \alpha$. Now heuristically, since in practice there is always a little bit of area excess, one can approximate ϕ by a piecewise affine function at a scale local enough that each piece is exempt from the area constraint; and we can then use the law of large numbers for the unconstrained geodesic (see Theorem 2.3) at every local scale and sum over them. This argument suggests that the approximating path will have length $2J(\phi)n + o(n)$ where

$$J(\phi) = \int_0^1 \sqrt{\dot{\phi}(s)} ds.$$

It thus seems that we should maximize $J(\phi)$ among all curves ϕ satisfying the area constraint, and the maximum will be the constant appearing in the required law of large numbers. We now proceed to make this precise. Let \mathcal{B} be the collection of all right-continuous non-decreasing functions from $[0, 1]$ to $[0, 1]$. Thus \mathcal{B} is in bijection with the set of all sub-probability measures on $[0, 1]$. Now, for any $\phi \in \mathcal{B}$, by the Lebesgue decomposition theorem we can write

$$\phi = \phi_{ac} + \phi_s \tag{6}$$

in a unique way as a sum of a pair of sub-probability measures, with ϕ_{ac} being the absolutely continuous part and ϕ_s the singular part (with respect to Lebesgue measure). Note that this implies ϕ_{ac} has a derivative $\dot{\phi}_{ac}$ almost everywhere² such that, for any $0 \leq x \leq 1$, we have

$$\int_0^x \dot{\phi}_{ac}(s) ds = \phi_{ac}(x)$$

while ϕ_s is almost surely flat and hence has derivative 0 almost surely. Thus $\dot{\phi} = \dot{\phi}_{ac}$ almost surely. Also define

$$J(\phi) = \int_0^1 \sqrt{\dot{\phi}(s)} ds. \quad (7)$$

Let

$$\mathcal{B}_\alpha = \left\{ \phi \in \mathcal{B} : \int_0^1 \phi(s) ds \geq \frac{1}{2} + \alpha \right\}. \quad (8)$$

We shall often omit the subscript α . Finally let

$$J_\alpha = \sup_{\phi \in \mathcal{B}_\alpha} J(\phi). \quad (9)$$

The first step is to show existence and uniqueness of the solution.

Proposition 3.1. *There exists a unique element $\psi = \psi_\alpha \in \mathcal{B}$ that attains the supremum in (9).*

Note that the function $\psi = \psi_\alpha$ in Proposition 3.1 will be the same in the statement of Theorem 1.10. The existence and uniqueness parts of Proposition 3.1 have separate proofs. We first state the existence result.

Lemma 3.2. *There exists $\psi \in \mathcal{B}$ that achieves the supremum in (9).*

The proof of Lemma 3.2 is technical: it uses a compactness argument on the space of probability measures following a similar argument from [12]. We postpone the proof until Section 7.

In the next part we show uniqueness.

Lemma 3.3. *Suppose ψ_1 and ψ_2 are in \mathcal{B}_α and satisfy $J(\psi_i) = J_\alpha$ for $i = 1, 2$. Then $\psi_1 = \psi_2$.*

The proof of Lemma 3.3 is rather straightforward once we establish the following technical lemma that rules out the possibility that any of the optimizing functions has a nontrivial singular part.

Lemma 3.4. *Let $\psi \in \mathcal{B}_\alpha$ be such that $J(\psi) = J_\alpha$. Then ψ corresponds to a probability measure which is absolutely continuous with respect to Lebesgue measure.*

Here is the basic idea of the proof of this lemma. From the results of [12] it follows that without the area constraint the optimizing curve is the diagonal line, i.e., the preferred gradient for the graph of the function is one. If the singular part is non-trivial, then the graph of the function ψ may be expected to have a flat piece. Because the gradient of the graph has to be one on average due to boundary conditions, it follows that there must be a piece of the graph having gradient away from one. We shall show that one can modify the flat and the steep parts of the curve locally by pieces with more moderate gradient in a way that increases not only the value of the functional J but while still satisfying the area constraint; and thereby optimality will be contradicted. Also, clearly in the case that ψ does not have total mass one, one can add to it another function to make it (or, more precisely of course, to bring it into correspondence with) a probability measure, while also increasing the value of $J(\psi)$, and thus contradict maximality in this case also. Thus ψ is also a probability measure. The details of the proof are postponed to Section 7.

We can now prove the uniqueness result using Lemma 3.4.

²Throughout the article, for any function f on $[0, 1]$, which is differentiable almost everywhere, we will denote its derivative by \dot{f} . Recall that this does not necessarily imply $\int_0^x \dot{f}(s) ds = f(x)$ for all x .

Proof of Lemma 3.3. We use the fact that the square-root function is strictly concave. Given any ψ_1 and ψ_2 as in the statement of the lemma, we consider $\psi = \frac{\psi_1 + \psi_2}{2}$. Clearly, ψ satisfies the area constraint and $(\frac{\psi_1 + \psi_2}{2})' = \frac{\psi_1' + \psi_2'}{2}$. Thus using Jensen's inequality $J(\psi)$ is necessarily larger than $J(\psi_1) = J(\psi_2)$ unless $\psi_1 = \psi_2$ almost surely. Since by Lemma 3.4, the absolutely continuous parts contain mass one, $\psi_1(0) = \psi_2(0) = 0$. Hence, for all $0 \leq x \leq 1$,

$$\psi_1(x) = \int_0^x \psi_1'(s) ds = \int_0^x \psi_2'(s) ds = \psi_2(x).$$

Thus we are done. \square

It still remains to identify the optimizer ψ in (9); in particular we need to show this is the same ψ as defined in Theorem 1.10. To this end, we shall now record some properties of the unique optimizer ψ that will be useful later. We show that the derivative is almost surely positive and decreasing. The proofs are easy and will be postponed to Section 7.

Lemma 3.5. *Let $\psi = \psi_\alpha$ denote the unique optimizer in Proposition 3.1. Then*

- (i) *The derivative $\dot{\psi}$ is almost surely decreasing i.e. there is a set of full measure on which it is decreasing.*
- (ii) *$\dot{\psi}$ is almost surely positive.*

3.1. Identifying the curve ψ_α . In this subsection we determine the curve ψ_α and show that it is the same as the curve in Theorem 1.10. We first start with the following proposition.

Proposition 3.6. *Given $\alpha \in (0, 1/2)$ there exists $c = c_\alpha > 0$, such that $\psi_\alpha(x) = \frac{(1+c)x}{1+cx}$.*

Proof. The proof proceeds by showing that, given any $0 < a < b < 1$,

$$\frac{1}{\sqrt{\dot{\psi}(x)}} = c_1 x + c_2 \text{ a.s. on } [a, b].$$

for some constants c_1 and c_2 . Fix $0 < a < b < 1$ and let h_* be a polynomial such that

$$\int_a^b h_*(x) dx = 0; \text{ and} \tag{10}$$

$$\int_a^b x h_*(x) dx = 0. \tag{11}$$

Define $h_*(x)$ on $[0, 1]$ by defining it to be 0 outside $[a, b]$. Let $g = \dot{\psi} + \varepsilon h_*$. Now since $\dot{\psi}$ is monotonically decreasing and positive almost surely (see Lemma 3.5), $\dot{\psi}$ is bounded above and below by constants on $[a, b]$ and hence for all ε (with sufficiently small absolute value depending on $[a, b]$), we conclude g is strictly positive. (We will shortly need to take ε to be negative as well as positive.) Moreover, by (10), $\int_0^1 g(x) dx = 1$. We now verify the area constraint. It is easy to see by Fubini's theorem (see (31) in the proof of Lemma 3.5) that $\int_0^1 (1-x)\dot{\psi}(x) dx \geq 1/2 + \alpha$. Thus

$$\int_0^1 (1-x)g(x) dx = \int_0^1 (1-x)\dot{\psi}(x) dx + \varepsilon \int_0^1 (1-x)h_*(x) dx \geq \frac{1}{2} + \alpha$$

by (10) and (11). Now, by Taylor expansion, for all small enough ε (depending on a, b, h and ψ),

$$\int_0^1 \left[\sqrt{g(x)} - \sqrt{\dot{\psi}(x)} \right] dx = \int_a^b \left(\sqrt{\dot{\psi}(x)} + \varepsilon \frac{h_*(x)}{2\sqrt{\dot{\psi}(x)}} - \varepsilon^2 O\left(\frac{h_*(x)^2}{\dot{\psi}(x)^{3/2}}\right) \right) dx, \tag{12}$$

where the constants in the O -notation depend only on $[a, b]$ and ψ (since $\dot{\psi}$ is in $[c_3, c_4]$ almost surely for some $c_3 < c_4$ on $[a, b]$ by the preceding discussion). Since $J(\psi) = J_\alpha$ (and ε can be both positive and negative), it follows that

$$\int_a^b \frac{h_*(x)}{\sqrt{\dot{\psi}(x)}} dx = 0. \quad (13)$$

Since $\frac{1}{\sqrt{\dot{\psi}}}$ is bounded above and below on $[a, b]$, one can find a linear function $L(x) := a_1x + b_1$ such that

$$\int_a^b \left[\frac{1}{\sqrt{\dot{\psi}(x)}} - L(x) \right] dx = 0 \quad \text{and} \quad \int_a^b \left[\frac{1}{\sqrt{\dot{\psi}(x)}} - L(x) \right] x dx = 0.$$

Indeed, such L can be found by solving the two linear equations in a_1 and b_1 given by the above two equations. To see that, for any $a < b$, the equations are non-degenerate, first observe that one can by a change of variable assume that $a = 0$ and $b = 1$. Then the matrix corresponding to the linear equations is nothing other than $\begin{bmatrix} 1/2 & 1 \\ 1/3 & 1/2 \end{bmatrix}$ and hence admits a unique solution. Together with (13) this now implies that

$$\int_a^b \left[\frac{1}{\sqrt{\dot{\psi}(x)}} - L(x) \right] P(x) dx = 0$$

for all polynomials $P(x)$. For this, observe that by Gram-Schmidt orthogonalization any polynomial P can be decomposed as $P(x) = L'(x) + h_*(x)$ where L' is a linear function and h_* is a polynomial satisfying (10) and (11). Since polynomials are dense in $\mathbb{L}_2[a, b]$ by the Stone-Weierstrass theorem, this implies

$$\frac{1}{\sqrt{\dot{\psi}(x)}} - L(x) = 0 \quad \text{a.e. on } [a, b].$$

Since $\frac{1}{\sqrt{\dot{\psi}}}$ is a linear function on all intervals $[a, b]$, it follows that it has to be the same linear function on all such intervals and hence $\frac{1}{\sqrt{\dot{\psi}(x)}} = L(x)$, almost everywhere on the entire interval $[0, 1]$. Thus $\dot{\psi}(x) = \frac{d}{(1+cx)^2}$ for some constants c, d . Integrating and using that $\psi(0) = 0$ and $\psi(1) = 1$ (see Lemma 3.4), we are done. \square

The next proposition completes the identification of ψ_α by showing that the constant c_α is given by (3).

Proposition 3.7. *The constant c_α is given implicitly by the following equation:*

$$\frac{1 + c_\alpha}{c_\alpha} \left[1 - \frac{\log(1 + c_\alpha)}{c_\alpha} \right] = \frac{1}{2} + \alpha$$

where c_α is the constant in Theorem 3.6 corresponding to α .

Proof. Clearly we have

$$\int_0^1 \psi(x) dx = \int_0^1 \frac{(1+c)x}{1+cx} = \frac{1+c}{c} \left[1 - \frac{\log(1+c)}{c} \right] = \frac{1}{2} + \beta$$

for some $\beta = \beta(\alpha) \geq \alpha$. We shall show that $\beta = \alpha$ which will establish the proposition. Suppose not. As we have pointed out, the left-hand side of this equation is increasing in c . Let c_* be the unique solution to the equation

$$\int_0^1 \psi(x) dx = \int_0^1 \frac{(1+c)x}{1+cx} = \frac{1+c}{c} \left[1 - \frac{\log(1+c)}{c} \right] = \frac{1}{2} + \alpha.$$

It is easy to check that $\beta > \alpha$ implies $c_* < c_\alpha$. Define ψ_* by $\psi_*(x) = \frac{(1+c_*)x}{1+c_*x}$. Clearly $\psi_* \in \mathcal{B}_\alpha$. A straightforward computation shows that

$$J(\psi_*) = \int_0^1 \sqrt{\dot{\psi}_*(x)} dx = \sqrt{1+c_*} \int_0^1 \frac{dx}{1+c_*x} = \sqrt{1+c_*} \frac{\log(1+c_*)}{c_*}. \quad (14)$$

The right-hand side here is readily seen to be decreasing in c_* and hence $J(\psi_*) > J(\psi)$. This contradicts the optimality of ψ , and completes the proof. \square

Observe that, $J_\alpha = \sqrt{1+c_\alpha} \frac{\log(1+c_\alpha)}{c_\alpha}$. This expression is the same as the constant w_α in Theorem 1.9 which is consistent with our heuristic explanation at the beginning of this section. We prove this theorem next.

4. LAW OF LARGE NUMBERS

With the preparation from the previous section we now turn to the proof of Theorem 1.9. Fix $\alpha \in (0, \frac{1}{2})$ as before. The idea of the proof is as follows. We first show that if a path follows approximately the blow-up $\psi_{\alpha,n}$ of the deterministic curve ψ (and also satisfies the area constraint) then it is overwhelmingly likely that the path has length at least $2(w_\alpha - \varepsilon)n$ for ε arbitrarily small. We further show that any path satisfying the area constraint is extremely likely to have length less than $2(w_\alpha + \varepsilon)n$. We begin with the following trivial but useful lemma that is an immediate consequence of continuity and monotonicity of w_α .

Lemma 4.1. *Given any $\alpha \in (0, \frac{1}{2})$, for all small enough $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|w_\alpha - w_{\alpha_1}| < \varepsilon$ if $|\alpha - \alpha_1| \leq \delta_1$ and $|w_\alpha - w_{\alpha_1}| > \varepsilon$ if $|\alpha - \alpha_1| \geq \delta_2$.*

Next we need a couple of preparatory lemmas about appropriate discretizations. Let $I_\delta = \{0, \delta, 2\delta, \dots, 1\}$ be the discretization of the unit interval; (we will take δ to be the reciprocal of a positive integer to avoid rounding errors). Consider any non-decreasing function $L : [0, 1] \rightarrow [0, 1]$ that corresponds to an absolutely continuous measure on $[0, 1]$: by identifying L with its graph, we shall interpret L as an increasing path on the unit square directed from $(0, 0)$ to $(1, 1)$. Also let L_δ be the piecewise affine function that agrees on I_δ with L .

Lemma 4.2. *We have $\left| \int_0^1 [L(x) - L_\delta(x)] dx \right| \leq \delta$.*

Proof. First notice that by integration by parts or by Fubini's theorem, for any absolutely continuous function f on $[0, 1]$, $\int_0^1 f(x) dx = \int_0^1 \dot{f}(x)(1-x) dx$. The lemma now follows from the observation that $\left| \int_0^1 x \dot{L}(x) dx - \int_0^1 x \dot{L}_\delta(x) dx \right| \leq \delta$. Let x_i be the midpoint of the interval $I_\delta^{(i)} = [i\delta, (i+1)\delta]$.

Notice that by definition \dot{L}_δ is constant on $I_\delta^{(i)}$ and is equal to $\delta^{-1} \int_{I_\delta^{(i)}} \dot{L}(x) dx$. Thus

$$\sum_{i=0}^{\delta^{-1}-1} \int_{I_\delta^{(i)}} x_i \dot{L}(x) dx = \sum_{i=0}^{\delta^{-1}-1} x_i \int_{I_\delta^{(i)}} \dot{L}(x) dx = \sum_{i=0}^{\delta^{-1}-1} x_i \int_{I_\delta^{(i)}} \dot{L}_\delta(x) dx = \sum_{i=0}^{\delta^{-1}-1} \int_{I_\delta^{(i)}} x \dot{L}_\delta(x) dx = \int_0^1 x \dot{L}_\delta(x) dx$$

The proof now follows by observing $\left| \int_0^1 x \dot{L}(x) dx - \int_0^1 x \dot{L}_\delta(x) dx \right| \leq \sum_{i=0}^{\delta^{-1}-1} \int_{I_\delta^{(i)}} |x - x_i| \dot{L}(x) dx \leq \delta$. \square

We now discretize the vertical direction as well. Let us choose $\eta \ll \delta$ (to be specified exactly later), and let $B_{\delta,\eta} = I_\delta \times I_\eta$. Given L as before let us now set $L_{\delta,\eta}$ to be the piecewise linear curve determined by the points $(i\delta, \eta \lfloor \frac{L(i\delta)}{\eta} \rfloor)$. Clearly for all x , $|\dot{L}_\delta - \dot{L}_{\delta,\eta}| \leq \frac{2\eta}{\delta}$, and hence for $\eta \leq \delta^2$,

$$\left| \int_0^1 [L(x) - L_{\delta,\eta}(x)] dx \right| \leq 4\delta. \quad (15)$$

Before proceeding further we make a few comments about notation. Recall that till now, our underlying noise space has been a point process Π on \mathbb{R}^2 and we have been concerned about geodesics in $[0, n]^2$. However, in the last section (while solving variational problems), we switched to a normalized picture where every path lives in $[0, 1]^2$. We will continue with this convention throughout this section. Equivalently, the noise space can be thought of as a Poisson point process of intensity n^2 in $[0, 1]^2$. In an abuse of notation, we will define $L(u, v)$ for two points $u, v \in [0, 1]^2$ to be the length of the geodesic $\gamma(u, v)$ between u and v , in the same noise space. In light of properties of Poisson process under scaling, all the estimates regarding $\gamma(u, v)$ stated in the previous sections continue to hold after the appropriate variable change and we do not elaborate this further. Next, we state a uniform version of Theorem 2.3 in the large deviation regime.

Lemma 4.3. *Fix δ, η as above. There exists $c = c(\delta, \eta) > 0$, such that, simultaneously for all $x \in I_\delta$, and $y_1, y_2 \in I_\eta$ such that $y_2 > y_1$, and for all large n ,*

(1) *With probability at least $1 - e^{-cn}$,*

$$L((x, y_1), ((x + \delta), y_2)) - 2n\sqrt{\delta(y_2 - y_1)} \leq \eta\delta n.$$

(2) *With probability at least $1 - e^{-cn^2}$,*

$$L((x, y_1), ((x + \delta), y_2)) - 2n\sqrt{\delta(y_2 - y_1)} \geq -\eta\delta n.$$

Proof. The proof follows from the large deviation probabilities for the length of unconstrained geodesics in [29]. In particular the theorem states that for any x, y_1, y_2 as in the statement of the theorem,

$$\begin{aligned} \mathbb{P}(L((x, y_1), ((x + \delta), y_2)) - 2n\sqrt{\delta(y_2 - y_1)} \geq \eta\delta n) &\leq e^{-cn}, \\ \mathbb{P}(L((x, y_1), ((x + \delta), y_2)) - 2n\sqrt{\delta(y_2 - y_1)} - \eta\delta n) &\leq e^{-cn^2}, \end{aligned}$$

for some constant $c = c(\delta, \eta)$. A union bound over points x, y_1, y_2 , the total number of which is $\frac{1}{\delta\eta^2}$, completes the proof. \square

We are now ready to prove Theorem 1.9. We start by showing the upper bound. Recall that $L_\alpha(n)$ denotes the length of the constrained geodesic.

Proposition 4.4. *Fix $\varepsilon > 0$. There exists $c = c(\alpha, \varepsilon) > 0$ such that, with probability at least $1 - e^{-cn}$, we have*

$$L_\alpha(n) \leq 2(w_\alpha + \varepsilon)n.$$

Proof. Recall that for any increasing path γ , we denote its length or the number of points of the Poisson process it passes through by $|\gamma|$. Let us now notice that

$$|\gamma| \leq \sum_{i \in I_\delta} L\left((i\delta, \gamma_{\delta,\eta}(i\delta)), ((i+1)\delta, \gamma_{\delta,\eta}((i+1)\delta) + \eta)\right).$$

Thus by the previous lemma, with probability at least $1 - e^{-cn}$, we have for all increasing paths γ (from $(0, 0)$ to $(1, 1)$ in the rescaled space)

$$\begin{aligned} |\gamma| &\leq \sum_{i \in I_\delta} 2\delta \sqrt{\frac{\gamma_{\delta,\eta}((i+1)\delta) + \eta - \gamma_{\delta,\eta}(i\delta)}{\delta}} n + 2\eta n, \\ &= \sum_{i \in I_\delta} 2n\delta \sqrt{\dot{\gamma}_{\delta,\eta} + \frac{\eta}{\delta}} + 2\eta n. \end{aligned}$$

An argument involving truncating at $\dot{\gamma}_{\delta,\eta} < \sqrt{\frac{\eta}{\delta}}$ shows that, with probability at least $1 - e^{-cn}$, for all increasing paths γ ,

$$\sum_{i \in I_\delta} n\delta \sqrt{\dot{\gamma}_{\delta,\eta} + \frac{\eta}{\delta}} \leq n \sum_i \delta \sqrt{\dot{\gamma}_{\delta,\eta}} + \left(\frac{\eta}{\delta}\right)^{1/4} O(n).$$

Thus, for $\eta < \delta^5$, with probability at least $1 - e^{-cn}$, for all γ ,

$$|\gamma| \leq n \int_0^1 2\sqrt{\dot{\gamma}_{\delta,\eta}(x)} dx + O(\delta n). \quad (16)$$

Now let γ be an increasing path that traps area at least $1/2 + \alpha$ (which corresponds to area $(1/2 + \alpha)n^2$ in the unscaled model). By (15), it follows that $\int_0^1 \gamma_{\delta,\eta}(x) dx \geq \frac{1}{2} + \alpha - O(\delta)$ and hence, by continuity of w_α , we have $\int_0^1 \sqrt{\dot{\gamma}_{\delta,\eta}(x)} dx \leq (w_\alpha + \varepsilon/2)$ by choosing δ sufficiently small. By choosing δ suitably small, this implies that, with probability at least $1 - e^{-cn}$, the bound $|\gamma| \leq 2(w_\alpha + \varepsilon)n$ for such increasing paths γ . This completes the proof of the proposition. \square

We now show the corresponding lower bound.

Proposition 4.5. *Fix $\varepsilon > 0$. There exists $c = c(\alpha, \varepsilon) > 0$ such that, with probability at least $1 - e^{-cn^2}$, we have*

$$L_\alpha(n) \geq 2(w_\alpha - \varepsilon)n.$$

Proof. The derivation of this lower bound is straightforward: we will take a discretization of ψ_α and then use the continuity of $\dot{\psi}$. Fix $\varepsilon > 0$. Choose $\theta > 0$ and consider $\psi_* = \psi_{\alpha+\theta}$. Then, by definition, $\int_0^1 \psi_*(x) dx \geq \frac{1}{2} + \alpha + \theta$. Choose θ small enough that $w_{\alpha+\theta} = \int_0^1 \sqrt{\dot{\psi}_*(x)} dx \geq w_\alpha - \varepsilon/2$.

By using the continuity of ψ and $\dot{\psi}$, we see that, for given ε , there exist sufficiently small choices of δ and η such that the piecewise affine function $\psi_{*,\delta,\eta}$ interpolating between the points $u_i := (i\delta, \eta \lfloor \frac{\psi_*(i\delta)}{\eta} \rfloor)$ satisfies

$$\int_0^1 \psi_{*,\delta,\eta}(x) dx \geq \frac{1}{2} + \alpha + \frac{\theta}{2}, \quad \int_0^1 \sqrt{\dot{\psi}_{*,\delta,\eta}(x)} dx \geq w_\alpha - \frac{3\varepsilon}{4}.$$

Now let $\gamma_{(i,i+1)}$ be the geodesic between the points u_i and u_{i+1} . Let γ be the path obtained by concatenating the paths $\gamma_{(i,i+1)}$ for $i = 0, \dots, \delta^{-1} - 1$. Using Lemma 4.3, it follows that, with probability at least $1 - e^{-cn^2}$,

$$|\gamma| \geq \sum_{i \in I_\delta} 2n\delta \sqrt{\dot{\psi}_{*,\delta,\eta}} - 2n\eta \geq 2(w_\alpha - \varepsilon)n,$$

where we take η and δ sufficiently small. Now, by Lemma 4.2, we find that

$$\left| \int_0^1 \gamma(x) dx - \int_0^1 \psi_{*,\delta,\eta}(x) dx \right| \leq 2\delta,$$

since γ and $\psi_{*,\delta,\eta}$ agree on I_δ . Hence, for δ small enough, $\int_0^1 \gamma(x)dx \geq \frac{1}{2} + \alpha$. Thus, $L_{\alpha,n}$ is at least as large as $|\gamma|$. Hence, we are done. \square

Proof of Theorem 1.9. Combining Proposition 4.4 and Proposition 4.5, we complete the proof of Theorem 1.9 by noting that an upper tail bound of the form $\mathbb{P}(L_\alpha(n) \geq k) \leq e^{-k}$ for all $k \geq 5n^2$ is easily obtained by bounding the upper tail of the Poisson-distributed total number of points. \square

4.1. Law of large numbers for the geodesic. We now prove Theorem 1.10. The proof is by contradiction: if $\text{dist}(\Gamma_\alpha, \psi_\alpha) \geq \Delta$, and the noise space is such that the events listed in Lemma 4.3 occur, then we would be able to construct an absolutely continuous function h that traps area at least $1/2 + \alpha$ and for which $\int_0^1 \sqrt{\dot{h}}(x)dx > \int_0^1 \sqrt{\dot{\psi}_\alpha}(x)dx$. This would contradict the extremality of ψ_α . To show the above inequality, we will employ a concavity argument. For notational brevity, let $\gamma = \Gamma_\alpha$ and $\psi = \psi_\alpha$. For $\delta, \eta > 0$, recall the definition of $\gamma_{\delta,\eta}$ from the previous section. We will consider the function $h = 2^{-1}(\gamma_{\delta,\eta} + \psi)$. Fixing parameters ε, δ and $\eta \leq \delta^5$, we will restrict attention to the event \mathcal{A} , on which:

$$(1) |\gamma| \leq 2n \int_0^1 \sqrt{\dot{\gamma}_{\delta,\eta}}(x)dx + O(\delta n),$$

$$(2) 2(w_\alpha - \varepsilon)n \leq |\gamma| \leq 2(w_\alpha + \varepsilon)n.$$

By Propositions 4.4 and 4.5, and (16),

$$\mathbb{P}(\mathcal{A}) \geq 1 - e^{-cn}. \quad (17)$$

Observe that Proposition 3.6 implies that $\dot{\psi}$ is bounded away from zero and infinity on $[0, 1]$. Thus, $\dot{\psi}$ lies in $[c_1, C_2]$ where $0 < c_1 < C_2 < \infty$. Suppose that $\sup_x |\gamma(x) - \psi(x)| \geq \Delta$. We now claim that if δ, η are small enough compared to Δ , then there exists $y \in [0, 1]$ such that

$$|\gamma_{\delta,\eta}(y) - \psi(y)| \geq \frac{\Delta}{2}. \quad (18)$$

To see this, we choose $y \in [0, 1]$ such that $|\gamma(y) - \psi(y)| \geq \Delta$. Let $y_1 \in I_\delta$ be such that $y_1 \leq y < y_1 + \delta$. Now we consider two cases:

(1) If $\gamma(y) \geq \psi(y) + \Delta$, then

$$\begin{aligned} \gamma_{\delta,\eta}(y_1 + \delta) &\geq \gamma(y) - \eta \geq \psi(y) + \Delta - \eta \geq \psi(y_1 + \delta) + \Delta - C_2\delta, \text{ and hence,} \\ \gamma_{\delta,\eta}(y_1 + \delta) &\geq \psi(y_1 + \delta) + \Delta - C_2\delta. \end{aligned}$$

These implications, excepting the last, follow by definition. The last implication is due to ψ having Lipschitz constant at most C_2 .

(2) If $\psi(y) \geq \gamma(y) + \Delta$, then, similarly,

$$\begin{aligned} \psi(y_1) &\geq \psi(y) - C_2\delta \geq \gamma(y) + \Delta - C_2\delta \geq \gamma_{\delta,\eta}(y_1) + \Delta - C_2\delta, \text{ and hence,} \\ \psi(y_1) &\geq \gamma_{\delta,\eta}(y_1) + \Delta - C_2\delta. \end{aligned}$$

Thus, by (18),

$$\int_0^1 |\dot{\gamma}_{\delta,\eta}(x) - \dot{\psi}(x)|dx \geq \left| \int_0^y [\dot{\gamma}_{\delta,\eta}(x) - \dot{\psi}(x)]dx \right| \geq \frac{\Delta}{2}.$$

Also, note that, since both ψ and $\gamma_{\delta,\eta}$ start at $(0, 0)$ and end at $(1, 1)$, $\int_0^1 \dot{\gamma}_{\delta,\eta}(x)dx = \int_0^1 \dot{\psi}(x)dx$. Hence,

$$\int_0^1 [\dot{\psi}(x) - \dot{\gamma}_{\delta,\eta}(x)]\mathbf{1}_S dx \geq \Delta/4, \quad (19)$$

where S denotes $\{x \in [0, 1] : \dot{\psi}(x) \geq \dot{\gamma}_{\delta,\eta}(x)\}$. Moreover, on S ,

$$\dot{\gamma}_{\delta,\eta} \leq \dot{\psi} \leq C_2 \quad (20)$$

for some $C_2 > 1$. Observing that $\dot{h} = \frac{\dot{\gamma}_{\delta,\eta} + \dot{\psi}}{2}$, we see that the concavity of $x \rightarrow \sqrt{x}$ implies that $\sqrt{\dot{h}} - \frac{\sqrt{\dot{\gamma}_{\delta,\eta}} + \sqrt{\dot{\psi}}}{2}$ is non-negative. Furthermore, simple algebra shows that, for all $x \in S$,

$$\begin{aligned} \sqrt{\dot{h}(x)} - \frac{[\sqrt{\dot{\psi}(x)} + \sqrt{\dot{\gamma}_{\delta,\eta}(x)}]}{2} &= \frac{(\sqrt{\dot{\psi}(x)} - \sqrt{\dot{\gamma}_{\delta,\eta}(x)})^2}{4[\sqrt{\dot{h}(x)} + \frac{[\sqrt{\dot{\psi}(x)} + \sqrt{\dot{\gamma}_{\delta,\eta}(x)}]}{2}]} \stackrel{(20)}{\geq} \frac{(\sqrt{\dot{\psi}(x)} - \sqrt{\dot{\gamma}_{\delta,\eta}(x)})^2}{8C_2}, \\ &\geq \frac{(\dot{\psi}(x) - \dot{\gamma}_{\delta,\eta}(x))^2}{8C_2[\sqrt{\dot{\psi}(x)} + \sqrt{\dot{\gamma}_{\delta,\eta}(x)}]^2} \stackrel{(20)}{\geq} \frac{(\dot{\psi}(x) - \dot{\gamma}_{\delta,\eta}(x))^2}{100C_2^2}. \end{aligned} \quad (21)$$

The proof is now completed by observing that a length gain has been realized:

$$\begin{aligned} \int_0^1 \left[\sqrt{\dot{h}(x)} - \frac{[\sqrt{\dot{\psi}(x)} + \sqrt{\dot{\gamma}_{\delta,\eta}(x)}]}{2} \right] dx &\geq \int_0^1 \left[\sqrt{\dot{h}(x)} - \frac{[\sqrt{\dot{\psi}(x)} + \sqrt{\dot{\gamma}_{\delta,\eta}(x)}]}{2} \right] \mathbf{1}_S dx \\ &\stackrel{(21)}{\geq} \int_0^1 \frac{(\dot{\psi}(x) - \dot{\gamma}_{\delta,\eta}(x))^2}{100C_2^2} \mathbf{1}_S dx \\ &\geq \frac{1}{100C_2^2} \left(\int |(\dot{\psi}(x) - \dot{\gamma}_{\delta,\eta}(x))| \mathbf{1}_S dx \right)^2 \\ &\stackrel{(19)}{\geq} \frac{1}{2000C_2^2} \Delta^2. \end{aligned}$$

On the event \mathcal{A} , we see, by means of this inequality and the definition of \mathcal{A} , that if $\sup_x |\gamma(x) - \psi(x)| \geq \Delta$ holds, then

$$\int_0^1 \sqrt{\dot{h}(x)} dx \geq [(w_\alpha - \varepsilon) + \frac{1}{C^2} \Delta^2]$$

for some constant C that depends only on α . Also, both $\int_0^1 \psi(x) dx$ and $\int_0^1 \gamma(x) dx$ are at least $\frac{1}{2} + \alpha$ by definition, and hence by (15), we have

$$\int_0^1 h(x) dx = \frac{\int_0^1 \psi(x) dx + \int_0^1 \gamma_{\delta,\eta}(x) dx}{2} \geq \frac{1}{2} + \alpha - \varepsilon/2,$$

where ε can be made arbitrarily small by choosing δ and hence η small enough. This inference contradicts the continuity of w_α in α . Hence we are done. \square

We now use similar arguments as those employed to prove the law of large numbers in order to establish a variant of Theorem 2.2 that is also uniform in α . This particular variant will be crucial in the proof of Theorem 1.7.

Theorem 4.6. *Fix $0 < \alpha_1 < \alpha_2 < \frac{1}{2}$. For any small enough $\delta > 0$, there exist $\omega > 0$ and $c > 0$ such that, with probability at least $1 - e^{-cn}$, all δ -interior facets of $\Gamma_{\alpha,n}$ for $\alpha \in [\alpha_1, \alpha_2]$, make an angle with the x -axis that lies in the interval $(\omega, \pi/2 - \omega)$.*

Proof. Recall that the proof of Theorem 2.2 used Theorem 1.10 and the strict convexity of the function ψ_α . Since ψ_α is uniformly convex for all $\alpha \in [\alpha_1, \alpha_2]$, this proof will be complete, using the same arguments as in the proof of Theorem 2.2, once we prove the following uniform version of Theorem 1.10: for any $\Delta > 0$, with probability at least $1 - e^{-cn}$,

$$\sup_{\alpha \in [\alpha_1, \alpha_2]} \sup_x |\Gamma_\alpha(x) - \psi_\alpha(x)| \leq \Delta,$$

where c depends on Δ and the interval $[\alpha_1, \alpha_2]$. We fix an ε to be specified later and discretize the interval $[\alpha_1, \alpha_2]$ to obtain the set $\mathcal{B} = \{\alpha_1, \alpha_1 + \varepsilon, \alpha_1 + 2\varepsilon, \dots, \alpha_2\}$. Fixing parameters ε, δ and $\eta \leq \delta^5$, again as before we restrict attention to the event \mathcal{A} on which, for all $\alpha \in \mathcal{B}$,

- (1) $|\Gamma_\alpha| \leq 2n \int_0^1 \sqrt{\dot{\Gamma}_{\alpha, \delta, \eta}(x)} dx + O(\delta n)$;
- (2) and $2(w_\alpha - \varepsilon)n \leq |\Gamma_\alpha| \leq 2(w_\alpha + \varepsilon)n$.

Similar to (17), by Propositions 4.4 and 4.5, and (16), followed by a union bound, $P(\mathcal{A}) \geq 1 - e^{-cn}$. Now if δ and η are chosen to be sufficiently small depending on Δ , Theorem 1.10, and a simple union bound, we obtain

$$\sup_{\alpha \in \mathcal{B}} \sup_x |\Gamma_\alpha(x) - \psi_\alpha(x)| \leq \Delta.$$

The proof will proceed along the same lines as the proof of Theorem 1.10 did. With the aim of arriving at a contradiction, let $\alpha \in [\alpha_1 + i\varepsilon, \alpha_1 + (i+1)\varepsilon]$ be such that

$$\sup_x |\Gamma_\alpha(x) - \psi_\alpha(x)| > \Delta. \quad (22)$$

Clearly, $|\Gamma_{\alpha_1 + (i+1)\varepsilon}| \leq |\Gamma_\alpha| \leq |\Gamma_{\alpha_1 + i\varepsilon}|$ holds by definition. Since ψ_β is a continuous function of β in the supremum norm, we see that, for small enough ε ,

$$\sup_x |\Gamma_\alpha(x) - \psi_{\alpha_1 + i\varepsilon}(x)| \geq \Delta/2.$$

Now, as we argued in the proof of Theorem 1.10, this implies that

$$\sup_x |\Gamma_{\alpha, \delta, \eta}(x) - \psi_{\alpha_1 + i\varepsilon}(x)| \geq \Delta/2. \quad (23)$$

Note then that

$$|\Gamma_\alpha| \geq |\Gamma_{\alpha_1 + (i+1)\varepsilon}| \geq 2n(w_{\alpha_1 + (i+1)\varepsilon} - \varepsilon) \geq 2n(w_\alpha - \varepsilon_1),$$

where ε_1 can be made small enough by choosing ε small enough. The first inequality follows by definition, and the second by the occurrence of \mathcal{A} .

Thus, using (16), we find that

$$\int_0^1 \sqrt{\dot{\Gamma}_{\alpha, \delta, \eta}(x)} dx \geq w_\alpha - \varepsilon_1.$$

This along with (23) allows us to apply the concavity argument that appears in the proof of Theorem 1.10: by choosing $\varepsilon, \delta, \eta$ much smaller than Δ , we thus contradict the continuity of w_β at $\beta = \alpha$. \square

5. LOWER BOUND FOR SCALING EXPONENTS

In this section we provide proofs of the lower bounds on $\text{MFL}(\Gamma_n)$ and $\text{MLR}(\Gamma_n)$, i.e., we prove Theorem 1.3 and Theorem 1.4. Most of the work goes into proving the MFL lower bound Theorem 1.3, since the lower bound for local roughness is a reasonably easy corollary of Theorem 2.4. Let $\alpha \in (0, \frac{1}{2})$ and $\varepsilon > 0$ be fixed for the rest of the section. Let \mathcal{A}_ε denote the event that $\text{MFL}(\Gamma_n) \leq n^{3/4-\varepsilon}$. We shall show that the event \mathcal{A}_ε is extremely unlikely. We start with an overview of the proof. We shall need a geometric definition.

Definition 5.1. *Let $C > 1, \kappa > 1$ be given constants. A sequence of points $u_0 \preceq u_1 \preceq \dots \preceq u_k$ is called a (C, κ) -regular sequence if the following conditions hold.*

- (i) *The union of line segments joining u_i to u_{i+1} for $i = 0, 1, \dots, k-1$, is the graph of a concave function.*
- (ii) *The gradient of all the line segments joining u_i to u_{i+1} is $\in (\frac{1}{\kappa}, \kappa)$.*
- (iii) *The distance $|u_k - u_0|$ between the first and last point in the sequence lies in $(\frac{1}{C}n^{3/4-\varepsilon/2}, Cn^{3/4-\varepsilon/2})$.*
- (iv) *The distance between the consecutive points in the sequence is small: $|u_i - u_{i+1}| \leq n^{3/4-\varepsilon}$.*

- (v) Let θ_1 and θ_2 be the angles that the line segments (u_0, u_1) and (u_{k-1}, u_k) make with the positive x -axis (clearly $\theta_1 \geq \theta_2$ by hypothesis). Then $\theta_1 - \theta_2 \leq 100C^2n^{-1/4-\varepsilon/2}$.

See Figure 3 for an illustration of this definition.

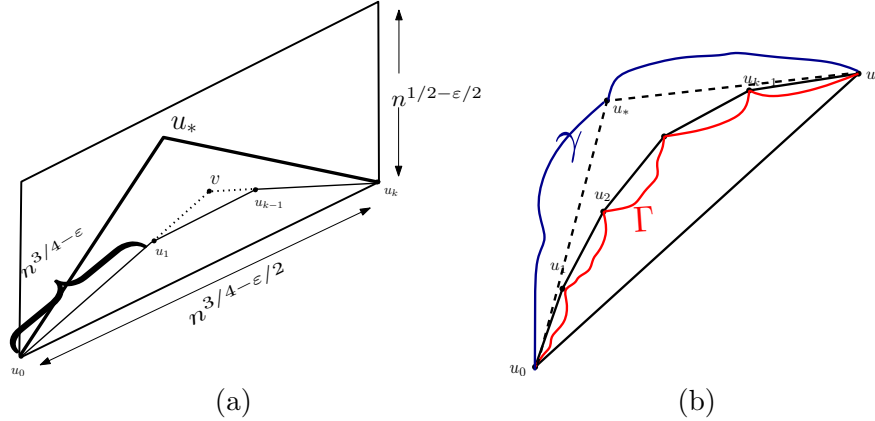


FIGURE 3. (a) A sequence of regular corners u_0, u_1, \dots, u_k . The triangle T is an isosceles triangle formed by vertices u_0, u_k and u_* such that the adjacent sides lie above the piecewise linear path passing through the points u_0, u_1, \dots, u_k . By definition of regularity, the triangle T is contained in the parallelogram R formed by the vertices u_0, v_0, v_k and u_k which has vertical height $n^{1/2-\varepsilon/2}$. Observe that the height of this parallelogram is much smaller than the transversal fluctuation of paths between u_0 and u_k . (b) On \mathcal{A}_ε , there is a regular sequence of corners u_0, u_1, \dots, u_k and Γ is the best constrained path. We show that there is an alternative path γ between u_0 and u_k above the triangle T that is extremely likely to be strictly larger in length than the restriction of Γ between u_0 and u_k . Then the path obtained by replacing Γ with γ between u_0 and u_k has a larger length and traps a larger area, thus leading to a contradiction.

Before proceeding, we try to motivate item (v). Consider the least concave majorant of an increasing path from $(0, 0)$ to (n, n) as described in Definition 1.2. The total change of angle made by the line segments constituting the concave majorant with the x -axis is roughly the ratio of $\pi/2$ and n . So over a distance of around $n^{3/4-\varepsilon/2}$, the change of angle should be roughly $n^{1/4-\varepsilon/2}$. Item (v) asserts that on a regular sequence the change of angle is not much more than this average.

The following basic geometric consequence will be useful.

Lemma 5.2. *Let $S = \{u_0, u_1, \dots, u_k\}$ be a (C, κ) -regular sequence for some $C, \kappa > 1$. Let L_S denote the union of line segments joining the consecutive points of S . Let R denote the parallelogram with two vertical sides of length $n^{1/2-\varepsilon/2}$ whose bottom side is the line segment joining u_0 and u_k ; see Figure 3(a). Then there exists a point u_* in R such that the triangle T formed by the points u_0, u_k and u_* is an isosceles triangle (with the sides adjacent to u_* being of equal length) contained in R such that the two sides of T adjacent to u_* lie above the piecewise affine curve obtained by joining the consecutive points of S .*

Proof. Consider the angles ω_1 and ω_2 made by the line segments (u_0, u_1) and (u_{k-1}, u_k) respectively with the line segment (u_0, u_k) . Also, let v be the point of intersection obtained by extending the line segments (u_0, u_1) and (u_{k-1}, u_k) . By convexity, the piecewise affine curve obtained by joining the points of S lies inside the triangle (u_0, u_k, v) . Moreover, it follows from elementary geometric arguments that $\theta_1 - \theta_2 = \omega_1 + \omega_2$. Now let us consider the isosceles triangle (u_0, u_k, u_*) where (u_0, u_k)

forms the base and $\theta_1 - \theta_2$ is the value of the two equal angles. Since $\theta_1 - \theta_2$ is at least as big as ω_1 and ω_2 , it follows that the triangle (u_0, u_k, v) is contained inside the triangle (u_0, u_k, u_*) . Moreover, the latter is clearly contained in a parallelogram of height $O(n^{3/4-\varepsilon/2}n^{-1/4-\varepsilon/2})$ where the constant in the $O(\cdot)$ notation depends on κ . Since for all large enough n , $n^{3/4-\varepsilon/2}n^{-1/4-\varepsilon/2} \ll n^{1/2-\varepsilon/2}$, we are done. \square

Let us now record the main steps of the proof of Theorem 1.3. Let Γ denote the constrained geodesic. Recall that $\Gamma_{\alpha,n}^*$ denotes least concave majorant of Γ and is a union of facets as described in Definition 1.2.

Step 1: We shall fix a (C, κ) -regular sequence $S = \{u_0, u_1, \dots, u_k\}$, and consider the best path γ_S from u_0 to u_k that passes through all the points in the sequence. We shall show that with very high probability the length of this path is much smaller than the length of the best path between u_0 and u_k . The reason that this event is likely is the following: because the lengths of the segments $\{u_i, u_{i+1}\}$ are much smaller than that of the segment $\{u_0, u_k\}$, it will turn out that the path γ will have a much smaller transversal fluctuation compared to the typical geodesic between u_0 and u_k . Thus, by Theorem 2.10, it is extremely likely that this path has a much smaller length.

Step 2: Next we will show that there exists a path between u_0 and u_k that lies above the line segments joining the consecutive points of S and whose length is comparable to that of the geodesic between u_0 and u_k . This will follow from the definition of regular sequence together with an application of Lemma 5.2 and Theorem 2.12.

Step 3: Finally, we will show that, on the event \mathcal{A}_ε , it is overwhelmingly likely that there exists a (C, κ) -regular sequence made out of consecutive corners of $\Gamma_{\alpha,n}^*$. Once we establish this, we will arrive at a contradiction: using the first two steps, we will construct a path that traps more area than does Γ and that also has a greater length.

The next proposition treats the first step.

Proposition 5.3. *Let $S = \{u_0, u_1, \dots, u_k\}$ be a (C, κ) -regular sequence. For $i = 0, 1, \dots, k-1$, let γ_i be the geodesic between u_i and u_{i+1} . Let γ_S denote the concatenation of the paths γ_i . Then there exists $c > 0$ such that, with probability at least $1 - e^{-n^c}$, we have*

$$|\gamma_S| \leq \mathbb{E}L(u_0, u_k) - n^{1/4-\varepsilon/12}.$$

Proof. See Figure 4(a). Let R' be the parallelogram with sides parallel to the sides of R such that u_0 and u_k are the midpoints of the vertical sides of R' ; and the height of the vertical sides is $4n^{1/2-\varepsilon/2}$. For each i , let R_i denote the parallelogram with the following properties:

- (1) R_i has one pair of vertical sides and the other pair of sides is parallel to the line segment joining u_i and u_{i+1} .
- (2) The vertical sides have midpoints u_i and u_{i+1} .
- (3) The height of the vertical sides is $n^{1/2-3\varepsilon/5}$.

It is straightforward to check from Definition 5.1 that all the rectangles R_i are contained in the rectangle R' . Now it follows from Theorem 2.5 that, with probability at least $1 - e^{-n^c}$, each geodesic γ_i is contained in the parallelogram R_i . In particular, on this event,

$$|\gamma_S| \leq L(u_0, u_k; R').$$

The result now follows using Theorem 2.10 and the assumed lower bound on $|u_0 - u_k|$. \square

Next we move on to Step 2, in which we show the existence of a “good” path between u_0 and u_k .

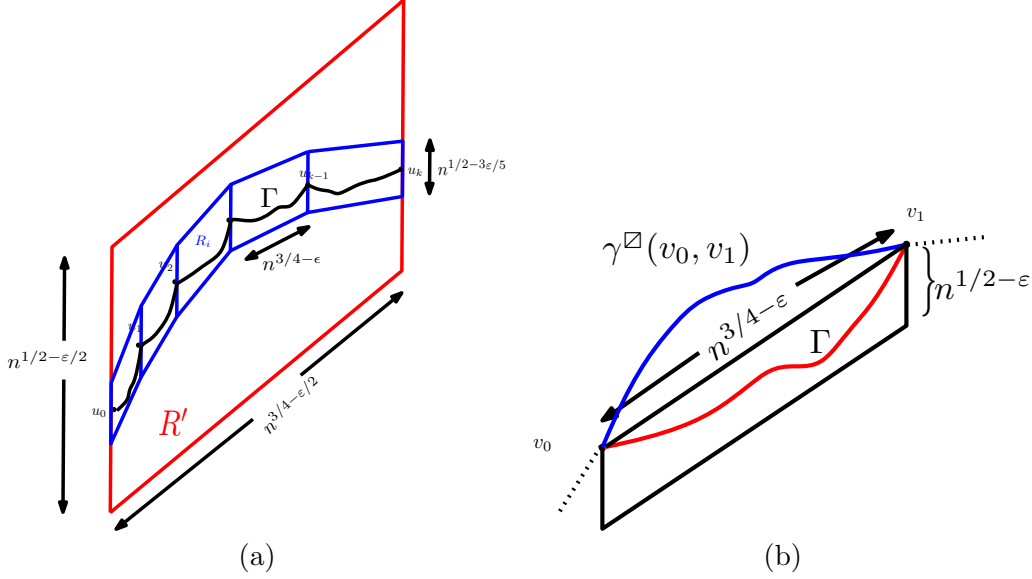


FIGURE 4. (a) Illustrates the situation when all the facets are small. Using a priori bounds on the transversal fluctuations of the unconstrained geodesic, it follows that the segment of $\Gamma_{\alpha,n}$ between u_0 and u_k lies in a thin parallelogram with width much smaller than the characteristic fluctuation scale. This then implies that the length of this segment is atypically low (see Theorem 2.10). This is used in the proof of Proposition 5.3. (b) Illustrates how a not too small facet (of length at least $n^{3/4-\epsilon}$) leading to an uncharacteristically small value of local roughness (of less than $n^{1/2-\epsilon}$) is unlikely. In this situation one can replace the segment of $\Gamma_{\alpha,n}$ between endpoints v_0 and v_1 of the facet by a path that lies strictly above the straight line joining the two points and fluctuates on a characteristic scale, and hence captures more area along with more length, which contradicts extremality of $\Gamma_{\alpha,n}$. This is used in the proof of Theorem 1.4.

Proposition 5.4. *Let $S = \{u_0, u_1, \dots, u_k\}$ be a (C, κ) -regular sequence. Let \mathbb{L}_S denote the union of line segments joining u_i to u_{i+1} . Then there exists $c > 0$ such that, with probability at least $1 - e^{-n^c}$, there exists a path γ_* from u_0 to u_k lying above \mathbb{L}_S such that*

$$|\gamma_*| > \mathbb{E}L(u_0, u_k) - n^{1/4-\epsilon/12}.$$

Proof. We will use the notation from Lemma 5.2. Let u_* be the point in R that satisfies the conclusion of that lemma. Let γ_1 denote a path from u_0 to u_* lying above the line segment joining u_0 and u_* that achieves the length $L^\square(u_0, u_*)$. Similarly, let γ_2 denote a path from u_* to u_k lying above the line segment joining u_* and u_k that achieves the length $L^\square(u_*, u_k)$. See Figure 3(b).

Let γ_* denote the concatenation of the paths γ_1 and γ_2 . Clearly, γ_* lies above \mathbb{L}_S . We shall show that, with overwhelming probability, γ_* also satisfies the other condition in the statement of the proposition. It follows by definition and some elementary geometry that both of the equal sides of the isosceles triangle T have gradient in $(\frac{1}{2\kappa}, 2\kappa)$, and also $|u_0 - u_*| = |u_* - u_k| = \Theta(n^{3/4-\epsilon/2})$. It follows using Theorem 2.12 that, with probability at least $1 - e^{-n^c}$, one has

$$L^\square(u_0, u_*) \geq \mathbb{E}L(u_0, u_*) - n^{1/4-\epsilon/8}; \quad L^\square(u_*, u_k) \geq \mathbb{E}L(u_*, u_k) - n^{1/4-\epsilon/8}.$$

Also observe that u_* is equidistant from u_0 and u_k and that this point lies within the parallelogram R whose height is $o(|u_0 - u_k|^{2/3})$. Thus, by a standard calculation,

$$\mathbb{E}L(u_0, u_*) + \mathbb{E}L(u_*, u_k) - \mathbb{E}L(u_0, u_k) = o(n^{1/4-\varepsilon/6}).$$

Combining these inferences, we find that, with probability at least $1 - e^{-n^c}$,

$$|\gamma_*| > \mathbb{E}L(u_0, u_k) - n^{1/4-\varepsilon/8}.$$

This completes the proof of the proposition. \square

In the final step, we establish that, on \mathcal{A}_ε , that is if $\text{MFL}(\Gamma_n)$ is smaller than $n^{3/4-\varepsilon}$, it is extremely likely that there exists a regular sequence whose points are consecutive corners of $\Gamma_{\alpha,n}^*$ (the least concave majorant of $\Gamma_{\alpha,n}$). Let $\text{Reg}(C, \kappa)$ denote the event that there exist consecutive corners u_0, u_1, \dots, u_k of $\Gamma_{\alpha,n}^*$ such that $\{u_0, u_1, \dots, u_k\}$ is a (C, κ) -regular sequence.

Proposition 5.5. *There exist C, κ such that, except for a sub-event of exponentially small probability, the event \mathcal{A}_ε is contained in the event $\text{Reg}(C, \kappa)$, i.e.,*

$$\mathbb{P}(\mathcal{A}_\varepsilon \setminus \text{Reg}(C, \kappa)) \leq e^{-cn},$$

for some universal constant $c > 0$.

On \mathcal{A}_ε , we shall find a regular sequence among interior facets. Before proceeding we state an easy geometric lemma that will be used. The proof is provided in Section 7.

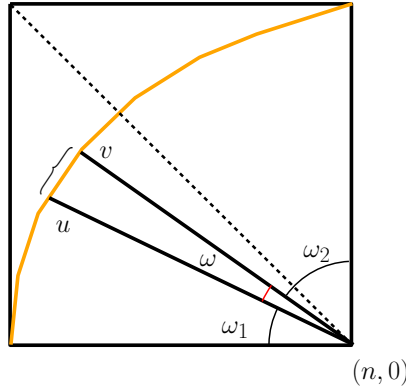


FIGURE 5. Crude estimates on length of facets which subtend an angle θ at the origin.

For any two points $p_1, p_2 \in [0, n]^2$, denote by $\theta(p_1, p_2)$ the acute angle between the lines joining the points p_1 and p_2 to the point $(n, 0)$, i.e., the bottom right corner of the square (see Figure 5).

Lemma 5.6. *There exists θ_0 , such that with probability at least $1 - e^{-cn}$, simultaneously, for all u, v on $\Gamma_{\alpha,n}^*$ with $\theta(u, v) \leq \theta_0$,*

$$|u - v| = \Theta(\theta(u, v)n).$$

Proof of Proposition 5.5. Observe that by definition, on \mathcal{A}_ε , any sequence $u_0 \preceq u_1 \preceq \dots \preceq u_k$ of consecutive corners of $\Gamma_{\alpha,n}^*$ satisfies condition (i) and (iv) of Definition 5.1. Fix $\delta > 0$. By Theorem 4.6, we see that it is with a failure probability of at most e^{-cn} that all the δ -interior facets have gradient s in the interval $(1/\kappa, \kappa)$ for some $\kappa(\delta) > 1$. For the remainder of the proof, we shall assume that this event occurs. Now we partition all the δ -interior vertices of $\Gamma_{\alpha,n}^*$ (it is easy to see they are non-empty on \mathcal{A}_ε) into consecutive segments

$$(v_1, v_2, \dots, v_{i_1}), (v_{i_1+1}, v_{i_1+2}, \dots, v_{i_2}), \dots, \quad (24)$$

(where v_i 's are the consecutive corners on $\Gamma_{\alpha,n}^*$) such that

$$\theta(v_{i_j}, v_{i_{j+1}-1}) < n^{-1/4-\varepsilon/2} \leq \theta(v_{i_{j+1}}, v_{i_{j+1}})$$

for all $j = 0, 1, \dots$, where $i_0 = 1$.

We denote the segment $(v_{i_j}, v_{i_{j+1}}, \dots, v_{i_{j+1}})$ by S_j . Thus if S_1, S_2, \dots, S_m are all such segments, then $m = \Theta(n^{1/4+\varepsilon/2})$. It follows from convexity and the boundedness of the gradients (see Lemma 5.6 and Figure 5) that $|v_{i_j} - v_{i_{j+1}}| = \Theta(n^{3/4-\varepsilon/2})$. Let ω_k denote the angle made by the line segment (v_k, v_{k+1}) with the positive x -axis. Let $\theta_j = \omega_{i_j} - \omega_{i_{j+1}-1}$. Notice that, by convexity, θ_j is nonnegative for all j . Also observe that

$$\sum_j \theta_j \leq \frac{\pi}{2}.$$

This together with Markov's inequality and the fact that the number of S_j 's is $\Theta(n^{1/4+\varepsilon/2})$ implies that, for some constant C , there exists at least one S_j such that $\theta_j \leq Cn^{-1/4-\varepsilon/2}$. Setting $S = (u_0, u_1, \dots, u_k)$ to be equal to such a sequence, we see that S satisfies all conditions in Definition 5.1 for some $C, \kappa > 1$. This completes the proof of the proposition. \square

With all this preparation, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\alpha \in (0, \frac{1}{2})$ and $\varepsilon > 0$ be fixed as before. Let (C, κ) be such that the conclusion of Proposition 5.5 holds. Let \mathcal{S} denote the event that there exists a (C, κ) -regular sequence for which at least one of the conclusions in Proposition 5.3 and Proposition 5.4 does not hold. Using the proofs of the said propositions, along with taking a union bound over all pairs of grid points obtained by coarse graining $[0, n]^2$ (exactly as Corollary 2.6 followed from Theorem 2.3), that $\mathbb{P}(\mathcal{S}) \leq e^{-n^c}$ for some $c > 0$. By Proposition 5.5, it suffices to show that

$$\mathcal{A}_\varepsilon \cap \mathcal{S}^c \cap \text{Reg}(C, \kappa) = \emptyset.$$

By way of contradiction, let us assume that the set is non-empty. Fix a configuration in the set $\mathcal{A}_\varepsilon \cap \mathcal{S}^c \cap \text{Reg}(C, \kappa)$, and consider the least concave majorant $\Gamma_{\alpha,n}^*$ of the constrained geodesic. On $\mathcal{A}_\varepsilon \cap \text{Reg}(C, \kappa)$, there exists a (C, κ) -regular sequence $S = \{u_0, u_1, \dots, u_k\}$ consisting of consecutive corners of $\Gamma_{\alpha,n}^*$. On \mathcal{S}^c , there exists a path γ_* from u_0 to u_k satisfying the conclusion of Proposition 5.4. Let γ_0 denote the restriction of Γ between u_0 and u_k . Clearly,

$$|\gamma_0| \leq \sum_i L(u_i, u_{i+1}).$$

It follows from Propositions 5.3 and 5.4 that, on \mathcal{S}^c , $|\gamma_0| < |\gamma_*|$. Also observe that γ_0 lies below the line segments joining u_i to u_{i+1} whereas γ_* lies above these segments. Hence the increasing path Γ' from $(0, 0)$ to (n, n) that coincides with Γ between $(0, 0)$ and u_0 and between u_k and (n, n) and coincides with γ_* between u_0 and u_k traps at least as much area as does Γ . However, by the above discussion, $|\Gamma'| > |\Gamma|$. This contradicts the extremality of Γ and completes the proof of the theorem. \square

5.1. Proof of lower bound of MLR. In this subsection we prove Theorem 1.4. That is, we show that the maximum local roughness of the facets has scaling exponent at least $\frac{1}{2}$. The basic idea of the proof is to use KPZ scaling to argue that, if there is a facet of length roughly $n^{3/4}$, then along that facet, the local roughness is likely to be at least $(n^{3/4})^{2/3} = n^{1/2}$. We now make this more precise.

Fix $\varepsilon > 0$. We have just proved that, with high probability, there exists a facet of length at least $n^{3/4-\varepsilon}$. In fact, by the proof of Theorem 1.3, more is true. Fix $\kappa \gg 1$. Let $\mathcal{B}_\varepsilon = \mathcal{B}_{n,\varepsilon,\kappa}$ denote the event that $\Gamma_{\alpha,n}^*$ (see Definition 1.2) has a facet of length at least $n^{3/4-\varepsilon}$ with endpoints v_0 and

v_1 such that the straight line joining v_0 and v_1 has gradient $\in (1/\kappa, \kappa)$. The following lemma is a consequence of the proof of Theorem 1.3.

Lemma 5.7. *For all large enough κ (depending on α), $\mathbb{P}(\mathcal{B}_\varepsilon) \geq 1 - e^{-n^c}$ for all n sufficiently large, where c is a constant depending only on ε and α .*

Proof. Note that the proof of Theorem 1.3 in fact showed that there must be a δ -interior facet of length at least $n^{3/4-\varepsilon}$ with sufficiently large probability. Since all the δ -interior facets satisfy the gradient requirement (see Theorem 4.6), we are done. \square

Proof of Theorem 1.4. Let \mathcal{C}_ε denote the event that $\text{MLR}(\Gamma_n) \leq n^{1/2-\varepsilon}$. Thus, to prove Theorem 1.4, it suffices to show that

$$\mathbb{P}(\mathcal{C}_\varepsilon \cap \mathcal{B}_\varepsilon) \leq e^{-n^c}.$$

Let Γ denote the constrained geodesic. On $\mathcal{C}_\varepsilon \cap \mathcal{B}_\varepsilon$, let v_0 and v_1 be corners of $\Gamma_{\alpha,n}^*$ satisfying the conditions in the definition of \mathcal{B}_ε . Consider the restriction $\Gamma(v_0, v_1)$ of Γ between v_0 and v_1 . As $\text{MLR}(\Gamma) \leq n^{1/2-\varepsilon}$, it follows that there exists $C = C(\kappa) > 0$ such that the restriction of Γ (call it γ) between v_0 and v_1 is contained in the parallelogram U that has a pair of vertical sides of height $Cn^{1/2-\varepsilon}$ and whose top side is the line segment joining v_0 and v_1 . Now observe that the height of the parallelogram is much smaller than the on-scale transversal fluctuation of geodesics between v_0 and v_1 , these fluctuations being at least

$$(n^{3/4-\varepsilon})^{2/3-\varepsilon/100} \geq n^{1/2-3\varepsilon/4}.$$

Thus, Corollary 2.7 implies that the bound

$$\mathbb{E}(L(v_0, v_1)) - |\gamma| \geq |v_0 - v_1|^{1/3-5\varepsilon/9}$$

fails with probability at most e^{-n^c} . By Corollary 2.13,

$$L^\square(v_0, v_1) - \mathbb{E}(L(v_0, v_1)) \geq -|v_0 - v_1|^{\frac{1}{3}+\varepsilon/100}$$

except, again, on a set of probability at most e^{-n^c} . Consider now the path Γ' that agrees with Γ outside the segment (v_0, v_1) and that between v_0 and v_1 equals $\gamma^\square(v_0, v_1)$, which recall is a path from v_0 to v_1 lying above the line segment that joins these two points and achieves length $L^\square(v_0, v_1)$. We see then that Γ' has greater length, and traps as much area as, Γ . This contradicts the assumption of length maximality for Γ ; (similarly to the proof of Theorem 1.3. See Figure 4(b).) \square

6. UPPER BOUND FOR SCALING EXPONENTS

This section is devoted to the proof of Theorem 1.7. That is, it will be shown here that, for a dense set of α that may depend on n , the maximum facet length of the constrained geodesic is very likely to be at most $n^{3/4+o(1)}$. Theorem 1.8 will then follow by the KPZ scaling enjoyed by transversal fluctuations.

We explain the main ideas before giving proofs. Recall Definition 1.6 and the interval $[\alpha_1, \alpha_2]$ in the hypothesis of Theorem 1.7. The basic idea is to show, using Corollary 2.13, that if a certain value of α is not ‘good’ (recall Definition 1.6), we may perform a ‘landgrab’ operation. Let us assume that there is a facet (v_0, v_1) which has length $n^{3/4+\varepsilon}$ and has a reasonable gradient bounded away from zero and infinity. Corollary 2.13 allows us to find a path γ from v_0 to v_1 which lies above the facet (v_0, v_1) at a characteristic height of $(n^{3/4+\varepsilon})^{2/3}$ and has characteristic fluctuations. Since we can afford to allow some error, we can assume

$$|\gamma| \geq \mathbb{E}(L(v_0, v_1)) - (n^{3/4+\varepsilon})^{1/3+\varepsilon/1000} \geq \mathbb{E}(L(v_0, v_1)) - n^{1/4+2\varepsilon/5}.$$

Similarly by Corollary 2.6 we can assume that

$$|\Gamma(v_0, v_1)| \leq \mathbb{E}(L(v_0, v_1)) + n^{1/4+2\varepsilon/5}$$

where $\Gamma(v_0, v_1)$ is the restriction of the constrained geodesic Γ between v_0 and v_1 . Thus the path Γ' which is obtained from Γ by replacing $\Gamma(v_0, v_1)$ by γ traps at least $n^{5/4+5\varepsilon/3}$ more area than does Γ and loses at most $n^{1/4+2\varepsilon/5}$ in length. Repeating this operation about $n^{3/4-5\varepsilon/3}$ times (a repetition which is possible, provided there is no good α in the interval $[\alpha_1, \alpha_2]$, one gains an area that is $\Theta(1)$ while losing at most $O(n^{1-\varepsilon})$ in length with very high probability. This contradicts the fact that $\mathbb{E}L_{\alpha_1}(n) - \mathbb{E}L_{\alpha_2}(n) = \Theta(n)$ for $\alpha_1 < \alpha_2$ which is an easy consequence of Theorems 1.9 and 2.1 along with the observation that w_α is strictly decreasing in α . Next we provide the details needed to make the above argument precise.

Fix $0 < \alpha_1 < \alpha_2 < \frac{1}{2}$ and also $\varepsilon > 0$ and $\delta \in (0, \pi/4)$. Let $\mathcal{G}_{\alpha_1, \alpha_2}$ denote the event that no $\alpha \in [\alpha_1, \alpha_2]$ is (n, ε, δ) -good. Let $B_n = B_{n, \varepsilon, \kappa}$ be the set of all pairs $(x, y) \in [0, n]^2$ satisfying the following conditions.

- (1) $x \preceq y$.
- (2) The straight line joining x and y has gradient $\in (\frac{1}{\kappa}, \kappa)$.
- (3) $|x - y| \geq n^{3/4}$.

For $(x, y) \in B_n$, let $A_{x, y}$ denote the event that $|L^\square(x, y) - \mathbb{E}L(x, y)| \leq |x - y|^{1/3+\varepsilon/1000}$ and $|L^*(x, y) - \mathbb{E}L(x, y)| \leq |x - y|^{1/3+\varepsilon/1000}$ where $L^*(x, y)$ denotes the best path between x and y that is constrained to stay below the straight line joining x and y . Let \mathcal{A}_ε denote the event that $A_{x, y}$ holds for all $(x, y) \in B_n$. Further, let S_δ denote the event in the statement of Theorem 4.6, i.e., for all $\alpha \in [\alpha_1, \alpha_2]$, all the δ interior facets of $\Gamma_{\alpha, n}^*$ make an angle with the x -axis that lies in the interval $(\omega, \pi/2 - \omega)$ where $\omega = \omega(\delta)$. We have the following proposition.

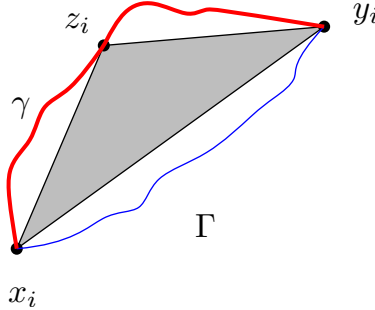


FIGURE 6. The landgrab operation. By replacing the path Γ by the path γ one gains at least the amount of the area in the shaded region, whereas the event \mathcal{A}_ε ensures that the length loss is not too much.

Proposition 6.1. *There exists $\kappa = \kappa(\delta, \varepsilon)$ sufficiently small such that, on $\mathcal{A}_\varepsilon \cap S_\delta \cap \mathcal{G}_{\alpha_1, \alpha_2}$, there exists $\alpha_* \geq \alpha_2$ for which*

$$L_{\alpha_1}(n) - L_{\alpha_*}(n) \leq (\alpha_* - \alpha_1)n^{1-6\varepsilon/5}.$$

Proof. We perform the following recursive construction. Let $\beta_0 = \alpha_1$. For $\beta_i \in [\alpha_1, \alpha_2]$ construct β_{i+1} recursively as follows: find the longest δ -interior facet in $\Gamma_{\beta_i, n}$. On $\mathcal{G}_{\alpha_1, \alpha_2}$, the longest δ -interior facet (x_i, y_i) has length at least $n^{3/4+\varepsilon}$. Note that on S_δ , the endpoints x_i and y_i of this facet satisfy $(x_i, y_i) \in B_n$ for some $\kappa = \kappa(\delta)$. Now pick a point z_i at orthogonal distance $|x_i - y_i|^{2/3}$ from the midpoint of the line segment joining x_i and y_i ; (see Figure 6). Thus the area of the triangle (x_i, y_i, z_i) is at least $c|x_i - y_i|^{5/3}$ for some constant c that does not depend on i . Set $\beta_{i+1} = \beta_i + cn^{-2}|x_i - y_i|^{5/3}$. Now consider the path γ that coincides with $\Gamma_{\beta_i, n}$ outside the facet (x_i, y_i) , and is formed by concatenating $\gamma^\square(x_i, z_i)$ and $\gamma^\square(z_i, y_i)$ between x_i and y_i . Clearly the area under the curve γ is at least $(\frac{1}{2} + \beta_{i+1})n^2$. Also, on \mathcal{A}_ε ,

$$|\Gamma_{\beta_i, n}| - |\gamma| \leq |x_i - y_i|^{1/3+\varepsilon/500}.$$

It follows that

$$L_{\beta_i}(n) - L_{\beta_{i+1}}(n) \leq |x_i - y_i|^{1/3+\varepsilon/500}.$$

Denote $\alpha_* = \beta_{i_0}$, where i_0 is the smallest index i for which $\beta_i \geq \alpha_2$. It follows that

$$L_{\alpha_1}(n) - L_{\alpha_*}(n) \leq \sum_{i=0}^{i_0-1} |x_i - y_i|^{1/3+\varepsilon/500} \leq c^{-1}(\alpha_* - \alpha_1)n^2 \left(\max_{i \leq i_0} |x_i - y_i|^{-4/3+\varepsilon/500} \right).$$

Since $|x_i - y_i| \geq n^{3/4+\varepsilon}$ for all i , it follows that the final term is at most $n^{-1-4\varepsilon/3}$. This completes the proof of the proposition. \square

We now complete the proof of Theorem 1.7.

Proof of Theorem 1.7. Let κ be large enough that the conclusion of Proposition 6.1 holds. Using Theorem 4.6 and Corollary 2.11, it follows that, for this choice of κ (and for given $\alpha_1, \alpha_2, \delta$ and ε), one has that, for some constant $c > 0$,

$$\mathbb{P}[\mathcal{A}_\varepsilon^c \cup S_\delta^c] \leq e^{-n^c}.$$

Hence, using Proposition 6.1, it suffices to show that

$$\mathbb{P}\left(L_{\alpha_1}(n) - L_{\alpha_2}(n) \leq (1/2 - \alpha_1)n^{1-6\varepsilon/5}\right) \leq e^{-n^c} \quad (25)$$

for some $c > 0$. Notice that in the above step we have used the trivial inequality $L_{\alpha_2}(n) \geq L_{\alpha_*}(n)$ for all $\alpha_* \in [\alpha_2, \frac{1}{2}]$. It follows now from Theorem 1.9 that $\mathbb{E}L_{\alpha_1}(n) - \mathbb{E}L_{\alpha_2}(n) \geq \frac{w_{\alpha_1} - w_{\alpha_2}}{2}n$ for n sufficiently large. The claimed bound (25) now follows from Theorem 2.1 and the fact that $w_{\alpha_1} - w_{\alpha_2} > 0$. This completes the proof. \square

6.1. Proof of Theorem 1.8. This derivation is similar to that by which Theorem 1.4 follows from Theorem 1.3. By Theorem 4.6, for all $\alpha \in [\alpha_1, \alpha_2]$ and all small enough δ , there is a probability at least $1 - e^{-cn}$ that all the δ -interior facets of $\Gamma_{\alpha,n}^*$ are κ -steep for some value of $1 \leq \kappa < \infty$. The set of good α 's is a function of the underlying point process and hence is random, but in any case, for any such α that lies in $[\alpha_1, \alpha_2]$, it holds by definition that all the interior facets of $\Gamma_{\alpha,n}^*$ have length at most $n^{3/4+\varepsilon}$. Moreover, by the uniform bounds discussed above, we may suppose that all the facets are κ -steep. Suppose that the maximum local roughness restricted to the δ -interior facets is at least $n^{1/2+\varepsilon}$. Let u and v be the endpoints of the facet that attains this maximum local roughness. Let $\gamma(u, v)$ be the segment of $\Gamma_{\alpha,n}$ between u and v . By using Corollary 2.7, we see that, with probability at least $1 - e^{-n^c}$,

$$|\gamma(u, v)| < \mathbb{E}(L(u, v)) - S(u, v).$$

On the other hand, by Corollary 2.13 again, we find that, except on an event of probability at most e^{-n^c} ,

$$L^\square(u, v) \geq \mathbb{E}(L(u, v)) - S(u, v).$$

Thus, the path that agrees with Γ from $(0, 0)$ to u and from v to (n, n) and equals the path $\gamma^\square(u, v)$ between the points u and v clearly captures at least as much area as does $\Gamma_{\alpha,n}$ and also contains more points than does $\Gamma_{\alpha,n}$. This contradicts the extremality of the latter path. \square

6.2. Possible extensions and difficulties. We end this section with a few remarks related to the first two open problems mentioned in Section 1.4. A natural approach to quantifying the proof of Theorem 1.7 is via Proposition 6.1. Namely, one can hope to bound below the density of those $\alpha \in [\alpha_1, \alpha_2]$ that are (n, ε) good (where here we ignore the parameter δ for the purpose of illustration.) This is because using similar arguments to those used in deriving these two results, one can hope to prove the following uniform bound: with high probability the noise space is such that for all $\alpha \in [\alpha_1, \alpha_2]$ we have $L_\alpha(n) - L_{\alpha_*}(n) \leq \tilde{O}(\ell^{1/3})$ for $\alpha_* = \alpha + \ell^{5/3}n^{-2}$, where $\ell = \max(\text{MFL}(\Gamma_{\alpha,n}), n^{3/4})$, and where the \tilde{O} -notation hides poly-logarithmic multiplicative factors (which arise due to the exceptional nature of the facet endpoints that contributes a polynomial entropy factor). Thus, the above implies that the length-to-area gradient in scaled coordinates (with length measured in units of n and area measured in units of n^2) is $\tilde{O}(\frac{n}{\ell^{4/3}}) = \tilde{O}(1)$ as by definition $\ell \geq n^{3/4}$. Also, we know that, for all α which are (n, ε) bad, the gradient is polynomially small (since by the land-grab argument, illustrated in Figure 6, the length loss is much smaller than the area gain). This coupled with the fact that $L_{\alpha_1} - L_{\alpha_2} = \Theta(n)$, should then allow to conclude that the density of good α is at least $\frac{1}{\tilde{O}(1)}$.

Notice that our approach for proving upper bounds relies on the joint coupling of the process for various values of α . This is quite different in spirit from the known proofs of similar statements in other contexts such as phase separation [20] where the upper bound was proven for a fixed value of α . The key ingredient there was an understanding of the excess area fluctuation (see Open question (2) in Section 1.4) which then allowed a resampling argument to work. However, since we do not have such bounds, we have to work across various values of α simultaneously.

A heuristic argument in our setting proving an upper bound of $n^{5/4+o(1)}$ on the excess area trapped by $\Gamma = \Gamma_\alpha$, assuming that α is $(n, \varepsilon/10)$ good (recall Definition 1.6) can be made as follows: Suppose that the excess area is more than $n^{5/4+\varepsilon}$. Since, by hypothesis the longest facet length is at most $n^{3/4+\varepsilon/10}$, we may find two points x and y on $\Gamma_{\alpha,n}^*$ such that $|x - y| \approx n^{3/4+\varepsilon/5}$ and segment of $\Gamma_{\alpha,n}^*$ between x and y has average curvature (the average distance of the curve from the line segment joining x and y is what it should be on for a circle; see Figure 2(b)). Now consider shortcutting Γ between x and y and hence replacing the subpath of $\Gamma_{\alpha,n}$ between x and y by the unconstrained geodesic between x and y . This operation will create an area loss (approximately $n^{5/4+3\varepsilon/5}$) which is not enough to violate the area constraint, because the excess area before the shortcut was at least $n^{5/4+\varepsilon}$. Moreover, the shortcut increases the overall weight and hence contradicts extremality of $\Gamma_{\alpha,n}$.

7. PROOFS OF SOME OF THE EARLIER STATEMENTS

It remains to provide the proofs of Lemmas 3.2, 3.4, 3.5, 5.6 and Theorem 2.1.

7.1. Proof of Lemma 3.2. This subsection follows closely the arguments presented in [12, Section 3]. Let ϕ_1, ϕ_2, \dots be a sequence of elements in \mathcal{B}_α such that

$$\lim_n J(\phi_n) = J_\alpha.$$

We equip the space of bounded signed measures on $[0, 1]$ with the weak topology generated by $\mathcal{C} = C[0, 1]$ (the space of continuous functions on $[0, 1]$).

Since for all $n \geq 1$, ϕ_n corresponds to a sub-probability measure, we may use compactness and pass to a subsequence (denoted again by $\{n\}$ for convenience), and thus suppose that ϕ_n converges to $\psi \in \mathcal{M}$, in the weak topology. This in particular implies $\phi_n(x)$ converges to $\psi(x)$ almost everywhere (since the convergence happens at all continuity points of ψ , see e.g., [16, Section 3.2])

). Since the ϕ_n 's are all bounded by 1 and belong to \mathcal{B}_α , it follows from the bounded convergence theorem that $\psi \in \mathcal{B}_\alpha$. Next we will show that

$$\lim_n J(\phi_n) \leq J(\psi). \quad (26)$$

Clearly this implies $J(\psi) = J_\alpha$ and hence we would be done. The claim in (26) follows from the upper semicontinuity of $J(\cdot)$ or equivalently the lower semicontinuity of $-J(\cdot)$. To show the latter, one represents $-J(\cdot)$ as an appropriate Legendre transform. To proceed we need some definitions: Clearly (7) extends naturally to all non-negative measures. We extend J to all of \mathcal{M} by setting it to be ∞ for all measures which are not non-negative. Moreover, for any $f \in \mathcal{C}$, define

$$\Lambda(f) = \begin{cases} \infty & \text{if } \int_0^1 \frac{1}{|f(s)|} ds = \infty \text{ or if } f \geq 0 \text{ on a set of positive Lebesgue measure.} \\ -\frac{1}{4} \int_0^1 \frac{1}{f(s)} ds & \text{otherwise} \end{cases}$$

Now for any $\phi \in \mathcal{B}$, define $\Lambda^*(\phi) = \sup_{f \in C[0,1]} \left[\int_0^1 f d\phi - \Lambda(f) \right]$. This function is lower semicontinuous, because whenever $f \in C[0,1]$ and ϕ_n is a sequence that converges to ϕ , the definition of weak convergence implies that

$$\lim_{n \rightarrow \infty} \int_0^1 f d\phi_n = \int_0^1 f d\phi.$$

Thus the proof is complete by [12, Lemma 5] which says $\Lambda^*(\phi) = -J(\phi)$. \square

7.2. Proof of Lemma 3.4. As already hinted right after the statement of Lemma 3.4, the proof is by contradiction. Assuming that the singular part of ψ , which we denote by ψ_s (see (6)) has positive mass, we create a modification ψ_1 such that $J(\psi_1) > J(\psi)$ (see (7)). However in the process, it is possible that $\int_0^1 \psi_1(x) dx < \frac{1}{2} + \alpha$ and hence the area constraint is violated. Thus we make a second modification to construct a function ψ_2 that satisfies $J(\psi_2) > J(\psi)$ as well as the area constraint. This contradicts the extremality of ψ which was a part of the hypothesis. The details follow.

Let us assume that ψ_s has total mass $k > 0$. Then clearly, without loss of generality, we can assume $\psi_s = k\delta_0$, i.e., there is an atom of mass k at zero, since

$$\int_0^1 (\psi_{ac}(x) + k) dx \geq \int_0^1 (\psi_{ac}(x) + \psi_s(x)) dx \geq \frac{1}{2} + \alpha.$$

Also, because the absolutely continuous part stays the same, $J(\psi_{ac} + k\delta_0) = J(\psi)$. We will now make some local modifications to contradict extremality of ψ .

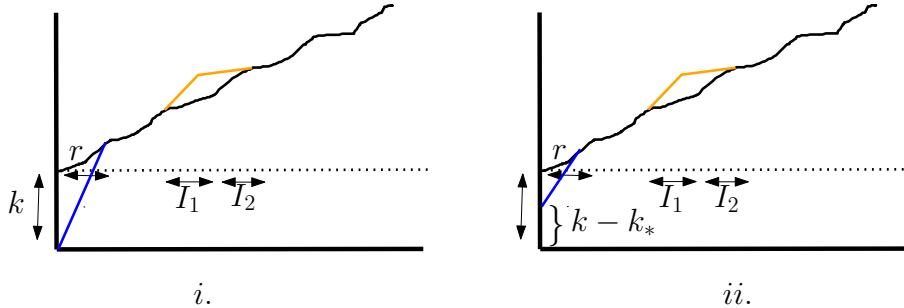


FIGURE 7. Local improvement

Let $r > 0$ be a small number to be chosen later. Consider a new function ψ_1 , as illustrated in Figure 7, which is a linear function interpolating $(0,0)$ and $(r, \psi(r))$ and which agrees with ψ on

$[r, 1]$. Since $\psi(r) = \psi_{ac}(r) + k$, it follows that

$$\int_0^1 \psi_1(x) dx \geq \int_0^1 \psi(x) dx - \frac{1}{2}kr - \psi_{ac}(r)r. \quad (27)$$

On the other hand, for all small enough r ,

$$\begin{aligned} \int_0^1 \sqrt{\psi_1(x)} dx &\geq \int_0^1 \sqrt{\dot{\psi}(x)} dx - \int_0^r \sqrt{\dot{\psi}(x)} dx + \sqrt{kr} \\ &\geq \int_0^1 \sqrt{\dot{\psi}(x)} dx - \sqrt{\psi_{ac}(r)r} + \sqrt{kr}, \\ &\geq \int_0^1 \sqrt{\dot{\psi}(x)} dx + \frac{\sqrt{kr}}{2}. \end{aligned} \quad (28)$$

The second inequality uses the bound $\int_0^r \sqrt{\dot{\psi}(x)} dx \leq \sqrt{r\psi_{ac}(r)}$, which follows from the Cauchy-Schwarz inequality. To see the last inequality, note that $\psi_{ac}(r)$ goes to zero as r goes to zero. However, it is easy to see that

$$\int_0^1 \psi_1(x) dx < \int_0^1 \psi(x) dx.$$

Thus a priori ψ_1 need not be an element of \mathcal{B}_α ; (see (8)). To ensure that indeed $\int_0^1 \psi_1(x) dx \geq \frac{1}{2} + \alpha$, we have to make another modification. Note that already in the proof of Lemma 3.3, we argued that, even if ψ is not unique, $\dot{\psi}$ is unique. Thus there exist $c_\alpha > 0$ and $0 < c_1 < c_2$ depending only on α such that the set $(\dot{\psi})^{-1}[c_1, c_2]$ has Lebesgue measure at least c_α . (Note that even though the proof of Lemma 3.3 relied on Lemma 3.4, the uniqueness of the density was argued independently and hence this is not a circular argument.)

For brevity, let us call the set $(\dot{\psi})^{-1}[c_1, c_2]$ by the name I . We now choose arbitrary disjoint subsets of $I \cap [\varepsilon, 1]$ for some $\varepsilon > 0$, denoted by I_1 and I_2 , both of measure k ; (we take the intersection with $[\varepsilon, 1]$ to ensure that both the sets are away from 0). Note that this can always be done if $\mu(I) > 2k$. In case $\mu(I) \leq 2k$, we can modify ψ_1 by taking it to be a linear function interpolating $(0, k - k_*)$ and $(r, \psi(r))$ for some k_* small enough so that $\mu(I) > 2k_*$; (see Figure 7 *ii*). Nothing changes in any of the arguments that follow as well as the conclusions and hence we will pretend that $k = k_*$ throughout the rest of the argument. Let us also choose I_1 and I_2 such that $\sup(I_1) < \inf(I_2)$, and let $\psi_2(x) := \int_0^x \dot{\psi}_2(y) dy$ where

$$\dot{\psi}_2 := \dot{\psi}_1 + \frac{r}{k} \mathbf{1}_{I_1} - \frac{r}{k} \mathbf{1}_{I_2}. \quad (29)$$

We will also choose $r < \varepsilon$. Note that since $\dot{\psi}_1 = \dot{\psi} \geq c_1$ on I_2 , for all small enough r , $\dot{\psi}_2$ is non-negative. Also, since the measures of I_1 and I_2 are the same,

$$\int_0^1 \dot{\psi}_2(x) dx = \int_0^1 \dot{\psi}_1(x) dx.$$

We now compute the area under the curve $\psi_2(\cdot)$ and see how it differs from that of $\psi_1(\cdot)$. For every $y \in [0, 1]$, we have

$$\psi_2(y) - \psi_1(y) = \frac{r}{k} [\mu(I_1 \cap [0, y]) - \mu(I_2 \cap [0, y])].$$

Thus,

$$\begin{aligned} \int_0^1 [\psi_2(y) - \psi_1(y)]dy &= \frac{r}{k} \int_0^1 [\mu(I_1 \cap [0, y]) - \mu(I_2 \cap [0, y])]dy, \\ &= \frac{r}{k} \int_0^1 (1-y)[\mathbf{1}_{I_1} - \mathbf{1}_{I_2}]dy, \\ &\geq rk. \end{aligned} \tag{30}$$

The last inequality follows since, as $\mu(I_1) = \mu(I_2) = k$ and $\sup(I_1) < \inf(I_2)$, the integral above is at least k^2 . Thus, the new function $\psi_2(x)$ traps at least as much area as does $\psi(x)$ since

$$\begin{aligned} \int_0^1 \psi_2(x)dx - \int_0^1 \psi(x)dx &= [\int_0^1 \psi_2(x)dx - \int_0^1 \psi_1(x)dx] + [\int_0^1 \psi_1(x)dx - \int_0^1 \psi(x)dx], \\ &\geq rk - \frac{rk}{2} - C\psi_{ac}(r)r > 0, \end{aligned}$$

for all small enough r , since $C > 0$ is an universal constant, (the last inequality uses (27)).

Moreover, by Taylor expansion (using (29) and $\dot{\psi}_1 \geq c_1$ on $I_1 \cup I_2$),

$$\begin{aligned} \int_0^1 \sqrt{\dot{\psi}_2(x)}dx - \int_0^1 \sqrt{\dot{\psi}_1(x)}dx &\geq \int_{I_1} \sqrt{\dot{\psi}_1(x)} \left(\frac{r}{2k\dot{\psi}_1(x)} - C \left(\frac{r}{k\dot{\psi}_1(x)} \right)^2 \right) dx \\ &\quad + \int_{I_2} \sqrt{\dot{\psi}_1(x)} \left(-\frac{r}{2k\dot{\psi}_1(x)} - C \left(\frac{r}{k\dot{\psi}_1(x)} \right)^2 \right) dx \\ &\geq -C \frac{r}{k} k = -Cr. \end{aligned}$$

We use the bound $\dot{\psi}_1 > c_1$ and that $\mu(I_1) = \mu(I_2) = k$ crucially in the last inequality, and all the constants depend only on c_1 through the Taylor expansion. Thus, by choosing $r \ll k$ we have $\sqrt{rk} \gg r$, and hence using (28),

$$\begin{aligned} \int_0^1 \sqrt{\dot{\psi}_2(x)}dx - \int_0^1 \sqrt{\dot{\psi}(x)}dx &= \int_0^1 \sqrt{\dot{\psi}_2(x)}dx - \int_0^1 \sqrt{\dot{\psi}_1(x)}dx + \int_0^1 \sqrt{\dot{\psi}_2(x)}dx - \int_0^1 \sqrt{\dot{\psi}_1(x)}dx, \\ &\geq -Cr + \sqrt{kr} - C\sqrt{\psi_{ac}(r)r}, \\ &\geq \sqrt{kr}/2, \end{aligned}$$

for all small enough r , since $C > 0$ is an universal constant. Hence, we obtain a contradiction to the extremality of ψ . \square

7.3. Proof of Lemma 3.5. (i) The proof follows by taking the monotone rearrangement and using uniqueness. Let $\dot{\psi}_{mon}$ be the monotone rearrangement of $\dot{\psi}$. Notice that, by Fubini's theorem,

$$\frac{1}{2} + \alpha \leq \int_0^1 \psi(x)dx = \int_0^1 \dot{\psi}(x)(1-x)dx \leq \int_0^1 (\dot{\psi})_{mon}(x)(1-x)dx = \int_0^1 \psi_1(x)dx \tag{31}$$

where $\psi_1 = (\dot{\psi})_{mon}$. Only the second inequality above needs justification and it is a consequence of a standard rearrangement inequality. Thus ψ_1 satisfies the area constraint. Also clearly $\int_0^1 \sqrt{\dot{\psi}(x)}dx = \int_0^1 \sqrt{\dot{\psi}_1(x)}dx$ as rearrangement keeps integrals unchanged. This contradicts the uniqueness of ψ unless $\psi = \psi_1$

(ii) Let $\dot{\psi}$ be zero on a set of positive measure. By the first part of the lemma, this implies that there exists a such that $\psi(y)$ is a constant on the interval $[a, 1]$. Note that it is easy to contradict

extremality of ψ if ψ is not one on this interval. In case it is one on this interval, consider the function $\psi_1(x) = 1 - \psi^{-1}(1 - x)$ (where $\psi^{-1}(1) = a$). We claim that

$$\int_0^1 \psi_1(y)dy = \int_0^1 \psi(x)dx, \quad \text{and} \quad \int_0^1 \sqrt{\psi_1(y)}dy = \int_0^1 \sqrt{\psi(x)}dx.$$

The first equality follows by Fubini's theorem. The second becomes clear after the change of variable $y = 1 - \phi(x)$ is made. Now, by uniqueness, $\psi_1 = \psi$, and by Lemma 3.4 ψ has no singular part. However, ψ_1 has an atom of mass $1 - a$ at 0 which implies $a = 1$. \square

7.4. Proof of Lemma 5.6. The reader may find it useful to refer to Figure 5. Let ω_1 and ω_2 be the acute angles that the line segments $((n, 0), u)$ and $((n, 0), v)$ make with the x -axis and the y -axis. Now, without loss of generality, we can assume $0 \leq \omega_1 \leq \pi/4$; otherwise, we could work with ω_2 , since clearly one of the two quantities is at most $\pi/4$. Now, as a simple consequence of Theorem 1.10, we see that, with probability at least $1 - e^{-cn}$, $(n, 0)$ is at a $\Theta(n)$ distance from $\Gamma_{\alpha, n}^*$. Also, since ψ_α is strictly concave, and $\omega_1 \leq \pi/4$, the line segment (u, v) makes at least an angle $c(\alpha) > 0$ (depending only on α and not on u and v) with the x -axis. The proof is now completed by considering the triangle $((n, 0), u, v)$ and simple geometric arguments. We omit the details. \square

7.5. Proof of Theorem 2.1. The proof will follow by constructing a suitable martingale and appealing to well known concentration results for martingales. However things are slightly complicated by the possibility that the martingale may not have bounded increments. A simple truncation argument will allow us to take the increments to be bounded. Our proof strategy follows closely arguments in [10]. The proof relies on some coarse graining. We start by introducing some notation. Let

$$A_i := \{(x, y) : i \leq x + y < i + 1\}.$$

Also let $B_{i,j} = [i - 1, i) \times [j - 1, j)$. Fix a number k to be specified soon. Recall the point process Π and the path $\Gamma = \Gamma_{\alpha, n}$ from Section 1.1.1. Let $\Gamma_k = \Gamma_{k, \alpha, n}$ be the increasing path between $(0, 0)$ and (n, n) with the same constraints as Γ along with the additional constraint that the cardinality of the intersection with A_i is less than k , for all $i \in [0, \dots, 2n - 1]$. Let us denote by $|\Gamma_k|$ the weight of Γ_k . Consider the Doob Martingale $\{\mathbb{E}[|\Gamma_k| \mid \mathcal{F}_i]\}_{0 \leq i \leq 2n-1}$, along the filtration $\mathcal{F}_i = \{\Pi \cap A_j : j \leq i\}$. Note that the martingale increments are deterministically bounded by k . The following is a standard consequence of the Azuma-Hoeffding inequality.

Lemma 7.1. $\mathbb{P}(||\Gamma_k| - \mathbb{E}(|\Gamma_k|)| \geq t) \leq e^{-\frac{t^2}{4k^2n}}$.

In the remainder of the section, we obtain a bound on $||\Gamma| - |\Gamma_k||$ and thus also on $|\mathbb{E}|\Gamma| - \mathbb{E}|\Gamma_k||$. Let Π_* be the point process obtained from Π by removing points arbitrarily if necessary from $\Pi \cap B_{i,j}$ to make sure that $|\Pi_* \cap B_{i,j}| \leq k/3$ for all $1 \leq i, j \leq n$. Let $\Gamma_* := \Gamma_{\alpha, *}$ be the longest path with the same constraints as Γ_α in the environment Π_* .

Lemma 7.2. *Deterministically as a consequence of the definitions it follows that $|\Gamma_k| \geq |\Gamma_*|$.*

Proof. Clearly it suffices to show that Γ_* intersects none of the A_i 's at more than k points. The proof is by contradiction: assume that there exists i such that $|\Gamma_* \cap A_i| \geq k + 1$. Since the points on Γ_* are totally ordered (recall the ordering introduced in Section 1.1), let (x_1, y_1) and (x_2, y_2) be the smallest and largest points in the set $\Gamma_* \cap A_i$. By definition of A_i ,

$$(x_2 - x_1) + (y_2 - y_1) \leq 1, \tag{32}$$

so that $\Gamma_* \cap A_i$ intersects at most three $B'_{i,j}$'s, since the number of different indices it can cross in each direction is at most one if (32) is to be satisfied. Thus $\Gamma_* \cap A_i$ has to intersect at least one of the boxes at more than $\frac{k+1}{3}$ points. This contradicts the definition of Γ_* . \square

Thus, it follows that $|\Gamma| - |\Gamma_k| \leq |\Gamma| - |\Gamma_*| \leq C$, where $C = \sum_{i,j} \max(|\Pi_{i,j}| - k/3, 0)$ and $\Pi_{i,j} = \Pi \cap B_{i,j}$. Taking $k = 6 \frac{\log n}{\log \log n}$, the proof is now complete in view of the next result.

Lemma 7.3. *The random variable C satisfies the following:*

- (1) $\mathbb{P}(C \geq \lambda n^{1/2} \frac{\log n}{\log \log n}) \leq 2\lambda^2 e^{-\lambda^2}$
- (2) $\mathbb{E}(C) \leq 1$.

This is a simple consequence of the observation that $|\Pi_{i,j}|$ are independent Poisson variables with mean one and the following tail bound of a standard Poisson variable X : $P(X > r) \leq e^{-r \log r + r}$. \square

REFERENCES

- [1] David Aldous and Persi Diaconis. Hammersley’s interacting particle process and longest increasing subsequences. *Probab. Th. Rel. Fields*, 103:199–213, 1995.
- [2] Kenneth S Alexander. Cube-root boundary fluctuations for droplets in random cluster models. *Communications in Mathematical Physics*, 224(3):733–781, 2001.
- [3] Antonio Auffinger, Michael Damron, and Jack Hanson. 50 years of first passage percolation. *arXiv preprint arXiv:1511.03262*, 2015.
- [4] Jinho Baik, Percy Deift, and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12:1119–1178, 1999.
- [5] Jinho Baik and Eric M. Rains. Algebraic aspects of increasing subsequences. *Duke Math. J.*, 109(1):1–65, 2001.
- [6] Jinho Baik and Eric M. Rains. Symmetrized random permutations. In *Random Matrix Models and Their Applications, volume 40 of Mathematical Sciences Research Institute Publications*, pages 1–19, 2001.
- [7] Riddhipratim Basu and Alan Hammond. Localization of near geodesics in Brownian Last Passage Percolation. In preparation.
- [8] Riddhipratim Basu, Vladas Sidoravicius, and Allan Sly. Last passage percolation with a defect line and the solution of the Slow Bond Problem. Preprint arXiv 1408.3464.
- [9] Marek Biskup, Oren Luidor, Eviatar B Procaccia, and Ron Rosenthal. Isoperimetry in two-dimensional percolation. *arXiv preprint arXiv:1211.0745*, 2012.
- [10] Béla Bollobás and Graham Brightwell. The height of a random partial order: concentration of measure. *The Annals of Applied Probability*, pages 1009–1018, 1992.
- [11] Sourav Chatterjee and Partha S. Dey. Central limit theorem for first-passage percolation time across thin cylinders. *Probability Theory and Related Fields*, 156(3):613–663, 2013.
- [12] Jean-Dominique Deuschel and Ofer Zeitouni. Limiting curves for iid records. *The Annals of Probability*, pages 852–878, 1995.
- [13] Jean-Dominique Deuschel and Ofer Zeitouni. On increasing subsequences of iid samples. *Combinatorics, Probability and Computing*, 8(03):247–263, 1999.
- [14] Partha S. Dey, Ron Peled, and Matthew Joseph. Longest increasing path within the critical strip. Preprint.
- [15] Roland L. Dobrushin, Roman Kotecký, and S Shlosman. *Wulff construction: a global shape from local interaction*, volume 104. American Mathematical Society Providence, Rhode Island, 1992.
- [16] Rick Durrett. *Probability: theory and examples*. Cambridge university press, 2010.
- [17] Patrik L Ferrari and Herbert Spohn. Constrained brownian motion: fluctuations away from circular and parabolic barriers. *Annals of probability*, pages 1302–1325, 2005.
- [18] Julian Gold. Isoperimetry in supercritical bond percolation in dimensions three and higher. *arXiv preprint arXiv:1602.05598*, 2016.
- [19] Alan Hammond. Phase separation in random cluster models iii: Circuit regularity. *Journal of Statistical Physics*, 142(2):229–276, 2011.
- [20] Alan Hammond. Phase separation in random cluster models i: Uniform upper bounds on local deviation. *Communications in Mathematical Physics*, 310(2):455–509, 2012.
- [21] Alan Hammond. Phase separation in random cluster models ii: The droplet at equilibrium, and local deviation lower bounds. *Ann. Probab.*, 40(3):921–978, 05 2012.
- [22] Alan Hammond and Yuval Peres. Fluctuation of a planar brownian loop capturing a large area. *Transactions of the American Mathematical Society*, 360(12):6197–6230, 2008.
- [23] Dmitry Ioffe and Roberto H Schonmann. Dobrushin–kotecký–shlosman theorem up to the critical temperature. *Communications in mathematical physics*, 199(1):117–167, 1998.
- [24] Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probability theory and related fields*, 116(4):445–456, 2000.

- [25] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56:889–892, 1986.
- [26] B.F. Logan and L.A. Shepp. A variational problem for random young tableaux. *Advances in Math.*, 26:206–222, 1977.
- [27] Matthias Löwe and Franz Merkl. Moderate deviations for longest increasing subsequences: The upper tail. *Comm. Pure Appl. Math.*, 54:1488–1519, 2001.
- [28] Matthias Löwe, Franz Merkl, and Silke Rolles. Moderate deviations for longest increasing subsequences: The lower tail. *J. Theor. Probab.*, 15(4):1031–1047, 2002.
- [29] Timo Seppäläinen. Large deviations for increasing sequences on the plane. *Probability Theory and Related Fields*, 112(2):221–244, 1998.
- [30] A.M. Vershik and S.V. Kerov. Asymptotics of the plancherel measure of the symmetric group and the limiting form of young tables. *Soviet Math. Dokl.*, 18:527–531, 1977. Translation of Dokl. Acad. Nauk. SSSR 233 (1977) 1024-1027.

RIDDHIPRATIM BASU, INTERNATIONAL CENTRE FOR THEORETICAL SCIENCES, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BANGALORE, INDIA

E-mail address: `rbasu@icts.res.in`

SHIRSHENDU GANGULY, DEPARTMENT OF STATISTICS, UC BERKELEY, BERKELEY, CA, USA

E-mail address: `sganguly@berkeley.edu`

ALAN HAMMOND, DEPARTMENTS OF MATHEMATICS AND STATISTICS, UC BERKELEY, BERKELEY, CA, USA

E-mail address: `alanmh@berkeley.edu`