GREEDY LATTICE ANIMALS: GEOMETRY AND CRITICALITY

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Abstract. Assign to each site of the integer lattice $\mathbb{Z}^d$ a real score, sampled according to the same distribution F, independently of the choices made at all other sites. A lattice animal is a finite connected set of sites, with its weight being the sum of the scores at its sites. Let N_n be the maximal weight of those lattice animals of size n that contain the origin. Denote by N the almost sure finite constant limit of $n^{-1}N_n$, which exists under a mild condition on the positive tail of F. We study certain geometrical aspects of the lattice animal with maximal weight among those contained in an n-box where n is large, both in the supercritical phase where $N > 0$, and in the critical case where $N = 0$.

1. Introduction

In this paper, a *lattice animal* is a connected set ξ of sites in $\mathbb{Z}^d$, where $d \geq 2$. To each ξ in the set $\mathcal{A}$ of all lattice animals assign the random *weight* $S(\xi) = \sum_{\mathbf{v} \in \xi} X_{\mathbf{v}}$, where $\{X_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^d\}$ are independent random variables, each having a common distribution F. With $|\xi|$ denoting the number of sites in a set $\xi \subset \mathbb{Z}^d$, a *greedy lattice animal of size* n is a set $\xi \in \mathcal{A}$ that contains the origin $\mathbf{0}$ with $|\xi| = n$ and whose weight N_n is maximal among all such sets. The study of greedy lattice animals was begun by [2, 6]. The authors of [2] present some optimization problems that motivate the definition of greedy lattice animals.

It is shown in [6, Theorem 1] that $n^{-1}N_n$ converges almost surely and in L^1 to a non-random finite constant N, in the case that the quantities $X_{\mathbf{v}}$ are non-negative and

$$\int_1^\infty x^d (\log x)^c dF(x) < \infty \text{ for some } c > d. \tag{1.1}$$

The same conclusion is derived in [20, Theorem 1.1] for non-negative $X_{\mathbf{v}}$ under the slightly weaker condition that

$$\int_0^\infty (1 - F(x))^{1/d} dx < \infty \tag{1.2}$$

(which in particular holds whenever $c > d - 1$ in (1.1)). By a subtle and involved argument, [3, Theorem 2.1] extends the almost sure convergence $n^{-1}N_n \to N$ to any real-valued $X_{\mathbf{v}}$ whose distribution satisfies (1.1). The condition (1.2) is almost optimal, as $\limsup_n n^{-1}N_n = \infty$ whenever $\int_0^\infty x^d dF(x) = \infty$ (see [3, Theorem 2.2]). In this paper, we continue the study, initaited in [3], of greedy lattice animals whose law F may have an arbitrary negative tail. We have used (1.2) as the condition that we require of the positive tail. We make use of some results in [3] in this paper,

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and note that the proofs on which we rely are valid with the condition (1.2) replacing (1.1), as is explained in the note on page 207 of [3].

We note in passing that a related object, the greedy lattice path of size n, in which the space $\mathcal{A}$ is replaced by the collection of finite self-avoiding paths, is studied in [6, 20]. These papers prove that the corresponding normalized weights $n^{-1}M_n$ converge to a non-random finite constant $M \leq N$ (subject to the same non-negativity and tail conditions on $X_{\mathbf{v}}$). It is further shown in [15] that $M = N$ only when $X_{\mathbf{0}}$ is of bounded support and the probability that $X_{\mathbf{0}}$ equals the right end point of its support is at least the critical probability p_c for site percolation on $\mathbb{Z}^d$.

One of the central themes in the study of greedy lattice animals is the phase transition that the model undergoes as the constant N changes from being negative to positive. It is true that independent site percolation, obtained by taking $X_{\mathbf{v}} \in \{-\infty, 1\}$, is excluded from the theory. (Indeed, in the supercritical phase, that is, when $\mathbb{P}(X_{\mathbf{0}} = 1) > p_c$, the limit N of $n^{-1}N_n$ is a non-degenerate random variable on $\{-\infty, 1\}$, with $\{N = 1\}$ being the event that $\mathbf{0}$ is in the infinite open percolation cluster.) It is helpful however to think of the natural objects of study in the theory of greedy lattice animals as counterparts of well-known objects in percolation. We will mention some of these parallels, as well as comparing and contrasting percolation with the current framework, throughout this Introduction.

Let $\mathcal{A}_C$ denote the collection of lattice animals contained in a set $C \subset \mathbb{Z}^d$. For positive integers n, let $B_{\mathbf{v},n} = \mathbf{v} + \{0, \ldots, n-1\}^d$ denote the n-box shifted by the vector $\mathbf{v} \in \mathbb{Z}^d$, with $B_n = B_{\mathbf{0},n}$. The paper [3] studies the limiting growth of the weight of the *greedy lattice animal in the n-box*,

$$G_n := \max\{S(\xi) \,:\, \xi \in \mathcal{A}_{B_n}\}\,,$$

and its size

$$L_n := \min\{|\xi| \,:\, \xi \in \mathcal{A}_{B_n} \text{ and } S(\xi) = G_n\},$$

the minimum being taken to break ties in the case where F is not atomless.

In the percolation model ($X_{\mathbf{v}} \in \{-\infty, 1\}$), there is a transition for the quantity $G_n = L_n$ from $O(\log n)$ at $p := \mathbb{P}\big(X_{\mathbf{v}} = 1\big) < p_c$ to $O(n^d)$ at $p > p_c$. This transition is analogous to the emergence of a giant component in the random graph model $G(n,p)$ for $p = c/n$ at $c = 1$. It is shown in [3, Theorems 3.1 and 3.2] that a similar transition occurs for any proper distribution F satisfying (1.2). That is, if $N < 0$, then $n^{-1}G_n$ is almost surely bounded in n, whereas, in the case that $N > 0$, there exists a constant $c \in (0,1)$ such that, almost surely, $n^{-d}G_n \in (c, c^{-1})$ for all n large enough.

Our main goals here are to understand more fully the transition of the weight and size of the greedy lattice animal in the n-box, its shape and the behaviour at criticality, that is, when $N = 0$.

Our first result sheds some light on the geometry of the greedy lattice animal in a large n-box in the supercritical case. We define an ℓ-box percolation process of parameter $p \in [0,1]$ to be the random collection of disjoint ℓ-boxes $\big\{B_{\ell\mathbf{a},\ell} : \mathbf{a} \in P\big\}$, where P is the collection of open sites for an independent site percolation in $\mathbb{Z}^d$ where each site is open with probability p.

Theorem 1.1. *Let F be a distribution satisfying (1.2) for which $N > 0$. For any $\epsilon > 0$ there exist $C, \ell \in \mathbb{N}$ and an ℓ-box percolation $\big\{B_{\ell\mathbf{a},\ell} : \mathbf{a} \in P\big\}$ of parameter at least $1 - \epsilon$ such that for all n*

sufficiently large, each greedy lattice animal in the n-box B_n intersects every ℓ-box from the largest connected component of $\{B_{\ell\mathbf{a},\ell} : \mathbf{a} \in P,\, B_{\ell\mathbf{a},\ell} \subseteq \{C\ell, \ldots, n-1-C\ell\}^d\}$.

Theorem 1.1 implies that L_n exceeds the number of ℓ-boxes in the largest connected component to which the theorem refers. Applying some well-known facts about the supercritical phase of percolation, that will later be stated in Lemma 2.7, the limiting fraction of the n-box occupied by the corresponding greedy lattice animal,

$$L = \liminf_{n\to\infty} n^{-d} L_n \,, \tag{1.3}$$

is therefore bounded away from zero when $N > 0$. In this way, the theorem removes the restriction of exponentially decaying positive tail of $X_{\mathbf{0}}$ under which this is proved in [3, Theorem 4.4]. Theorem 1.1 shows how pervasive any greedy lattice animal in a large box must be: it reaches into all but a small fraction of the array of ℓ-boxes. The limiting density $\lim_n n^{-d}|\gamma_n|$ of the largest cluster γ_n in B_n of a supercritical percolation is the density of the unique infinite cluster, $\theta(p) = \mathbb{P}(|C(\mathbf{0})| = \infty)$, by [7, Lemma 7.89]. For this reason, the counterpart in the framework of greedy lattice animals of the density of the infinite cluster is the limiting fraction L. At least in principle, this quantity may be random. Our next result advances the treatment of [3, Theorem 3.2] by resolving the corresponding question for the quantity G.

Theorem 1.2. *For any distribution F that satisfies condition (1.2), there exists a non-random finite constant G such that almost surely*

$$G = \lim_{n\to\infty} n^{-d} G_n.$$

The proof of Theorem 1.2 that we present addresses only the case when $N > 0$, because, as we shall discuss in comments after the statements of Theorem 1.3 and Proposition 1.5, results from [3] and other arguments settle the case in which $N \leq 0$ (the constant G then being equal to zero).

In common with [3], the assumption that F may have an arbitrary negative tail has created the need for more intricate techniques in the proof of Theorem 1.1, and still more so in that of Theorem 1.2, than those that would work were these results to suppose conditions on the negative tail. We next prove a relation between L and the constants G and N.

Theorem 1.3. *If F is a ditribution satisfying condition (1.2) for which $N > 0$, then the inequality $G \leq LN$ holds almost surely.*

Given Theorem 1.1, the proof of Theorem 1.3 is simple. To outline the argument, whose details appear in Section 4, let ξ_n be a greedy lattice animal in the box B_n for which $|\xi_n| = L_n$. If ξ_n happens to contain $\mathbf{0}$, then

$$N_{L_n} \geq S(\xi_n) = G_n. \tag{1.4}$$

The quantity N_{L_n} behaves like NL_n for high values of n by [3, Theorem 2.1], from which the inequality $G \leq LN$ follows. In general, of course, the origin may not lie in ξ_n. However, it follows from Theorem 1.1 that ξ_n reaches to within a distance of $\mathbf{0}$ that is bounded above, uniformly in n. This means that (1.4) holds up to a constant term, implying Theorem 1.3.

Consider a distribution F for which $N = 0$. We may apply Theorem 1.3 in the supercritical case of the law F_ϵ, defined as the distribution of the random variable $X_{\mathbf{0}} + \epsilon$, where $X_{\mathbf{0}}$ has the law F, and where $\epsilon > 0$ is arbitrarily small. It readily follows that $G = 0$ when $N = 0$. The authors of [3] comment that they do not address the critical case. The next theorem does so, providing a more precise estimate on the growth of G_n at criticality than the statement that $G = 0$ in this case.

Theorem 1.4. *Suppose F satisfies (1.2) and is such that $N = 0$. Then, for any $c > d/(d-1)$, we have that almost surely*

$$\lim_{n\to\infty} (\log n)^{-c} n^{-1} G_n = 0 \,. \tag{1.5}$$

The result stands in contrast to that valid for critical percolation in $\mathbb{Z}^2$, for which [10] proves that the size of the largest open cluster in the box B_n grows at a rate exceeding $n^{1+\delta}$, for some $\delta > 0$. Theorem 1.4 is an optimal result up to logarithmic corrections, because for choices of F that come close to violating the positive tail condition (1.2), the maximum weight of sites in B_n is typically of power order n. Indeed, for any $\alpha > 0$, $F(x) = 1 - x^{-d}(\log x)^{-d(1+\alpha)}$ satisfies (1.2), with

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{\mathbf{v}\in B_n} X_{\mathbf{v}} > n(\log n)^{-1-\alpha} \Big) = 1, \tag{1.6}$$

where $X_{\mathbf{0}}$ has law F. For each $\lambda \in \mathbb{R}$, the random variables given by $Y_{\mathbf{v}}(\lambda) = X_{\mathbf{v}} \mathbb{1}\{X_{\mathbf{v}} > \lambda\} - \lambda$ satisfy (1.6) (or, more precisely, they satisfy this condition if the factor of n in (1.6) is replaced by $n/2$). Provided that λ is high enough, the corresponding value of N may be zero or even negative. Thus, the bound in Theorem 1.4 cannot be improved by more than a logarithmic correction. Nonetheless, it may be that for some less contrived choices of F for which $N = 0$, the growth rate of G_n is sublinear.

Finally, we mention some results that arise from considering 'greedy' lattice animals that are constrained to occupy a given fraction of the sites of a large box.

For $n \in \mathbb{N}$ and $\alpha \in (0,1)$, let

$$\tilde{G}_n(\alpha) := \max\{ S(\xi) \, : \, \xi \in \mathcal{A}_{B_n}, \, |\xi| = \lfloor n^d \alpha \rfloor \} \,,$$

denote the maximal weight among lattice animals of specified size that are also contained inside the n-box. The constant N can be obtained as the limit of the weight per site of low density maximal weight animals in a large box:

Proposition 1.5. *Suppose that F satisfies (1.2). Then*

$$\tilde{G}(\alpha) = \lim_{n\to\infty} n^{-d} \tilde{G}_n(\alpha)$$

exists almost surely for each $\alpha \in (0,1)$, with $\tilde{G} : (0,1) \to \mathbb{R}$ a concave, non-random function. If further $X_{\mathbf{0}}$ is bounded below then $\alpha^{-1}\tilde{G}(\alpha) \to N$ as $\alpha \downarrow 0$.

Amir Dembo proposed the method of proof of Proposition 1.5 as a means to derive the inequality $G \leq LN$. Given that we obtain this inequality by a different proof, we have omitted the proof of Proposition 1.5 from this paper. The proof is presented in the Appendix of [11]. The consequence that $G \leq LN$ holds in the case where $X_{\mathbf{0}}$ is bounded below is also proved in [11, Corollary 1]. This alternative approach to proving this inequality has the minor merit that it works even in the case that $N \leq 0$, because Proposition 1.5 requires no hypothesis on the value of N. We know from [3] that $G = 0$ when $N < 0$ and $G > 0$ when $N > 0$, so that [11, Corollary 1] implies that $L = 0$ when $N < 0$ and $X_{\mathbf{0}}$ is bounded below. (In the case where $X_{\mathbf{0}}$ has exponentially decaying positive tail and $N < 0$, [3, Theorems 4.3 and 4.4] prove that $G_n = O(\log n)$ and $L_n = O(\log n)$, similarly to the case of percolation.)

If the distribution F is chosen so that $N = 0$, with $X_{\mathbf{0}}$ bounded below and the function $\tilde{G} : (0,1) \to \mathbb{R}$ strictly concave, then Proposition 1.5 implies that the right-derivative at 0 of the concave function $\tilde{G} : (0,1) \to \mathbb{R}$ vanishes, so that $\tilde{G}(0) = \sup_{\alpha \in (0,1)} \tilde{G}(\alpha)$. It is plausible from the definition of $\tilde{G}$, and is verified during the course of the proof of Proposition 1.5 in [11, Corollary 1], that the constant G is equal to $\sup_{\alpha \in (0,1)} \tilde{G}(\alpha)$. For such a law F, then, we find that $\tilde{G}(0) = G = \lim_n n^{-d} G_n = \lim_n n^{-d} \tilde{G}_n(n^{-d} L_n)$, from which we learn that $L = 0$. Informally, we might say that, in many cases of a critical choice of law F, the greedy lattice animal in a large box comprises a negligible fraction of the sites in that box. The corresponding statement for percolation is that $\theta(p_c) = 0$, which amounts to the absence of any infinite cluster at the critical value. One analogue of continuity of the percolation probability is trivially false: if $X_{\mathbf{0}}$ is equal to ϵ almost surely, then L is the almost sure constant 1, whatever the value of $\epsilon > 0$. However, the map $F \mapsto G[F]$ does not have a jump discontinuity as the law F is increased through choices for which $N = 0$. This follows from [16], which proves, under uniform stochastic dominance and moment assumptions, that $F \mapsto N[F]$ is continuous with respect to weak convergence of measures, and the upper bound $G \leq LN \leq N$ asserted by Theorem 1.3.

The global geometry of the greedy lattice animal: two examples.

Theorem 1.1 prompts the question: if $N > 0$, how closely does the geometry of a greedy lattice animal in a large box resemble that of the infinite component of a supercritical site percolation? Two examples illustrate how the answer depends on the choice of the distribution F. We do not rely on the assertions made about these examples later in the paper, and do not provide proofs of them, although these proofs are straightforward.

In the first example, $X_{\mathbf{v}} \in \{-\lambda, 1\}$ with $\mathbb{P}(X_{\mathbf{0}} = 1) = 1 - \mathbb{P}(X_{\mathbf{0}} = -\lambda) = p$ for a pair $(p, \lambda) \in (0,1) \times [0, \infty)$ for which $N > 0$. If the parameter $p > p_c$ is supercritical for site percolation, then a greedy lattice animal in B_n contains, for n large enough, the largest connected component Γ of $\{\mathbf{v} \in B_n : X_{\mathbf{v}} = 1\}$. Moreover, a smaller cluster γ of one-valued sites in B_n would lie in the greedy animal by forming a path into Γ unless it is isolated from Γ by a region of $-\lambda$-valued sites which requires a path of length about $|\gamma|/\lambda$ sites to cross. Connected sets of one-valued sites are therefore much more prone to be part of the greedy lattice animal in a large box than connected sets of open sites are liable to form part of the largest connected component of open sites in the same box for a supercritical percolation. This means that, in this case, the global geometry of a greedy lattice animal is at least as connected as that of the largest component of a percolation.

This choice of law for $X_{\mathbf{0}}$ has in fact been studied previously. The behaviour of the greedy lattice path in this model is closely related (by setting $\lambda = \rho/(1+\rho)$) to the model of ρ-percolation, introduced in [22]. Taking $\lambda > 0$ fixed, [17] explores the behaviour of $N[p]$ as $p \downarrow 0$ and that of $1 - N[p]$ as $p \uparrow p_c$.

As a contrast, consider the case where $X_{\mathbf{v}} \in \{-1, \lambda\}$ with $\mathbb{P}(X_{\mathbf{0}} = -1) = 1 - \mathbb{P}(X_{\mathbf{0}} = \lambda) = p$, with λ high and p close to one. Supposing that for a greedy lattice animal γ in a large box B_n, the collection $\gamma \cap \{\mathbf{v} \in B_n : X_{\mathbf{v}} = \lambda\}$ is some given set $\phi \subseteq B_n$, then γ will minimize the size of the set of connection costs $\{\mathbf{v} \in B_n : X_{\mathbf{v}} = -1\}$ subject to joining together the sites of ϕ by paths in B_n. The choice for the set ϕ is made by then optimizing over subsets of $\{\mathbf{v} \in B_n : X_{\mathbf{v}} = \lambda\}$. Thinking of λ-valued sites as cities that benefit from travel between them but must pay some given cost per unit distance of connecting road (or -1-valued site), the greedy lattice animal is the network of cities that renders the greatest benefit over road cost. As such, the greedy animal in this case is closely related to the tree with minimal total edge length subject to the constraint that its vertices contain some fixed fraction of the points of a constant rate Poisson process in a large box in $\mathbb{R}^d$, with the edges being line segments between such points. The global geometry of this object would seem to be starkly different from that of the largest connected component of a supercritical percolation in a large box, having a much higher graphical distance between a typical pair of distant points.

We mention that, in the current case, we may interpret Theorems 1.4 and 1.1 as asserting that if the benefit λ per city is high enough that there exists a network of cities in B_n whose collective benefit exceeds road cost by an amount that is super-linear in n, then the optimal network in fact comprises a positive proportion of all the cities in B_n.

We remark also that the greedy lattice path for this choice of law F corresponds to the variant of the travelling salesman problem for the Poisson process of points, where the salesman need only visit a high but fixed fraction of the points in a large box.

Organization.

In Section 2, we firstly define notations and prove some lemmas that will be of use throughout the paper, and then give the proof of Theorem 1.1. The key to this proof is Lemma 2.4, which shows that it is probable that large boxes, of sidelength ℓ, contain weighty lattice animals that may readily be joined to moderately sized animals in their surroundings. Any greedy lattice animal in a much larger box Γ fails to avoid most such animals in the array of ℓ-boxes that lie in Γ, because each of these animals is liable to increase the weight of any animal that runs nearby by joining up with it.

In Section 3, we prove Theorem 1.2, beginning with an outline of the argument. The proof of Theorem 1.3 is given in Section 4.

In Section 5, we prove Theorem 1.4 by showing that the negation of (1.5) implies that $N > 0$: animals that witness the violation of the bound (1.5) at finite values of n are concatenated by reasonably short paths the weight of whose sites is uniformly bounded below. All sufficiently large boxes are therefore highly likely to contain animals whose weight is a high multiple of the sidelength of the box. A further argument which involves concatenating these animals establishes the conclusion that $N > 0$.

2. The geometry of the maximal weight animal: proof of Theorem 1.1

We shall examine in this section some aspects of the geometry of the greedy lattice animals in the n-box B_n for large n, in the case where $N > 0$. We show below that such animals intersect well the largest connected component in the n-box for supercritical ℓ-box percolation, provided that ℓ is some fixed large value and $n \geq n(\ell)$ is high enough. The proof of Theorem 1.1, in common with several later proofs, will require numerous definitions and lemmas, which we now state and prove.

Definition 2.1. *Let G_B and L_B denote the weight and size of a greedy lattice animal of minimal size in a given m-box $B = B_{\mathbf{x},m}$. For $\lambda \in \mathbb{R}$, say that a site $\mathbf{v} \in \mathbb{Z}^d$ is λ-white (or white, for short) if $X_{\mathbf{v}} \geq -\lambda$, and is black otherwise. The set of λ-white sites is a percolation: we will define numerous site percolations on $\mathbb{Z}^d$, so that each percolation process is an independent site percolation, unless stated otherwise, and so that $p_c = p_c(d)$ denotes the critical value for site percolation in $\mathbb{Z}^d$. The minimal length among all λ-white paths in $\mathbb{Z}^d$ from some white site in B to some white site in A is denoted by $D(B, A)$ (or by $D(\mathbf{v}, A)$ or $D(\mathbf{v}, \mathbf{u})$, in the case that $B = \{\mathbf{v}\}$ and possibly $A = \{\mathbf{u}\}$). We write $B \leftrightarrow A$ (in C) in case such a path (in C) exists. We also write $\ell_\infty(\mathbf{v}, A) = \inf_{\mathbf{u} \in A} \|\mathbf{v} - \mathbf{u}\|$ for the minimal sup-norm distance from $\mathbf{v}$ to a site in A. Further, for an m-box $B = B_{\mathbf{x},m}$ and for any $q \in \mathbb{N}$, set*

$$B[q] = \bigcup_{0 \leq \|\mathbf{a}\| \leq q} B_{\mathbf{x}+m\mathbf{a},m},$$

noting that $B \subseteq B[q]$.

Definition 2.2. *The boundary of $A \subset \mathbb{Z}^d$, denoted ∂A, is the collection of sites $\mathbf{u} \notin A$ adjacent in $\mathbb{Z}^d$ to some $\mathbf{v} \in A$. For $B \subseteq \mathbb{Z}^d$, the B-boundary $\partial_B A$ of $A \subseteq B$ is given by $B \cap \partial A$. The white cluster $\mathcal{W}(\mathbf{v})$ of $\mathbf{v} \in \mathbb{Z}^d$ is the collection of sites $\mathbf{u}$ such that $\mathbf{u} \leftrightarrow \mathbf{v}$ (in particular $\mathcal{W}(\mathbf{v})$ is empty in case $\mathbf{v}$ is black). In an analogous manner, we define the black cluster of $\mathbf{v}$, denoted by $\mathcal{B}(\mathbf{v})$.*

Let $\mathcal{L}$ be the graph with vertex set $\mathbb{Z}^d$ and with an edge between any pair $\mathbf{u} \neq \mathbf{v} \in \mathbb{Z}^d$ with $\|\mathbf{u} - \mathbf{v}\| = \max_{1 \leq i \leq d} |u(i) - v(i)| = 1$. We use the notation $\mathcal{W}_{\mathcal{L}}(\mathbf{v})$ and $\mathcal{B}_{\mathcal{L}}(\mathbf{v})$ to denote the white and black clusters of the site $\mathbf{v}$ with respect to the graph $\mathcal{L}$. For $\mathbf{v} \in B_n$, we write $\mathcal{W}_n(\mathbf{v})$ or $\mathcal{B}_n(\mathbf{v})$ for the white or black clusters of $\mathbf{v}$ in the graph B_n, and $\mathcal{W}_{n,\mathcal{L}}(\mathbf{v})$ or $\mathcal{B}_{n,\mathcal{L}}(\mathbf{v})$ for these clusters in the induced subgraph of $\mathcal{L}$ with vertex set B_n. By 'path' or '$\mathcal{L}$-path', we mean a finite self-avoiding path in the nearest neighbour or $\mathcal{L}$-topology. We will occasionally write '$\mathbb{Z}^d$-path' to emphasise that the nearest-neighbour topology is being used.

The set C separates $A, B \subseteq \mathbb{Z}^d$ (in D) if any path (in D) from A to B intersects C. If each such path intersects C at a location not lying in $A \cup B$, then we say that C properly separates A and B. The set C separates A from infinity if any infinite path from A intersects C.

For $C \subseteq \mathbb{Z}^d$, the exterior boundary of C is given by

$$\partial_{ext}(C) = \Big\{ \mathbf{v} \in \mathbb{Z}^d : \mathbf{v} \text{ is adjacent in } \mathcal{L} \text{ to some } \mathbf{w} \in C, \; \exists \text{ a path from } \infty \text{ to } \mathbf{v} \text{ disjoint from } C \Big\}. \quad (2.1)$$

For $C \subseteq B_n$ and $\mathbf{x} \in B_n \setminus C$, the boundary of C visible from $\mathbf{x}$ in B_n is given by

$$\partial_{vis(\mathbf{x}),n}(C) \;=\; \Big\{ \mathbf{v} \in B_n : \mathbf{v} \text{ is adjacent in } \mathcal{L} \text{ to some } \mathbf{w} \in C, \; \exists \text{ a path in } B_n \text{ from } \mathbf{x} \text{ to } \mathbf{v} \text{ disjoint from } C \Big\}. \tag{2.2}$$

Lemma 2.1. *If $C \subseteq \mathbb{Z}^d$ is finite and $\mathcal{L}$-connected, then $\partial_{ext}(C)$ is $\mathbb{Z}^d$-connected. If $C \subseteq B_n$ is $\mathcal{L}$-connected, with $\mathbf{x} \in B_n$ and $\mathbf{x} \notin C$, then $\partial_{vis(\mathbf{x}),n}(C)$ is $\mathbb{Z}^d$-connected in B_n.*

Proof: The first part of the lemma is [9, Lemma 2.23]. We prove the second part by altering Kesten's proof. The topological setting is that of a closed d-ball instead of $\mathbb{R}^d$, making the changes more than merely notational. We write

$$\overline{U} = \Big\{ x \in \mathbb{R}^d : |x(i)| \leq 1/2, \, i \in \{1, \ldots, d\} \Big\}$$

and

$$U^\epsilon = \Big\{ x \in \mathbb{R}^d : |x(i)| < 1/2 + \epsilon, \, i \in \{1, \ldots, d\} \Big\}$$

for some $\epsilon \in (0, 1/8]$, and we set $N = \bigcup_{\mathbf{v} \in C} (\mathbf{v} + U^\epsilon) \cap [0, n-1]^d$. Kesten first proves, by invoking Alexander and Poincaré duality, that, if $N' \subseteq \mathbb{R}^d$ is bounded and connected with its topological boundary $\partial N'$ a topological $(d-1)$-manifold, then the set

$$\partial_{ext}(N') \;:=\; \Big\{ y \in \partial N' : \exists \text{ a continuous path } p : (0,\infty) \to \mathbb{R}^d \setminus N', \; p(t) \to y \text{ as } t \to 0, \; |p(t)| \to \infty \text{ as } t \to \infty \Big\} \tag{2.3}$$

is path connected. We define for any sets $F, O \subseteq \mathbb{R}^d$ such that $O \cap F$ is open in the subspace topology in F, and for each point $x \in F \setminus \mathrm{clos}(O)$,

$$\partial_{vis(x),F}(O) \;=\; \big\{ y \in F \cap \partial O : \exists \text{ continuous path } p : [0,T] \to F : \; p(0) = x, p(T) = y, \, p[0,T] \cap O = \emptyset \big\}.$$

(Note that the use of the ∂_{ext} and ∂_{vis} notations causes no conflict with the discrete case (2.1) or (2.2), because we are considering $O \subseteq \mathbb{R}^d$). The analogue of the path-connectedness of the set (2.3) that we require is that, for each $x \in [0, n-1]^d \setminus \mathrm{clos}(N)$,

$$\text{the set } \partial_{vis(x),[0,n-1]^d}(N) \text{ is } [0,n-1]^d\text{-path connected.} \tag{2.4}$$

To establish (2.4), let $x \in [0, n-1]^d \setminus \mathrm{clos}(N)$ be fixed. We assume in the first instance that $N \cap \partial([0, n-1]^d) \neq \emptyset$, and will prove (2.4) in this instance by reducing to the $\mathbb{R}^d$ case by a doubling trick. We let closed d-balls $B_d(1)$ and $B_d(2)$ denote two homeomorphic images of $[0, n-1]^d$, writing x and N for the images in $B_d(1)$ of x and N by a harmless abuse of notation. We glue the two balls by identifying their boundaries to form $\Gamma = B_d(1) \cup_{S^{d-1}} B_d(2)$, which is a homeomorphic image of the sphere S^d. Let $\phi : \Gamma \to \Gamma$ map each point in $B_d(i)$ to the corresponding one in $B_d(3-i)$ for $i \in \{1, 2\}$ (so that ϕ fixes the common boundary of $B_d(1)$ and $B_d(2)$), and let $\hat{\phi} : \Gamma \to \Gamma$ be given by

$$\begin{aligned} \hat{\phi}(y) &= y \text{ if } y \in B_d(1) \\ &= \phi(y) \text{ if } y \notin B_d(1). \end{aligned}$$

Define $\hat{N} = N \cup_Z \phi(N)$, where $Z = N \cap \partial B_d(1)$ is the part of the boundary of $B_d(1)$ along which N and its mirror $\phi(N)$ are glued. If $\chi : \Gamma \setminus \{x\} \to \mathbb{R}^d$ is a homeomorphism, then $\partial_{vis(x),\Gamma}(\hat{N}) = \chi^{-1}\big(\partial_{ext}(\chi(\hat{N}))\big)$. We now check that the conditions on $\chi(\hat{N})$ that permit us to apply (2.3) with the choice $N' = \chi(\hat{N})$. Note that $\chi(\hat{N}) \subseteq \mathbb{R}^d$ is bounded, because $x \notin \mathrm{clos}(\hat{N})$, and that $\chi(\hat{N})$ is connected because the fact that $N \cap \partial[0, n-1]^d \neq \emptyset$ implies that $\hat{N}$ is connected. We must also show that $\partial_{\mathbb{R}^d}(\chi(\hat{N}))$ is a topological $(d-1)$-manifold. Kesten proved that

$$\partial_{\mathbb{R}^d} \bigcup_{\mathbf{v} \in C} (\mathbf{v} + U^\epsilon) \text{ is a topological } (d-1)\text{-manifold.} \tag{2.5}$$

Recalling that we consider N as a subset of $B_d(1)$, it follows easily from the definition of N that $Z = N \cap \partial B_d(1)$ is a submanifold of $\partial B_d(1)$ with boundary. This is a sufficient condition for the doubled object $\hat{N}$ to be a topological d-manifold and furthermore, for its boundary in Γ to be given by

$$\partial_\Gamma(\hat{N}) = \big(\partial_{B_d(1)} N \setminus \dot{\mathrm{Z}}\big) \cup_{\partial_{\partial B_d(1)} Z} \big(\partial_{B_d(2)} \phi(N) \setminus \dot{\mathrm{Z}}\big), \tag{2.6}$$

where $\dot{\mathrm{Z}} = Z \setminus \partial Z$, with ∂Z denoting the boundary in $\partial B_d(1)$ of Z (in (2.6), $\dot{\mathrm{Z}}$ is respectively embedded in $B_d(1)$ or $B_d(2)$). See [12, Chapter 8.2] for a discussion of gluing of such manifolds. We must now check that the right-hand-side of (2.6) is a $d-1$-manifold. We will do so by applying the same sufficient condition already mentioned, which in this case means that ∂Z is a $d-2$ manifold. We show in fact that each of the finitely many connected components of ∂Z is a $d-2$ manifold. To this end, let Z' denote one of these components. Note that $U \cap Z'$ is homeomorphic to an open subset of a set of the form appearing in (2.5), with d replaced by $d-1$, provided that U is an open set lying in a part of $\partial B_d(1)$ corresponding to an open face of $[0, n-1]^d$. If U is a neighbourhood of a point in the boundary of at least two such faces, then an initial homeomorphism is required to flatten $\partial B_d(1) \cap U$. The image of $U \cap Z'$ under this map is then homeomorphic to the same type of set as in the other case. We thus learn from (2.5) (applied with d replaced by $d-1$) that Z' is indeed a manifold. We find that the glued object $\partial_\Gamma(\hat{N})$ is a $(d-1)$-manifold, so that $\partial_{\mathbb{R}^d}\big(\chi(\hat{N})\big)$ is, as well.

We may apply (2.3) to $\chi(\hat{N})$ as we sought, learning by doing so that $\partial_{ext}(\chi(\hat{N}))$ is path connected, so that $\partial_{vis(x),\Gamma}(\hat{N})$ is Γ-path connected. We now use this fact to verify that the set in (2.4) is $[0, n-1]^d$-path connected. To this end, let $y, z \in \partial_{vis(x),B_d(1)}(N)$. As $N \subseteq B_d(1)$, we may consider y and z as members of $\partial_{vis(x),\Gamma}(\hat{N})$. We use the Γ-path connectedness of this last set to find a path $p : [0,1] \to \partial_{vis(x),\Gamma}(\hat{N})$ such that $p(0) = y$ and $p(1) = z$. We claim that the path $\hat{\phi} \circ p : [0,1] \to B_d(1)$ satisfies $\hat{\phi} \circ p[0,1] \subseteq \partial_{vis(x),B_d(1)}(N)$. Indeed, to any $t \in [0,1]$, let $q : [0,1] \to \Gamma$ denote a continuous map satisfying $q(0) = x$, $q(1) = p(t)$ and $q[0,1] \subseteq \Gamma \setminus \hat{N}$ (such a map existing because $p(t) \in \partial_{vis(x),\Gamma}(\hat{N})$.) The map $\hat{\phi} \circ q : [0,1] \to B_d(1) \setminus N$ demonstrates that $\hat{\phi} \circ p(t) \in \partial_{vis(x),B_d(1)}(N)$ for each $t \in [0,1]$. Since $y = (\hat{\phi} \circ p)(0)$ and $z = (\hat{\phi} \circ p)(1)$, we have proved that the set in (2.4) is $[0, n-1]^d$-path connected, in the case that $N \cap \partial([0, n-1]^d) \neq \emptyset$.

If $N \cap \partial([0, n-1]^d) = \emptyset$, let Y denote the topological space formed from $[0, n-1]^d$ by identifying all points in $\partial([0, n-1]^d$. Define a homeomorphism $\xi : Y \setminus \{x\} \to \mathbb{R}^d$, and note that we may apply (2.3) with the choice $N' = \xi(N)$ by a similar argument to that which permitted the choice $N' = \chi(\hat{N})$.

We learn that, in this case also, the set $\partial_{vis(x),[0,n-1]^d}(N) = \xi^{-1}\big(\partial_{ext}(\xi(N))\big)$ is $[0,n-1]^d$-path connected, as we sought.

Secondly, Kesten proves that if $\mathbf{v}', \mathbf{v}'' \in \mathbb{Z}^d$ are connected by a path $\phi : [0,1] \to \mathbb{R}^d \setminus N$, then $\mathbf{v}'$ and $\mathbf{v}''$ may be connected by a $\mathbb{Z}^d$-path that is disjoint from N, and intersects only such cubes $\mathbf{v} + \overline{U}$, $\mathbf{v} \in \mathbb{Z}^d$, that also contain a point of ϕ. Correspondingly, our pair of sites $\mathbf{v}', \mathbf{v}''$ lie in $[0,n-1]^d$, and the path ϕ that connects them in $[0,n-1]^d \setminus N$, and we require that the $\mathbb{Z}^d$ path lies in $[0,n-1]^d \setminus N$. For this, the path produced in the $\mathbb{R}^d$-case does the job.

There are two more steps in Kesten's argument. The use of $[0,n-1]^d$ in place of $\mathbb{R}^d$ makes no difference to the proofs of these steps, and we only state the form they take in our case. The third step asserts that, for any $\mathbf{x} \in B_n \setminus C$, and for each $\mathbf{v} \in B_n$, we have that $\mathbf{v} \in \partial_{vis(\mathbf{x}),B_n}(C)$ if and only if $\mathbf{v} \notin C$ and $(\mathbf{v} + \overline{U}) \cap \partial_{vis(\mathbf{x}),[0,n-1]^d}(N) \neq \emptyset$. The fourth step claims that each pair $\mathbf{v}', \mathbf{v}'' \in \partial_{vis(\mathbf{x}),B_n}(C)$ can be connected by a path in $[0,n-1]^d$ which lies in $(\mathbf{v}' + \overline{U}) \cup (\mathbf{v}'' + \overline{U}) \cup \partial_{vis(\mathbf{x}),[0,n-1]^d}(N)$. Similarly to Kesten, this last path may be deformed by the procedure in the second step into a path in B_n that contains only sites of $\partial_{vis(\mathbf{x}),B_n}(C)$. Thus, $\partial_{vis(\mathbf{x}),B_n}(C)$ is connected in B_n, as required. □

Lemma 2.2. *Suppose that the connected sets $C, D \subseteq B_n$ are disjoint, and that $E \subseteq B_n$ separates C and D in B_n, and is disjoint from D. Then there exists an $\mathcal{L}$-connected subset of E that also separates C and D in B_n. Suppose instead that C,D and E are subsets of $\mathbb{Z}^d$ that satisfy the same conditions of disjointness, and that C and D are connected. If E separates C and D, then, similarly, an $\mathcal{L}$-connected subset of E separates C and D.*

Proof: We comment briefly on the content of what is asserted in the lemma, before proving it. In the case of the first part of the lemma, set

$$\tilde{C} = \bigcup_{\mathbf{v} \in C \setminus E} \big\{ \mathbf{x} \in B_n : \exists \text{ a path in } E^c \text{ from } \mathbf{x} \text{ to } \mathbf{v} \big\}, \tag{2.7}$$

and

$$\hat{C} = (C \cap E) \cup \partial_{B_n} \tilde{C}, \tag{2.8}$$

Note that $\hat{C} \subseteq E$, and that $\hat{C}$ separates C and D in B_n: indeed, the first element of E encountered in any path in B_n from C to D lies in $\hat{C}$. From this fact and the disjointness of D and E, it is straightforward that, for any $\mathbf{y} \in D$, the set

$$\Big\{ \mathbf{v} \in \hat{C} : \mathbf{v} \text{ is } \mathbb{Z}^d\text{-adjacent to some } \mathbf{w} \in B_n \setminus \hat{C}, \exists \text{ a path in } B_n \text{ from } \mathbf{y} \text{ to } \mathbf{w} \text{ disjoint from } \hat{C} \Big\}. \tag{2.9}$$

separates C and D in B_n. A variant of the second part of Lemma 2.1 asserting that the set (2.9) is $\mathcal{L}$-connected would therefore suffice for our purpose. We mention that a sketch of an elementary proof of a similar assertion appears in the Appendix of [21]. The variant might also be obtained by re-examining the proof of [9, Lemma 2.23]. Instead of doing this, we will prove the first assertion of the lemma by finding an $\mathcal{L}$-connected component of $\hat{C}$ that separates C and D in B_n, making direct use of the second part of Lemma 2.1 in the course of the argument .

Let $\{\gamma_1, \ldots, \gamma_k\}$ denote the $\mathcal{L}$-connected components of $\hat{C}$, and let

$$\phi_i = \{\mathbf{v} \in B_n : \gamma_i \text{ separates } \mathbf{v} \text{ from } D \text{ in } B_n\}.$$

We claim that

$$\phi_i \cap \phi_j \neq \emptyset \text{ implies that either } \phi_i \subseteq \phi_j, \text{ or } \phi_j \subseteq \phi_i. \tag{2.10}$$

For $\mathbf{x} \in \phi_i \cap \phi_j$, we consider the case that there exists a path τ in B_n from $\mathbf{x}$ to D that intersects γ_i before it intersects γ_j. We will show that $\phi_i \subseteq \phi_j$, and, to this end, we let $\mathbf{y} \in \phi_i$ be arbitrary. Let $(\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{r_1})$ denote the segment of the path τ until its first intersection with γ_i (so that $\mathbf{x}_{r_1} \in \gamma_i$). Let $(\mathbf{y} = \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{r_2})$ denote any path in B_n from $\mathbf{y}$ to D. There exists $r_3 \in \{0, \ldots, r_2\}$ for which $\mathbf{y}_{r_3} \in \gamma_i$, since $\mathbf{y} \in \phi_i$. Note that

$$\Big\{\mathbf{v} \in B_n : \ell_\infty(\mathbf{v}, \gamma_i) \leq 1\Big\} \cap \gamma_j = \emptyset \text{ for } j \neq i, \tag{2.11}$$

because the sets γ_j are $\mathcal{L}$-connected components of a larger set. There exists an $\mathcal{L}$-path from $\mathbf{x}_{r_1}$ to $\mathbf{y}_{r_3}$ in γ_i. Any consecutive members $\mathbf{u}$ and $\mathbf{v}$ of this path each lie in a unit cube contained in B_n. We may find a $\mathbb{Z}^d$-path in this cube from $\mathbf{u}$ to $\mathbf{v}$. In this way, we may find a path $(\mathbf{x}_{r_1} = \mathbf{z}_1, \ldots, \mathbf{z}_{r_4} = \mathbf{y}_{r_3})$ in B_n such that $\ell_\infty(\mathbf{z}_l, \gamma_i) \leq 1$ for $l \in \{1, \ldots, r_4\}$, and thus, by (2.11), such that $\mathbf{z}_l \notin \gamma_j$ for these l and for $j \neq i$. Note that the path $(\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \ldots \mathbf{x}_{r_1} = \mathbf{z}_1, \ldots, \mathbf{z}_{r_4} = \mathbf{y}_{r_3}, \ldots \mathbf{y}_{r_2} = \mathbf{y})$ connects $\mathbf{x}$ to D in B_n. Note also that its initial segment $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{r_1}, \mathbf{z}_2, \ldots \mathbf{z}_{r_4})$ is disjoint from γ_j by construction. The fact that $\mathbf{x} \in \phi_j$ implies that $\mathbf{y}_r \in \gamma_j$ for some $r \in \{r_3 + 1, \ldots, r_2\}$. The site $\mathbf{y}_r$ lies in the path $(\mathbf{y}_0, \ldots, \mathbf{y}_{r_2})$, which was chosen to be an arbitrary path from $\mathbf{y}$ to D. Thus, $\mathbf{y} \in \phi_j$, and $\phi_i \subseteq \phi_j$, as we claimed. The conclusion $\phi_j \subseteq \phi_i$ arises in the other case, where there exists a path from $\mathbf{x}$ to D that intersects γ_j before it intersects γ_i. This establishes (2.10).

Note that (2.10) implies that to each $i \in \{1, \ldots, k\}$, there is an unique $j \in \{1, \ldots, k\}$ such that $\phi_i \subseteq \phi_j$ and for which $\phi_j \subseteq \phi_l \implies j = l$. Let $\{j_1, \ldots, j_s\}$ denote the collection of $j \in \{1, \ldots, k\}$ arising in this way for some $i \in \{1, \ldots, k\}$. Note that, by (2.10), $\phi_{j_i} \cap \phi_{j_l} = \emptyset$ for $i, l \in \{1, \ldots, s\}$ with $i \neq l$. We claim that:

$$C \subseteq \bigcup_{i=1}^{s} \phi_{j_i}. \tag{2.12}$$

To derive this, suppose that $\mathbf{x} \notin \bigcup_{i=1}^{s} \phi_{j_i} = \bigcup_{i=1}^{k} \phi_i$ for some $\mathbf{x} \in B_n$. Let $T = (\mathbf{x} = \mathbf{v}_0, \ldots \mathbf{v}_{r_5})$ denote an arbitrary path in B_n from $\mathbf{x}$ to D. If $T \cap \hat{C} = \emptyset$, then $x \notin C$, because $\hat{C}$ separates C and D in B_n. We may assume then that $T \cap \hat{C} \neq \emptyset$, and that T intersects γ_1 before it intersects any γ_i for $i \in \{2, \ldots k\}$. In this case, let $0 < r_6 \leq r_7 < r_5$ be such that $\mathbf{v}_{r_6}$ is the first visit of T to γ_1, while $\mathbf{v}_{r_7}$ is the last such. (That $r_6 > 0$ follows from $\mathbf{x} \notin \phi_1$, and thus $\mathbf{x} \notin \gamma_1$). We next show that

$$\mathbf{v}_{r_6-1}, \mathbf{v}_{r_7+1} \in \bigcup_{\mathbf{y} \in D} \partial_{vis(\mathbf{y}),n}(\gamma_1) : \tag{2.13}$$

firstly, each of these two sites is adjacent to a member of γ_1. Note also that $(\mathbf{v}_{r_7+1}, \ldots, \mathbf{v}_{r_5})$ is a path in $B_n \setminus \gamma_1$ from $\mathbf{v}_{r_7+1}$ to D. Since $\mathbf{x} \notin \phi_1$, we may find a path in B_n from $\mathbf{x}$ to some site $\mathbf{y} \in D$ that is disjoint from γ_1. Concatenating this path to $(\mathbf{v}_{r_6-1}, \mathbf{v}_{r_6-2}, \ldots, \mathbf{v}_1)$ forms a path in B_n from $\mathbf{v}_{r_6-1}$ to $\mathbf{y}$ that is disjoint from γ_1. Thus, $\mathbf{v}_{r_6-1} \in \partial_{vis(\mathbf{y}),n}(\gamma_1)$, and (2.13).

Note that $\partial_{vis(\mathbf{y}),n}(\gamma_1)$ is independent of $\mathbf{y} \in D$ because the connected set D satisfies $D \cap E = \emptyset$ and thus $D \cap \gamma_1 = \emptyset$ (because $\gamma_1 \subseteq \hat{C} \subseteq E$). By the second assertion of Lemma 2.1, we may choose a path $\big(\mathbf{v}_{r_6-1} = \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{r_8} = \mathbf{v}_{r_7+1}\big)$ that lies in $\partial_{vis(\mathbf{y}),n}(\gamma_1)$ for any given $\mathbf{y} \in D$. It follows from $\partial_{vis(\mathbf{y}),n}(\gamma_1) \subseteq \big\{\mathbf{v} \in \mathbb{Z}^d : \ell_\infty(\mathbf{v}, \gamma_1) \leq 1\big\}$ and (2.11) that

$$\big\{\mathbf{u}_0, \ldots, \mathbf{u}_{r_8}\big\} \cap \bigcup_{i=2}^{k} \gamma_i = \emptyset.$$

We alter the path T to form

$$\big(\mathbf{x} = \mathbf{v}_0, \ldots, \mathbf{v}_{r_6-1} = \mathbf{u}_0, \ldots, \mathbf{u}_{r_8} = \mathbf{v}_{r_7+1}, \ldots, \mathbf{v}_{r_5}\big).$$

This new path reaches D from $\mathbf{x}$, is disjoint from γ_1 and intersects $\bigcup_{i=2}^k \gamma_i$ only at points where the path T does. By performing alterations that similarly remove the intersections of the new path with the other sets γ_i, we produce a path in B_n from $\mathbf{x}$ to D that is disjoint from $\cup_{i=1}^k \gamma_i = \hat{C}$. Since $\hat{C}$ separates C and D as noted after (2.8), $\mathbf{x} \not\in C$. We have proved (2.12).

We now claim that

$$\text{there exists } i \in \big\{1, \ldots, s\big\} \text{ for which } C \subseteq \phi_{j_i}. \tag{2.14}$$

Were this not the case, there would exist by (2.12) adjacent sites $\mathbf{w}_1, \mathbf{w}_2 \in C$ such that $\mathbf{w}_1 \in \phi_{j_{i_1}}$ and $\mathbf{w}_2 \in \phi_{j_{i_2}}$ for indices $i_1, i_2 \in \{1, \ldots, s\}$ satisfying $i_1 \neq i_2$. By (2.11), one of $\mathbf{w}_1 \in \gamma_{j_{i_1}}$ and $\mathbf{w}_2 \in \gamma_{j_{i_2}}$ fails. We may assume that $\mathbf{w}_1 \not\in \gamma_{j_{i_1}}$. The sets ϕ_{j_i} being disjoint, $\mathbf{w}_2 \in \phi_{j_{i_2}}$ implies that there exists a path σ from $\mathbf{w}_2$ to D that is disjoint from $\gamma_{j_{i_1}}$. The fact that $\mathbf{w}_1 \not\in \gamma_{j_{i_1}}$ implies that the path formed by prefixing $\mathbf{w}_1$ to σ reaches D from $\mathbf{w}_1$ and is disjoint from $\gamma_{j_{i_1}}$. We have reached the contradiction that $\mathbf{w}_1 \not\in \phi_{j_{i_1}}$ and have proved (2.14). From (2.14) follows the first statement of the lemma, because γ_{j_i} is $\mathcal{L}$-connected.

The second assertion has the same proof, with the first part of Lemma 2.1 being applied, instead of the second part. The notational changes consist of omitting each reference to 'in B_n', including in the term $\partial_{vis(\mathbf{y})}(\cdot)$ (which is independent of $\mathbf{y} \in D$). $\square$

Lemma 2.3. *Let $B \subseteq \mathbb{Z}^d$ denote the collection of sites that a finite set $A \subseteq \mathbb{Z}^d$ separates from infinity. Then*

$$\big|A\big| \geq \big|B\big|^{1-\frac{1}{d}}.$$

Let $C, D, E \subseteq B_n$, with C and D connected, and $C \cap D = \emptyset$. Suppose that E properly separates C and D in B_n. Then

$$\big|E\big| \geq \frac{1}{2d} \min\Big\{\big|C\big|^{1-\frac{1}{d}}, \big|D\big|^{1-\frac{1}{d}}\Big\}.$$

Proof: To prove the first part of the lemma, note that, for each $i \in \{1, \ldots, d\}$ and $\mathbf{x} \in B$, $\big\{\mathbf{x} + n\mathbf{e}_i : n \in \mathbb{Z}\big\} \cap A \neq \emptyset$, where $\big\{\mathbf{e}_i : i \in \{1, \ldots, d\}\big\}$ denote unit vectors in the directions of the co-ordinate axes of $\mathbb{R}^d$. It follows that

$$|A| \geq \max_{i \in \{1, \ldots, d\}} |\mathrm{proj}_i(B)|, \tag{2.15}$$

where

$$\mathrm{proj}_i(B) = \big\{(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}) \in \mathbb{Z}^{d-1} : \exists\, \mathbf{x} \in \mathbb{Z} \text{ such that } (\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_i, \ldots, \mathbf{a}_{d-1}) \in B\big\}.$$

By the Loomis-Whitney inequality [19],

$$\max_{i\in\{1,\dots,d\}} |\mathrm{proj}_i(B)| \geq |B|^{1-1/d}, \tag{2.16}$$

so that we obtain the first part of the lemma from (2.15) and (2.16).

To begin the proof of the second part of the lemma, note that the first part of Lemma 2.2 permits us to assume that E is $\mathcal{L}$-connected. We may also assume that $E\cap(C\cup D)=\emptyset$, because $E\setminus(C\cup D)$ properly separates C and D. Recalling the definition (2.7), and defining $\tilde{D}$ analogously, note that

$$\tilde{C}\cap\tilde{D}=\emptyset. \tag{2.17}$$

This is because, otherwise, we might construct a path in E^c from C to D. By (2.17), at least one of the inequalities $|\tilde{C}|\leq\frac{1}{2}n^d$ and $|\tilde{D}|\leq\frac{1}{2}n^d$ holds. We suppose the former for the time being. It follows from [14, Theorem 19] that, for any $A\subseteq B_n$ for which $|A|\leq n^d/2$, we have that

$$\left|\partial_{B_n}A\right| \geq \frac{1}{2d}|A|^{1-\frac{1}{d}}. \tag{2.18}$$

Note that $\partial_{B_n}\tilde{C}\subseteq E$. Indeed, it was noted after (2.8) that $\hat{C}\subseteq E$, and $\partial_{B_n}\tilde{C}=\hat{C}$ in the current case, because $C\cap E=\emptyset$. We find that

$$|E|\geq|\partial_{B_n}\tilde{C}|\geq\frac{1}{2d}|\tilde{C}|^{1-\frac{1}{d}}\geq\frac{1}{2d}|C|^{1-\frac{1}{d}},$$

where (2.18) with the choice $A=\tilde{C}$ was applied in the second inequality, this choice being valid because $|\tilde{C}|\leq\frac{1}{2}n^d$. The third inequality follows from $C\subseteq\tilde{C}$, which is implied by $C\cap E=\emptyset$.

In the case where $|\tilde{D}|\leq\frac{1}{2}n^d$, we deduce that $|E|\geq\frac{1}{2d}|D|^{1-\frac{1}{d}}$. This completes the proof of the second part of the lemma. □

Having assembled these preliminaries, we now state and prove a lemma that is central to the proof of Theorem 1.1.

Lemma 2.4. *For given constants $c>0$ and $\lambda,\rho<\infty$, we say that an m-box $B=B_{\mathbf{x},m}$ satisfies condition $\mathbf{A}^{\rho}_{\lambda,c}$ if there exists $\gamma^\star\in\mathcal{A}_B$ such that $S(\gamma^\star)=G_B\geq cm^d$, $|\gamma^\star|\geq(\log m)^\rho+1$, and $D(\gamma,\gamma^\star)\leq\rho m$ for all $\gamma\in\mathcal{A}_{B[2]}$ such that $|\gamma|\geq(\log m)^\rho$. We set $q_{m,\lambda,c,\rho}:=\mathbb{P}(B$ satisfies condition $\mathbf{A}^{\rho}_{\lambda,c})$, which is independent of the choice of the m-box B. Suppose that the distribution F satisfies (1.2) and is such that $N>0$. Then, for $\lambda,\rho<\infty$ sufficiently large and $c\in(0,\liminf n^{-d}G_n)$, we have that*

$$\lim_{m\to\infty} q_{m,\lambda,c,\rho}=1.$$

Remark Theorem 3.2 of [3] states in the case that $N>0$, there exists a constant $c>0$ such that $\liminf n^{-d}G_n>c>0$ almost surely.

Proof: Fixing $\epsilon>0$, (1.2) implies that $\int_0^\infty x^d dF(x)<\infty$ (this can be derived directly or by contrasting [3, Theorem 2.2] with the note in page 207 of that paper). Following [2, proof of (3.17)], this implies that $X_{\mathbf{v}}\geq\|\mathbf{v}\|$ for at most finitely many $\mathbf{v}\in\mathbb{Z}^d$ almost surely, which implies, in view of $N>0$ and [3, Theorem 3.2], that, for some $c_1>0$,

$$L_m\geq c_1 m^{d-1} \text{ for all sufficiently high } m. \tag{2.19}$$

Thus, for all m large enough,

$$\mathbb{P}(L_B \geq c_1 m^{d-1}) > 1 - \epsilon \tag{2.20}$$

for any m-box B.

For λ such that

$$p := \mathbb{P}(X_{\mathbf{0}} \geq -\lambda) > \max\{p_c, 1 - p_c(\mathbb{Z}^d, \mathcal{L})\}, \tag{2.21}$$

let $\mathcal{W}$ denote the unique infinite cluster of λ-white sites in $\mathbb{Z}^d$ (which exists by [7, Theorem 8.1], since the λ-white sites form a supercritical percolation). Note that this choice of λ ensures that the process of black $\mathcal{L}$-clusters is subcritical. Increasing λ as needed, we next show that for any $\rho > d/(d-1)$, all m large enough and any m-box B,

$$\mathbb{P}\Big(\exists \gamma \in \mathcal{A}_{B[2]} \text{ such that } |\gamma| \geq (\log m)^\rho \text{ and } \gamma \cap \mathcal{W} = \emptyset\Big) \leq \epsilon\,. \tag{2.22}$$

For such a γ as in (2.22), there exists, by choosing $C = \gamma$, $D = \mathcal{W}$ and $E = \{\text{black sites}\}$ in the second part of Lemma 2.2, an $\mathcal{L}$-connected set $\hat{\gamma}$ of black sites that separates γ and $\mathcal{W}$. Applying the first part of Lemma 2.3, we find that

$$\big|\hat{\gamma}\big| \geq \big(\log m\big)^{\rho(1-\frac{1}{d})}, \tag{2.23}$$

because $\hat{\gamma}$ separates γ from ∞. We set $A = A_\gamma$ according to $A = \inf\big\{q \geq 2 : \hat{\gamma} \cap B[q] \neq \emptyset\big\}$. Note that if $A > 2$, then $B[A-1]$ is separated from $\mathcal{W}$, and thus from ∞, by $\hat{\gamma}$: indeed, a path from $B[A-1]$ to $\mathcal{W}$ disjoint from $\hat{\gamma}$ could be extended in $B[A-1]$ to such a path from γ because $B[A-1] \cap \hat{\gamma} = \emptyset$. By Lemma 2.3 again,

$$\big\{A = q\big\} \subseteq \Big\{\big|\hat{\gamma}\big| \geq \big((2q-1)m\big)^{d-1}\Big\}, \text{ for } q \geq 3, \tag{2.24}$$

since $\big|B[q-1]\big| = \big[(2q-1)m\big]^d$. Thus,

$$\begin{aligned}
& \mathbb{P}\Big(\exists \gamma \in \mathcal{A}_{B[2]} \text{ such that } |\gamma| \geq (\log m)^\rho \text{ and } \gamma \cap \mathcal{W} = \emptyset\Big) \\
\leq \quad & (5m)^d \mathbb{P}\Big(\big|\mathcal{B}_{\mathcal{L}}(\mathbf{0})\big| \geq (\log m)^{\rho(1-1/d)}\Big) \\
& + \sum_{q=3}^{\infty} \big[(2q+1)m\big]^d \mathbb{P}\Big(\big|\mathcal{B}_{\mathcal{L}}(\mathbf{0})\big| \geq (2q-1)^{d-1} m^{d-1}\Big) \\
\leq \quad & (5m)^d \exp\big\{-c_2\big(\log m\big)^{\rho(1-1/d)}\big\} + \sum_{q=3}^{\infty} \big[(2q+1)m\big]^d \exp\big\{-c_2(2q-1)^{d-1}m^{d-1}\big\},
\end{aligned} \tag{2.25}$$

for all m and any m-box B. The first term after the first inequality in (2.25) corresponds to the case where $A = 2$, in which case, $\gamma \cap B[2] \neq \emptyset$, and thus, $|\mathcal{B}_{\mathcal{L}}(\mathbf{x})| \geq (\log m)^{\rho(1-1/d)}$ for some $\mathbf{x} \in B[2]$ by (2.23). The term indexed by q in the sum after the first inequality corresponds to the case where $A = q$, with (2.24) being used in place of (2.23). That the constant c_2 is positive follows from the Aizenman-Newman Theorem in [8, Section 2.4.2], which proves an exponential rate of decay for the probability of a large subcritical cluster containing a given site in any homogeneous lattice where each vertex has finite degree, including $\mathcal{L}$. From (2.25), we obtain (2.22) for all but finitely many m.

Taking $\gamma^\star$ to be a greedy lattice animal of minimal size in B, we find from (2.20) and (2.22) that

$$\mathbb{P}\Big(\big\{\gamma^* \cap \mathcal{W} \neq \emptyset\big\} \cap \big\{ \text{if } \gamma \in \mathcal{A}_{B[2]} \text{ satisfies } |\gamma| \geq (\log m)^\rho, \text{ then } \gamma \cap \mathcal{W} \neq \emptyset\big\}\Big) \geq 1 - 2\epsilon\,, \tag{2.26}$$

for all m large enough and any m-box B.

We apply [3, Lemma 2.14], which is a variant of a result in [1], to the supercritical percolation of λ-white sites. Using a union bound, we find that there exists $\rho = \rho(\lambda, d) > 1$ and $c_3 > 0$ such that, for all m and any m-box B,

$$\mathbb{P}(\exists \mathbf{v}, \mathbf{w} \in B[2] : \mathbf{v} \leftrightarrow \mathbf{w} \text{ and } D(\mathbf{v}, \mathbf{w}) \geq \rho m) \leq e^{-c_3 m}. \tag{2.27}$$

By (2.26) and (2.27),

$$\mathbb{P}\Big(\forall \gamma \in \mathcal{A}_{B[2]} \text{ such that } |\gamma| \geq (\log m)^\rho : \;\; D(\gamma, \gamma^\star) < \rho m\Big) \geq 1 - 3\epsilon\,, \tag{2.28}$$

for all m large enough and any m-box B.

By the definition of c, we have that $\mathbb{P}\big(S(\gamma^\star) = G_B \geq cm^d\big) \geq 1 - \epsilon$ for all m large enough and any m-box B. By (2.20), we have that $\mathbb{P}\big(|\gamma^\star| = L_B \geq (\log m)^\rho + 1\big) \geq 1 - \epsilon$ for such boxes B_m. So, in view of the assertion (2.28), the proof of the lemma is complete. □

Proof of Theorem 1.1: By Lemma 2.4, we may and shall set $\lambda < \infty$, $2 < \rho < \infty$ and $c > 0$ such that $\lim_{m\to\infty} q_{m,\lambda,c,\rho} = 1$.

Definition 2.3. *For $\ell \in \mathbb{N}$, we say that $\mathbf{a} \in \mathbb{Z}^d$ is ℓ-active if the ℓ-box $B_{\ell\mathbf{a},\ell}$ satisfies the condition $\mathbf{A}^\rho_{\lambda,c}$, defined in Lemma 2.4.*

Definition 2.4. *A random process τ taking values in subsets of $\mathbb{Z}^d$ is said to be a ρ-near percolation of parameter $p \in (0,1)$ provided that for any $\mathbf{x} \in \mathbb{Z}^d$, $\mathbb{P}(\mathbf{x} \in \tau) = p$, and for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ for which $\|\mathbf{x} - \mathbf{y}\| > \rho$, the events $\big\{\mathbf{x} \in \tau\big\}$ and $\big\{\mathbf{y} \in \tau\big\}$ are independent.*

Lemma 2.5. *For any $\ell \in \mathbb{N}$, the collection of ℓ-active sites forms a $(2\rho+1)$-near percolation.*

Proof: Note that, for given $\mathbf{a} \in \mathbb{Z}^d$, the event

$$\Big\{ B_{\ell\mathbf{a},\ell} \, \text{satisfies condition } \mathbf{A}^\rho_{\lambda,c} \Big\} \tag{2.29}$$

is measurable with respect to $\sigma\big\{X_\mathbf{v} : \ell_\infty(\mathbf{v}, B_{\ell\mathbf{a},\ell}) \leq \rho\ell\big\}$. Indeed, the event that there exists a lattice animal $\gamma^\star \subseteq B_{\ell\mathbf{a},\ell}$ such that $S(\gamma^\star) = G_{B_{\ell\mathbf{a},\ell}} \geq c\ell^d$ is measurable with respect to $\sigma\big\{X_\mathbf{v} : \mathbf{v} \in B_{\ell\mathbf{a},\ell}\big\}$. The event E that $D(\gamma, \gamma^*) \leq \rho\ell$ whenever $\gamma \in \mathcal{A}_{B_{\ell\mathbf{a},\ell}[2]}$ satisfies $|\gamma| \geq (\log m)^\rho$ occurs if and only if for each such γ, there exists a path τ_{γ,γ^*} of white sites of length at most $\rho\ell$ that has one site $\mathbf{w} \in \gamma^\star$ and another in γ. Each site $\xi \in \tau_{\gamma,\gamma^\star}$ satisfies $\ell_\infty(\xi, B_{\ell\mathbf{a},\ell}) \leq \ell_\infty(\xi, \mathbf{w}) \leq \rho\ell$ because $\mathbf{w} \in \gamma^\star \subseteq B_{\ell\mathbf{a},\ell}$, whereas each site $\mathbf{v}$ of γ satisfies $\ell_\infty(\mathbf{v}, B_{\ell\mathbf{a},\ell}) \leq 2\ell$. Thus, the event E is measurable with respect to $\sigma\big\{X_\mathbf{v} : d(\mathbf{v}, B_{\ell\mathbf{a},\ell}) \leq \max\{\rho, 2\}\ell\big\}$. As $\rho > 2$, this establishes (2.29).

For any pair $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{Z}^d$ that satisfy $\|\mathbf{a}_1 - \mathbf{a}_2\| > 2\rho + 1$,

$$\Big\{\mathbf{v} \in \mathbb{Z}^d : \ell_\infty\big(\mathbf{v}, B_{\ell\mathbf{a}_1,\ell}\big) \leq \rho\ell\Big\} \cap \Big\{\mathbf{v} \in \mathbb{Z}^d : \ell_\infty\big(\mathbf{v}, B_{\ell\mathbf{a}_2,\ell}\big) \leq \rho\ell\Big\} = \emptyset. \tag{2.30}$$

From (2.29) and (2.30), it follows that the events $\big\{\mathbf{a}_1 \,\text{is } \ell\text{-active}\big\}$ and $\big\{\mathbf{a}_2 \,\text{is } \ell\text{-active}\big\}$ are independent. Noting that $\mathbb{P}\big(\mathbf{a} \text{ is active}\big)$ is independent of $\mathbf{a}$ completes the proof of the lemma. □

Lemma 2.6. *Let $\sigma > 0$ be given. For any $q \in (0,1)$, there exists $\epsilon > 0$ such that if τ is any σ-near percolation in $\mathbb{Z}^d$ of parameter exceeding $1-\epsilon$, then there exists an independent percolation τ' of parameter exceeding q such that $\tau' \subseteq \tau$ almost surely.*

Proof: The statement of the lemma is implied by [18, Theorem 0.0(i)]. $\square$
Recall from [7, page 23] that the percolation probability $\theta(p)$ of a percolation of parameter p is given by $\mathbb{P}_p(|C(\mathbf{0})| = \infty)$, where $C(\mathbf{0})$ denotes the cluster of open sites containing the origin. We make use of the fact that

$$\theta(p) \to 1 \text{ as } p \to 1, \tag{2.31}$$

which is implied by [7, Theorem 8.8] (there, the result is being asserted for bond percolation, but the arguments used are valid for the current case of site percolation). By Lemma 2.6 and (2.31), we may choose $\delta > 0$ such that any $(2\rho+1)$-near percolation of parameter exceeding $1-\delta$ contains a percolation P whose parameter p is such that

$$\theta(p) > 1 - \epsilon. \tag{2.32}$$

We now fix $\ell \in \mathbb{N}$ such that $q_{\ell,\lambda,c,\rho} \geq 1 - \delta/2$. (We will also be requiring that ℓ is sufficiently high relative to λ, ρ and c, but we prefer to state the particular bounds that are needed as they arise.) By Lemma 2.5, we may find such a percolation P satisfying $P \subseteq \{\mathbf{a} \in \mathbb{Z}^d : \mathbf{a} \text{ is active}\}$, where, now that ℓ is fixed, we write active in place of ℓ-active for the rest of the proof. For any $n \in \mathbb{N}$, we write throughout the proof $n = F\ell + r$ with $F \in \mathbb{N}$ and $r \in \{0, \ldots, \ell - 1\}$ so that F implicitly depends on n. For each greedy lattice animal ξ in B_n, let W_ξ denote the collection of ℓ-boxes of the form $B_{\ell\mathbf{a},\ell}$ that are contained in B_n (i.e. $\mathbf{a} \in \{0, \ldots, F-1\}^d$) and which ξ intersects. On several later occasions, we will use the following definition and lemma.

Definition 2.5. *For P a percolation on $\mathbb{Z}^d$, we write $P_{F,C}$ for the largest connected component of $P \cap \{C, \ldots, F-1-C\}^d$.*

Lemma 2.7. *For any $j \in \mathbb{N}$, and P a percolation on $\mathbb{Z}^d$ of supercritical parameter $p > p_c$, we have that*

$$\liminf_{n\to\infty} \frac{|P_{n,j}|}{n^d} \geq \theta(p) \text{ almost surely.} \tag{2.33}$$

Proof: We prove the lemma in the case where $j = 0$, the other cases being no different. Let P_∞ denote the unique infinite component of P. Given $\alpha \in (0,1)$, let the event $Q_n(\alpha)$ be given by

$$\begin{aligned} Q_n(\alpha) \quad = \quad & \Big\{ \exists \text{ a connected component } C_n = C_n(\alpha) \text{ of } P \cap B_n \text{ such that} \\ & \text{if } D \subseteq P \cap B_n \text{ has radius exceeding } \alpha n, \text{ then } D \subseteq C_n. \Big\} \end{aligned} \tag{2.34}$$

(Recall that the radius of a connected set $C \subseteq \mathbb{Z}^d$ is the maximum over pairs of sites $\mathbf{x}, \mathbf{y} \in C$ of the minimal number of edges in a path in C from $\mathbf{x}$ to $\mathbf{y}$.) By (2.24) of [1], there exists, for each $\alpha \in (0,1)$, a constant $c(\alpha) > 0$ such that

$$\mathbb{P}\big(Q_n(\alpha)\big) \geq 1 - \exp\{-cn\}, \tag{2.35}$$

for all sufficiently high values of n. (Note that the arguments in [1] are performed for bond percolation. The authors note however that they may be applied to site percolation. Note also that their argument is applied for the choice $\alpha = 1/25$, but is valid for each $\alpha \in (0,1)$.) The Borel-Cantelli lemma applied to (2.35) implies that $Q_n(\alpha)$ occurs for all but finitely many n. We claim that, for each $\alpha \in (0,1)$,

$$P_\infty \cap \big\{ \lfloor 2\alpha n \rfloor, \ldots, n-1-\lfloor 2\alpha n \rfloor \big\}^d \subseteq C_n(\alpha), \tag{2.36}$$

for all n sufficiently high. To derive (2.36), consider $\mathbf{x} \in P_\infty \cap \big\{ \lfloor 2\alpha n, \ldots, n-1-\lfloor 2\alpha n \rfloor \big\}^d$. Note that the connected component of $P \cap B_n$ in which the site $\mathbf{x}$ lies has a radius of at least $\lfloor 2\alpha n \rfloor > \alpha n$. Thus, if $Q_n(\alpha)$ occurs, then $\mathbf{x} \in C_n$. Hence, we obtain (2.36) for high values of n.

We bound

$$|C_n| \geq \big| P_\infty \cap \big\{ \lfloor 2\alpha n \rfloor, \ldots, n-1-\lfloor 2\alpha n \rfloor \big\}^d \big| \geq \big| P_\infty \cap B_n \big| \, - \, 4d\alpha n^d,$$

so that

$$\liminf_{n\to\infty} n^{-d} |C_n(\alpha)| \geq \liminf_{n\to\infty} n^{-d} |P_\infty \cap B_n| \, - \, 4d\alpha = \theta(p) - 4d\alpha, \tag{2.37}$$

the latter an almost sure equality that is due to an application of the ergodic theorem to the process P. By the definition of the set $C_n(\alpha)$ appearing in (2.34), we have that $C_n(\alpha) = P_{n,0}$, provided that $P_{n,0}$ has radius at least αn. If $\alpha \in (0,1)$ is chosen to be small enough that $\theta(p) > 4d\alpha + 2\alpha^d$, then (2.37) implies that $|C_n(\alpha)| > \alpha^d n^d$ for high n, whence $|P_{n,0}| > \alpha^d n^d$ for such n. Noting that, for any finite connected set $B \subseteq \mathbb{Z}^d$,

$$B \subseteq \Big[\min_{\mathbf{x}\in B} \mathbf{x}_1, \max_{\mathbf{x}\in B} \mathbf{x}_1 \Big] \times \ldots \times \Big[\min_{\mathbf{x}\in B} \mathbf{x}_d, \max_{\mathbf{x}\in B} \mathbf{x}_d \Big],$$

and that $\max_{\mathbf{x}\in B} \mathbf{x}_i - \min_{\mathbf{x}\in B} \mathbf{x}_i \leq \mathrm{rad}(B)$ for each $i \in \{1,\ldots,d\}$, we find that $|B| \leq \mathrm{rad}(B)^d$. We deduce that the radius of $P_{n,0}$ exceeds αn, for high n, so that indeed $C_n(\alpha) = P_{n,0}$ for such n. Given that (2.37) holds for each $\alpha \in (0,1)$, we deduce that

$$\liminf_{n\to\infty} n^{-d} |P_{n,0}| \geq \theta(p),$$

as we sought. □

We define

$$\begin{aligned} E_1 \quad = \quad & \Big\{ \text{for some greedy lattice animal } \xi \text{ in } B_n, \\ & \qquad \text{there exist } \mathbf{a}_1, \mathbf{a}_2 \in P_{F, \lfloor \rho \rfloor + 1}, \text{ such that } B_{\ell \mathbf{a}_1, \ell} \in W_\xi \text{ and } B_{\ell \mathbf{a}_2, \ell} \notin W_\xi \Big\} \end{aligned}$$

and

(2.38)

$$E_2 = \Big\{ \text{for some greedy lattice animal } \xi \text{ in } B_n, \text{ we have that } W_\xi \cap \big\{ B_{\ell\mathbf{a},\ell} : \mathbf{a} \in P_{F,\lfloor\rho\rfloor+1} \big\} = \emptyset \Big\}.$$

We will now prove that the events E_1 and E_2 occur for finitely values of n almost surely. In the case of E_1, we firstly show that, for all n sufficiently large,

$$\big\{ L_n > \big(\log \ell \big)^\rho \big\} \cap E_1 = \emptyset. \tag{2.39}$$

To derive (2.39), suppose that the event on the left-hand-side occurs. The set $P_{F,\lfloor\rho\rfloor+1}$ being connected, we may suppose that $\mathbf{a}_1$ and $\mathbf{a}_2$ are adjacent. Let $\gamma_{\mathbf{a}_2}$ denote a lattice animal playing the role of $\gamma^\star$ in the condition $\mathbf{A}^\rho_{\lambda,c}$ that $B_{\ell\mathbf{a}_2,\ell}$ satisfies. We may locate a connected set $\phi \subseteq \xi$ satisfying $|\phi| = \lfloor (\log \ell)^\rho \rfloor + 1$ and $\phi \cap B_{\ell\mathbf{a}_1,\ell} \neq \emptyset$, because $L_n > (\log \ell)^\rho$. Requiring that $\ell \geq \lfloor (\log \ell)^\rho \rfloor + 1$, we see that $\phi \in B_{\ell\mathbf{a}_2,\ell}[2]$. The fact that $B_{\ell\mathbf{a}_2,\ell}$ satisfies condition $\mathbf{A}^\rho_{\lambda,c}$ implies that there exists a white path $\tau_{\gamma_{\mathbf{a}_2},\phi}$ of length at most $\rho\ell$ from some site of $\gamma_{\mathbf{a}_2}$ to some site of $\phi \subseteq \xi$. Consider the lattice animal $\xi^+ = \xi \cup \tau_{\gamma_{\mathbf{a}_2},\phi} \cup \gamma_{\mathbf{a}_2}$. Note firstly that ξ^+ is connected, because ξ and $\gamma_{\mathbf{a}_2}$ are, and $\tau_{\gamma_{\mathbf{a}_2},\phi}$ is a path between them. Secondly, note that $\xi^+ \subseteq B_n$: indeed, $\xi \subseteq B_n$, $\gamma_{\mathbf{a}_2} \subseteq B_{\ell\mathbf{a}_2,\ell} \subseteq B_n$, while $\tau_{\gamma_{\mathbf{a}_2},\phi}$ is a path of length at most $\rho\ell$, with $\tau_{\gamma_{\mathbf{a}_2},\phi} \cap B_{\ell\mathbf{a}_2,\ell} \neq \emptyset$. Since $\mathbf{a}_2 \in P_{F,\lfloor\rho\rfloor+1}$,

$$\ell_\infty\big(B_{\ell\mathbf{a}_2,\ell}, B_n^c\big) \geq \big(\lfloor\rho\rfloor + 1\big)\ell \geq \rho\ell,$$

and thus, $\tau_{\gamma_{\mathbf{a}_2},\phi} \subseteq B_n$.

We bound from below the weight of ξ^+:

$$\begin{aligned} S\big(\xi^+\big) &= S\big(\xi\big) + S\big(\gamma_{\mathbf{a}_2}\big) + S\Big(\tau_{\gamma_{\mathbf{a}_2},\phi} \setminus \big(\xi \cup \gamma_a\big)\Big) \\ &\geq S\big(\xi\big) + S\big(\gamma_{\mathbf{a}_2}\big) - \lambda\big|\tau_{\gamma_{\mathbf{a}_2},\phi}\big| \geq S\big(\xi\big) + S\big(\gamma_{\mathbf{a}_2}\big) - \lambda\rho\ell, \end{aligned} \tag{2.40}$$

where, in the equality, we used $\gamma_{\mathbf{a}_2} \cap \xi = \emptyset$, which follows from $\gamma_{\mathbf{a}_2} \subseteq B_{\ell\mathbf{a}_2,\ell} \notin W_\xi$. In the first inequality, the fact that the path $\tau_{\gamma_{\mathbf{a}_2},\phi}$ is white was used. From $S(\gamma_{\mathbf{a}_2}) \geq c\ell^d$ and (2.40), we deduce that $S(\xi^+) > S(\xi)$, provided that ℓ was chosen so that $\ell \geq (\lambda\rho/c)^{1/(d-1)}$. We have however shown that ξ^+ is a lattice animal in B_n, so this contradicts the fact that ξ is a greedy lattice animal in B_n. We have proved (2.39). That E_1 may occur for only finitely many values of n almost surely follows from (2.19).

It remains to rule out the implausible scenario that a greedy animal in B_n might fail to meet any ℓ-box $B_{\ell\mathbf{a},\ell}$ with $\mathbf{a} \in P_{F,\lfloor\rho\rfloor+1}$. To prove that E_2 may occur for only finitely many values of n almost surely, we firstly show, for any $\epsilon > 0$, for χ satisfying

$$\chi > (1+\epsilon)d/(d-1), \tag{2.41}$$

and for all n sufficiently high, that

$$\big\{L_n \geq \big(\log n\big)^\chi\big\} \cap \big\{\mathbf{x} \in B_n \implies \big|B_{\mathcal{L}}(\mathbf{x})\big| < \big(\log n\big)^{1+\epsilon}\big\} \cap E_2 = \emptyset. \tag{2.42}$$

To derive (2.42), suppose now that the event on its left-hand-side occurs. We will connect a greedy lattice animal ξ in B_n satisfying the condition in (2.38) to a weighty lattice animal Ψ in B_n formed from animals in boxes corresponding to active sites. To construct Ψ, note that for each pair of adjacent sites $\mathbf{a}_1, \mathbf{a}_2 \in P_{F,\lfloor\rho\rfloor+1}$, we may find a white path $\phi_{\mathbf{a}_1,\mathbf{a}_2}$ from $\gamma_{\mathbf{a}_1}$ to $\gamma_{\mathbf{a}_2}$ of length at most $\rho\ell$. This is because the condition $\mathbf{A}^\rho_{\lambda,c}$ satisfied by $B_{\ell\mathbf{a}_2,\ell}$ ensures that $|\gamma_{\mathbf{a}_2}| \geq (\log\ell)^\rho + 1$, so that the path $\phi_{\mathbf{a}_1,\mathbf{a}_2}$ may be found by putting $\gamma = \gamma_{\mathbf{a}_2}$ in the condition $\mathbf{A}^\rho_{\lambda,c}$ satisfied by $B_{\ell\mathbf{a}_1,\ell}$. We set

$$\Psi = \bigcup_{\mathbf{a} \in P_{F,\lfloor\rho\rfloor+1}} \gamma_{\mathbf{a}} \cup \bigcup_{\mathbf{a}_1,\mathbf{a}_2 \in P_{F,\lfloor\rho\rfloor+1} : |\mathbf{a}_1 - \mathbf{a}_2| = 1} \phi_{\mathbf{a}_1,\mathbf{a}_2}. \tag{2.43}$$

We now check that $\Psi \in \mathcal{A}_{B_n}$. It is connected, because each $\gamma_{\mathbf{a}}$ is, and to each adjacent pair $(\mathbf{a}_1, \mathbf{a}_2)$ of sites in the connected set $P_{F,\lfloor\rho\rfloor+1}$, there corresponds a path $\phi_{\mathbf{a}_1,\mathbf{a}_2}$ that joins $\gamma_{\mathbf{a}_1}$ and $\gamma_{\mathbf{a}_2}$. Note that for $\mathbf{a}_1, \mathbf{a}_2 \in P_{F,\lfloor\rho\rfloor+1}$, we have that $\ell_\infty(\phi_{\mathbf{a}_1,\mathbf{a}_2}, B_n^c) \geq \ell_\infty(B_{\ell\mathbf{a}_1,\ell}, B_n^c) - |\phi_{\mathbf{a}_1,\mathbf{a}_2}| \geq (\lfloor\rho\rfloor + 1)\ell - \rho\ell > 0$, so that $\phi_{\mathbf{a}_1,\mathbf{a}_2} \subseteq B_n$, and thus, $\Psi \subseteq B_n$.

Note also that

$$\big|\Psi\big| \geq \big|P_{F,\lfloor\rho\rfloor+1}\big| \geq F^d\big(1-\epsilon\big) \geq \frac{(n-\ell)^d}{\ell^d}(1-\epsilon), \tag{2.44}$$

the second inequality following for high choices of n from Lemma 2.7 and (2.32).

Let ξ be a greedy lattice animal in B_n. We are aiming to find a path from Ψ to ξ that is white except perhaps for its endpoints. We may thus assume that $\Psi \cap \xi = \emptyset$, the other case being trivial. Any set F properly separating Ψ and ξ in B_n satisfies

$$\begin{aligned} \big|F\big| &\geq \frac{1}{2d} \min\Big\{ \big|\xi\big|^{1-\frac{1}{d}}, \big|\Psi\big|^{1-\frac{1}{d}} \Big\} \\ &\geq \frac{1}{2d} \min\Big\{ \big(\log n\big)^{\chi(1-1/d)}, (1-\epsilon)^{1-1/d} \frac{(n-\ell)^{d-1}}{\ell^{d-1}} \Big\} \geq \big(\log n\big)^{1+\epsilon}, \end{aligned} \tag{2.45}$$

for high values of n. In the first inequality here, we made use of the second assertion in Lemma 2.3 (which requires that $\Psi \cap \xi = \emptyset$), while the second is valid by the occurrence of the first event on the left-hand-side of (2.42), and by (2.44). The third is due to (2.41). By the occurence of the second event in (2.42), and (2.45), no black $\mathcal{L}$-cluster properly separates ξ and Ψ in B_n. Applying the first part of Lemma 2.2 with the choices $C = \xi$, $D = \Psi$ and $E = \{\text{black sites}\} \setminus (\xi \cup \Psi)$, we deduce that the black sites do not properly separate ξ and Ψ in B_n, and thus, we may locate the desired path $T_{\mathbf{x},\mathbf{y}}$ in B_n from $\mathbf{x} \in \xi$ to $\mathbf{y} \in \Psi$ that is white with the possible exception of its endpoints.

We need a reasonably short white path from ξ to Φ. The path $T_{\mathbf{x},\mathbf{y}}$ in principle could be very long. We can make use of it, however, in constructing a short white path. To do this, consider for now the case where $\mathbf{x}$ and $\mathbf{y}$ are white. Let $\tau_{\mathbf{x},\mathbf{y}} = \big(\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r = \mathbf{y}\big)$ denote some path from $\mathbf{x}$ to $\mathbf{y}$ in B_n for which $r \leq dn$ and whose sites may or may not be white. We now modify $\tau_{\mathbf{x},\mathbf{y}}$ to form a white path σ from $\mathbf{x}$ to $\mathbf{y}$ in B_n such that

$$|\sigma| \leq dn(3^d - 1)(\log n)^{1+\epsilon}. \tag{2.46}$$

If $\tau_{\mathbf{x},\mathbf{y}}$ is white, we are done. Otherwise, let $r_1 \in \{0, \ldots, r-1\}$ denote the smallest value for which $\mathbf{x}_{r_1}$ is white and $\mathbf{x}_{r_1+1}$ is black. Note that $\mathbf{x}_{r_1} \in \partial_{vis(\mathbf{x}),n}\big(\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})\big)$: indeed, $(\mathbf{x}_{r_1}, \mathbf{x}_{r_1-1}, \ldots, \mathbf{x}_0 = \mathbf{x})$ is disjoint from $\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})$. We claim that there exists $r_2 \in \{r_1+2, \ldots, r\}$ for which

$$\mathbf{x}_{r_2} \in \partial_{vis(\mathbf{x}),n}\big(\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})\big). \tag{2.47}$$

For, taking $r_2 = 1 + \sup\big\{r' \in \{r_1+1, \ldots, r-1\} : \mathbf{x}_{r'} \in \mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})\big\}$, the path $(\mathbf{x}_{r_2}, \ldots, \mathbf{x}_r)$ is disjoint from $\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})$ and may be prefixed to the reversal of $T_{\mathbf{x},\mathbf{y}}$ (which is white and hence disjoint from $\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})$). The result is a path from $\mathbf{x}_{r_2}$ to $\mathbf{x}$ in B_n that does not intersect $\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})$. Thus, (2.47).

By the second assertion of Lemma 2.1, we may find a path $(\mathbf{x}_{r_1} = \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{r_3} = \mathbf{x}_{r_2}) \subseteq \partial_{vis(\mathbf{x}),n}\mathcal{B}_{n,\mathcal{L}}(\mathbf{x}_{r_1+1})$. The same path-altering procedure may be applied to

$$\big(\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{r_1} = \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{r_3} = \mathbf{x}_{r_2}, \ldots, \mathbf{x}_r = \mathbf{y}\big),$$

and then iterated. The effect of each iteration is to replace the passage of the original path through a black $\mathcal{L}$-cluster by one in its visible boundary with respect to $\mathbf{x}$. After at most $r \leq dn$ applications, the iteration produces a white path from $\mathbf{x}$ to $\mathbf{y}$ in B_n. The length of the path increases by at most $\max\{|\partial_{vis(\mathbf{x}),n}(B)| - 1 : B \text{ a black } \mathcal{L}\text{-cluster in } B_n\}$ at each step. Note that $|\partial_{vis(\mathbf{x}),n}(B)| \leq (3^d - 1)|B| \leq (3^d - 1)(\log n)^{1+\epsilon}$, the latter inequality by the occurence of the second event on the left-hand-side of (2.42). Thus, setting σ to be equal to the white path that the iteration produces, we have obtained (2.46). In the case where at least one of $\mathbf{x}$ and $\mathbf{y}$ is black, we may recolour them white for the course of the argument to produce a path σ which is white except for its endpoints $\mathbf{x}$ and $\mathbf{y}$.

We form the lattice animal $\Phi = \Psi \cup \sigma \cup \xi$. We showed after (2.43) that $\Psi \subseteq B_n$, from which $\Phi \subseteq B_n$ immediately follows. We compute

$$\begin{aligned} S(\Phi) &= S(\xi) + \sum_{\mathbf{a} \in P_{F,\lfloor \rho \rfloor + 1}} S(\gamma_{\mathbf{a}}) + S\Big(\Big(\sigma \cup \bigcup_{\mathbf{a}_1, \mathbf{a}_2 \in P_{F,\lfloor \rho \rfloor +1} : |\mathbf{a}_1 - \mathbf{a}_2| = 1} \phi_{\mathbf{a}_1, \mathbf{a}_2} \Big) \setminus \Big(\bigcup_{\mathbf{a} \in P_{F,\lfloor \rho \rfloor + 1}} \gamma_{\mathbf{a}} \cup \xi \Big) \Big) \\ &\geq S(\xi) + \big| P_{F,\lfloor \rho \rfloor + 1} \big| c \ell^d - \lambda \Big(|\sigma| + \rho \ell \Big| \Big\{ \{ \mathbf{a}_1, \mathbf{a}_2 \} : \mathbf{a}_1, \mathbf{a}_2 \in P_{F,\lfloor \rho \rfloor + 1}, |\mathbf{a}_1 - \mathbf{a}_2| = 1 \Big\} \Big| \Big) \\ &\geq S(\xi) + (n - \ell)^d (1 - \epsilon) c - \lambda \Big(dn (3^d - 1)(\log n)^{1+\epsilon} + d \big| P_{F,\lfloor \rho \rfloor + 1} \big| \rho \ell \Big) \\ &\geq S(\xi) + n^d \Big[(1-\epsilon) c - \frac{d \lambda \rho}{\ell^{d-1}} \Big] - C n^{d-1} \ell - \lambda d n (3^d - 1)(\log n)^{1+\epsilon}, \end{aligned} \tag{2.48}$$

where, in the equality, we used the fact that $\gamma_{\mathbf{a}} \cap \xi = \emptyset$ for each $\mathbf{a} \in P_{F,\lfloor \rho \rfloor + 1}$, which follows from $W_\xi \cap \{ B_{\ell \mathbf{a}, \ell} : \mathbf{a} \in P_{F,\lfloor \rho \rfloor + 1} \} = \emptyset$. Regarding the first inequality, note that, if either of the endpoints $\mathbf{x}$ and $\mathbf{y}$ of σ is black, then that endpoint lies in $\bigcup_{\mathbf{a} \in P_{F,\lfloor \rho \rfloor + 1}} \gamma_{\mathbf{a}} \cup \xi$. In the second inequality, we made use of (2.44) and (2.46). The third follows from the fact that $|P_{F,\lfloor \rho \rfloor + 1}| \leq F^d$. Provided that ℓ was chosen so that $\ell > \big(d \lambda \rho c^{-1} (1-\epsilon)^{-1} \big)^{1/(d-1)}$, the inequality (2.48) is admissible for at most finitely many values of $n \in \mathbb{N}$, because $S(\Phi) > S(\xi)$ would contradict ξ being a greedy lattice animal in B_n. Thus, (2.42).

We now check that the two events other than E_2 that appear on the left-hand-side of (2.42) may occur for only finitely many values of n. That $L_n \geq (\log n)^\chi$ for all high n is implied by (2.19). Note also that

$$\mathbb{P}\Big(\exists \, \mathbf{x} \in B_n : \big| \mathcal{B}_{\mathcal{L}}(\mathbf{x}) \big| > \big(\log n \big)^{1+\epsilon} \Big) \leq n^d \mathbb{P}\Big(\big| \mathcal{B}_{\mathcal{L}}(\mathbf{0}) \big| > \big(\log n \big)^{1+\epsilon} \Big) \leq n^d \exp \Big\{ - c \big(\log n \big)^{1+\epsilon} \Big\},$$

with the constant $c = c(\lambda) > 0$ by [8, Theorem 2.4.2] and (2.21). Since $\epsilon > 0$, we see from the Borel-Cantelli lemma that, for all high n, every black $\mathcal{L}$-cluster intersecting B_n has at most $(\log n)^{1+\epsilon}$ sites.

We conclude that the event E_2 may occur for only finitely many values of n almost surely. Note that $E_1^c \cap E_2^c$ occurs if and only if for each greedy lattice animal $\xi \in B_n$, $W_\xi \supset \big\{ B_{\ell \mathbf{a}, \ell} : \mathbf{a} \in P_{F,\lfloor \rho \rfloor + 1} \big\}$. This completes the proof of Theorem 1.1. □

3. Existence of G; proof of Theorem 1.2

In this section, we strengthen a result of [3], with hypotheses as general as those of that paper.

Proof of Theorem 1.2: We set $y = \liminf n^{-d} G_n$ and $y + E = \limsup n^{-d} G_n$. Note that $y, E \in \cap_{n \geq 1} \sigma\{ Y_m : m \geq n \}$, where Y_m is a vector whose components comprise $\big\{ X_{\mathbf{v}} : \mathbf{v} \in B_n - B_{n-1} \big\}$ in an arbitrary order. The family of random variables $\big\{ Y_n : n \in \mathbb{N} \big\}$ being independent, Kolmogorov's zero-one law [5, Theorem 1.8.1] implies that y and E are almost sure constants. It thus suffices for proving the theorem to show that $E > \epsilon > 0$ results in a contradiction. To this end, we aim to produce a box percolation whose members contain lattice animals whose weight per unit volume is close to the value y and which may be connected by paths of negligible weight. If the sidelength of the boxes is chosen to be large, then the percolation has a high density. As such, the animals lying in members of the largest connected component of the box percolation inside any very large box may be joined to form a well-spread lattice animal of weight per unit volume close to y. The assumption that $E > \epsilon$ should allow us to identify lattice animals whose weight per unit volume exceeds $y + \epsilon$, which may replace parts of the constructed animal, thereby increasing its weight per unit volume. These heavier animals must be found in a uniform fraction of space and be capable of being joined to nearby structure at negligible cost if a sufficient increase in weight is to result from the proposed modification.

By Lemma 2.4, we may and shall set $\lambda < \infty$, $2 < \rho < \infty$ and $c \in \big(y - \epsilon/5^{6d}, y\big)$ such that $\lim_{m \to \infty} q_{m,\lambda,c,\rho} = 1$. By the argument presented after (2.31) in the proof of Theorem 1.1, we may find $\ell \in \mathbb{N}$ so that there exists a percolation P whose parameter p satisfies

$$\theta(p) > 1 - \epsilon/(2^{8d} y) \tag{3.1}$$

and for which $P \subseteq \big\{ \mathbf{a} \in \mathbb{Z}^d : \mathbf{a} \text{ is active} \big\}$, where, now that ℓ is fixed, we write active in place of ℓ-active for the rest of the proof. (We also require that ℓ be chosen high relative to λ, ρ and also to ϵ. We state the precise bounds as each one arises.) We will use $n \in \mathbb{N}$ to denote the large scale in the proof, that of the patchwork of joined animals, and, similarly to the proof of Theorem 1.1, will write $n = F\ell + r$ with $F \in \mathbb{N}$ and $r \in \{0, \ldots, \ell - 1\}$.

By Lemma 2.7 and (3.1), we find that, for any given $C \in \mathbb{N}$, and all n (and thus F) sufficiently high,

$$\big| P_{F,C} \big| \geq \Big(1 - \frac{\epsilon}{2^{8d} y} \Big) F^d. \tag{3.2}$$

Each member of the ℓ-box percolation $\{ B_{\ell \mathbf{a}, \ell} : \mathbf{a} \in P \}$ satisfies condition $\mathbf{A}^{\rho}_{\lambda,c}$, and thus contains a lattice animal of weight exceeding $(y - \epsilon/5^{6d})\ell^d$ that may be connected to another such in an adjacent box. We will obtain a backdrop lattice animal by joining together such animals that lie in ℓ-boxes $B_{\ell \mathbf{a}, \ell}$ for sites $\mathbf{a}$ belonging to a large connected component of $P \cap B_F$. We now make precise the notion of the heavier animal instances of which we seek to stitch into this patchwork. The following definition is convenient.

Definition 3.1. *For any $m \in \mathbb{N}$, $m \geq \ell$ and an m-box $\Gamma = B_{\mathbf{x},m}$, we set, for $q \in \mathbb{N}$,*

$$\Gamma(\ell, q) = \bigcup \big\{ B_{\ell \mathbf{v}, \ell}[q] : \mathbf{v} \text{ such that } B_{\ell \mathbf{v}, \ell} \cap \Gamma \neq \emptyset \big\}$$

equal to the union of those ℓ-boxes whose sup-norm distance from some ℓ-box intersecting Γ is at most q. We also write

$$w_\Gamma = \big\{ \mathbf{a} \in \mathbb{Z}^d : B_{\ell \mathbf{a}, \ell} \cap \Gamma \neq \emptyset \big\}, \text{ and } D_\infty(w_\Gamma, q) = \big\{ \mathbf{v} \in \mathbb{Z}^d : l_\infty(\mathbf{v}, w_\Gamma) \leq q \big\},$$

so that $\Gamma(\ell, q) = \bigcup \big\{ B_{\ell \mathbf{v}, \ell} : \mathbf{v} \in D_\infty(w_\Gamma, q) \big\}$.

Definition 3.2. *For $m \geq \ell$, an m-box $\Gamma = B_{\mathbf{x},m}$ is said to be (c_1, λ, ρ)-high provided that*

- *there exists $\gamma^\star \in \mathcal{A}_\Gamma$ such that $S(\gamma^\star) = G_\Gamma \geq (y + c_1) m^d$ and $D(\gamma, \gamma^\star) \leq \rho m$ for all $\gamma \in \mathcal{A}_{\Gamma[2]}$ such that $|\gamma| \geq (\log m)^\rho$,*

- *any two sites* $\mathbf{v}_1, \mathbf{v}_2 \in D_\infty\big(w_\Gamma, \lfloor \log(m/\ell) \rfloor\big) \setminus D_\infty\big(w_\Gamma, \lfloor \log(m/\ell) \rfloor - 1\big)$ *that are connected by a path in* $P \cap D_\infty\big(w_\Gamma, \lfloor \log(m/\ell) \rfloor\big)$, *are connected by a path in* $P \cap \big(D_\infty\big(w_\Gamma, \lfloor \log(m/\ell) \rfloor\big) \setminus w_\Gamma\big)$.

We write 'high' for $(\epsilon, \lambda, \rho)$*-high.*

We now construct a disjoint collection of high boxes that fill out a uniform fraction of a large box in $\mathbb{Z}^d$.

Lemma 3.1. *For any* $m_1 \geq \ell$, *there exists* $m_2 \geq m_1$, *and a collection* κ_{m_1,m_2} *of high boxes in* $\mathbb{Z}^d$ *such that if* $\Gamma = B_{\mathbf{x},m} \in \kappa_{m_1,m_2}$, *then* $m \in \{m_1, \ldots, m_2\}$, *if* $\Gamma_1, \Gamma_2 \in \kappa_{m_1,m_2}$, *then* $\Gamma_1[1] \cap \Gamma_2[1] = \emptyset$, *while*

$$\liminf_m m^{-d} \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma \Big) \cap B_m \Big| > \frac{1}{2.7^d} \tag{3.3}$$

and

$$\limsup_m m^{-d} \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma \Big) \cap B_m \Big| \leq \frac{1}{3^d}. \tag{3.4}$$

Proof: We claim firstly that

$$\mathbb{P}\big(B_m \text{ is high for infinitely many } m \in \mathbb{N}\big) = 1. \tag{3.5}$$

To derive (3.5), note that the box B_m satisfies the first requirement in the definition of high provided that it satisfies condition $\mathbf{A}^\rho_{\lambda,y+\epsilon}$. The fact that $\limsup_m m^{-d} G_m = y + E > y + \epsilon$ implies that there are almost surely infinitely many values of $m \in \mathbb{N}$ for which there exists $\xi \in \mathcal{A}_{B_m}$ such that $S(\xi) = G_m \geq (y+\epsilon) m^d$. The proof of Lemma 2.4 may be applied to show that condition $\mathbf{A}^\rho_{\lambda,y+\epsilon}$ holds for all but finitely many of those m for which such ξ exists.

To handle the second requirement, suppose that for a pair of sites $\mathbf{v}_1, \mathbf{v}_2 \in D_\infty\big(w_{B_m}, \lfloor \log(m/\ell) \rfloor\big) \setminus D_\infty\big(w_{B_m}, \lfloor \log(m/\ell) \rfloor - 1\big)$, we may find a path $\big(\mathbf{v}_1 = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r = \mathbf{v}_2\big)$ with

$$\mathbf{x}_i \in P \cap D_\infty\big(w_{B_m}, \lfloor \log(m/\ell) \rfloor\big)$$

for $i \in \{1, \ldots, r\}$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are not connected by a path in $P \cap \big(D_\infty\big(w_{B_m}, \lfloor \log(m/\ell) \rfloor\big) \setminus w_{B_m}\big)$, then the path $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$ visits w_{B_m}, and we can set $r_1 = \inf\big\{i \in \{1, \ldots, r\} : \mathbf{x}_i \in w_{B_m}\big\}$ and $r_2 = \sup\big\{i \in \{1, \ldots, r\} : \mathbf{x}_i \in w_{B_m}\big\}$. The fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not connected by such a path implies that the sets $C = \{\mathbf{x}_1, \ldots, \mathbf{x}_{r_1-1}\}$ and $D = \{\mathbf{x}_{r_2+1}, \ldots, \mathbf{x}_r\}$ are separated by P^c in $D_\infty(w_{B_m}, \lfloor \log(m/\ell) \rfloor) \setminus w_{B_m}$. Put differently, the sets C and D are separated by $E = P^c \cup w_{B_m}$ in the box $B(w_{B_m}, \lfloor \log(m/\ell) \rfloor)$. Noting the pairwise disjointness of C,D and E, we may apply the first part of Lemma 2.2, to deduce that there is an $\mathcal{L}$-connected set $\chi \subseteq P^c \cup w_{B_m}$ that separates $\{\mathbf{x}_1, \ldots, \mathbf{x}_{r_1-1}\}$ and $\{\mathbf{x}_{r_2+1}, \ldots, \mathbf{x}_r\}$ in $D_\infty(w_{B_m}, \lfloor \log(m/\ell) \rfloor)$. Note that

$$\chi \cap \big(D_\infty(w_{B_m}, i) \setminus D_\infty(w_{B_m}, i-1)\big) \neq \emptyset \tag{3.6}$$

for each $i \in \big\{1, \ldots, \lfloor \log(m/\ell) \rfloor\big\}$. To see this, note that $l_\infty(\mathbf{x}_1, w_{B_m}) = \lfloor \log(m/\ell) \rfloor$ and $\mathbf{x}_{r_1} \in w_{B_m}$ imply that for any $i \in \big\{1, \ldots, \lfloor \log(m/\ell) \rfloor\big\}$, there exists $j_1 \in \{1, \ldots, r_1 - 1\}$ for which $\ell_\infty(\mathbf{x}_{j_1}, w_{B_m}) = i$. Similarly, there exists $j_2 \in \{r_2+1, \ldots, r\}$ for which $\ell_\infty(\mathbf{x}_{j_2}, w_{B_m}) = i$. Let $(\mathbf{x}_{j_1} = \mathbf{y}_1, \ldots, \mathbf{y}_{r_3} = \mathbf{x}_{j_2})$ denote a path from $\mathbf{x}_{j_1}$ to $\mathbf{x}_{j_2}$ in the connected set $D_\infty(w_{B_m}, i) \setminus D_\infty(w_{B_m}, i-1)$. This path must intersect χ by the separation property that χ satisfies. We have proved (3.6).

We have seen that if the second requirement for the box B_m to be high fails, then there exists $\mathbf{x} \in D_\infty(w_{B_m}, \lfloor \log(m/\ell) \rfloor)$ such that $|C_{P^c,\mathcal{L}}(\mathbf{x})| \geq \lfloor \log(m/\ell) \rfloor$, where $C_{P^c,\mathcal{L}}(\mathbf{x})$ denotes the $\mathcal{L}$-cluster of P^c that contains $\mathbf{x}$. By applying a union bound, we see that, for high values of m, this latter event has probability at most

$$\Big| D_\infty(w_{B_m}, \lfloor \log(m/\ell) \rfloor) \Big| \, \mathbb{P}\Big(|C_{P^c,\mathcal{L}}(\mathbf{0})| \geq \lfloor \log(m/\ell) \rfloor \Big). \tag{3.7}$$

Note that

$$\mathbb{P}\Big(|C_{P^c,\mathcal{L}}(\mathbf{0})| \geq \lfloor \log(m/\ell) \rfloor \Big) \leq \sum_{n=\lfloor \log(m/\ell) \rfloor}^{\infty} \sigma_n (1-p)^n ,$$

where σ_n denotes the number of $\mathcal{L}$-clusters containing $\mathbf{0}$ and having n sites. It is easy to show that $\big\{ \sigma_n : n \in \mathbb{N} \big\}$ grows exponentially. From this, we find that

$$\mathbb{P}\Big(|C_{P^c,\mathcal{L}}(\mathbf{0})| \geq \lfloor \log(m/\ell) \rfloor \Big) \leq \exp\Big\{ - c \log(m/\ell) \Big\},$$

where $c = c(p)$ is a positive constant satisfying $c(p) \to \infty$ as $p \uparrow 1$. For this reason, we may suppose that p has been chosen so close to one that the sequence in m of terms (3.7) is summable. (This lower bound on p is ensured by fixing ℓ at a high enough value, in the same way that (3.1) was obtained.) We deduce from the Borel-Cantelli lemma that the second condition in the definition of 'high' applies to all but finitely many of the boxes B_m almost surely. We have shown (3.5).

Given $m_1 \in \mathbb{N}$, there exists by (3.5) $m_2 \geq m_1$ for which

$$\mathbb{P}\big(B_m \text{ is high for some } m \in \{m_1, \ldots, m_2\} \big) > 1/2. \tag{3.8}$$

Let $\Delta = \big\{ \mathbf{v} \in \mathbb{Z}^d : B_{\mathbf{v},m} \text{ is high for some } m \in \{m_1, \ldots, m_2\} \big\}$. Let $\{\mathbf{z}_j : j \in \mathbb{N}\}$ denote an ordering of $\mathbb{N}^d$ that enumerates each shell $B_j \setminus B_{j-1}$ in turn. We claim that

$$\big\{ \mathbf{x} \in \Delta \big\} \in \Big\{ X_{\mathbf{v}} : \ell_\infty(\mathbf{v}, B_{\mathbf{x},m_2}) \leq \rho m_2 \Big\}, \tag{3.9}$$

provided that m_1 (and thus m_2) is high enough relative to ℓ. To verify (3.9), note that the first condition that defines a high box is determined by the values of these random variables by (2.29). The second is measurable with respect to the sigma-algebra $\sigma\big\{ \mathbf{a} \in \overline{P} \cap D_\infty\big(\omega_{B_{\mathbf{x},m_2}, \log(m_2/\ell)}\big) \big\}$. Recalling the propery (2.29) of measurability satisfies by the event $\mathbf{a} \in \overline{P}$, we find that this second condition is measurable with respect to $\sigma\big\{ X_{\mathbf{v}} : \ell_\infty(\mathbf{v}, B_{\mathbf{x},m_2}) \leq \ell\big(\log(m_2/\ell) + 1 + \rho \big\}$. A sufficiently high choice of m_1 relative to m_2 therefore ensures that (3.9) does indeed hold.

Setting $\hat{m} = \lfloor \rho m_2 \rfloor + 1$ and using the fact that $B_{\mathbf{x},\hat{m}} \cap B_{\mathbf{x}',\hat{m}} = \emptyset$ if $\|\mathbf{x} - \mathbf{x}'\| \geq \hat{m}$, it follows that the sequence of events $\big\{ \mathbf{a} + \hat{m}\mathbf{z}_j \in \Delta : j \in \mathbb{N} \big\}$ is independent, for any given $\mathbf{a} \in B_{\hat{m}}$. This sequence is identically distributed because the law of $\{X_{\mathbf{v}} : \mathbf{v} \in \mathbb{N}^d\}$ is translation invariant. As such, the strong law of large numbers [5, Theorem 1.7.1] may be applied to the sequence of random variables $\mathbb{1}\big\{ \mathbf{a} + \hat{m}\mathbf{z}_j \in \Delta : j \in \mathbb{N} \big\}$. By considering the sequence of partial sums corresponding to those j at which the enumeration of the set B_j is completed by $\mathbf{z}_j$, we deduce from (3.8) that

$$\text{the limit } \lim_m \frac{\big| B_m \cap \Delta \cap \{\mathbf{a} + \hat{m}\mathbb{N}^d\} \big|}{\big| B_m \cap \{\mathbf{a} + \hat{m}\mathbb{N}^d\} \big|} \text{ exists and exceeds } 1/2. \tag{3.10}$$

From (3.10) and the fact that $\big|B_m \cap \{\mathbf{a} + \hat{m}\mathbb{N}^d\}\big| = m^d \hat{m}^{-d} + \hat{m}^{1-d} O(m^{d-1})$, it follows that

$$\begin{aligned} \lim_m m^{-d} \big|B_m \cap \Delta\big| &= \lim_m m^{-d} \sum_{\mathbf{a} \in B_{\hat{m}}} \big|B_m \cap \Delta \cap \{\mathbf{a} + \hat{m}\mathbb{N}^d\}\big| \\ &= \lim_m \big[\hat{m}^{-d} + \hat{m}^{1-d} O(m^{-1})\big] \sum_{\mathbf{a} \in B_{\hat{m}}} \frac{\big|B_m \cap \Delta \cap \{\mathbf{a} + \hat{m}\mathbb{N}^d\}\big|}{\big|B_m \cap \{\mathbf{a} + \hat{m}\mathbb{N}^d\}\big|} > 1/2. \end{aligned} \tag{3.11}$$

For $\mathbf{x} \in \Delta$, let $\Gamma_{\mathbf{x}} = B_{\mathbf{x},m}$ with $m \in \{m_1, \ldots, m_2\}$ maximal such that $B_{\mathbf{x},m}$ is high. Let $\Delta' = \{\Gamma_{\mathbf{x}} : \mathbf{x} \in \Delta\}$. It remains to disjointify the collection of boxes Δ' while retaining enough of its members so that their union has positive density. To do this, enumerate

$$\Delta' = \{B_{\mathbf{x}_{m,j},m} : m \in \{m_1, \ldots, m_2\}, j \in \mathbb{N}\},$$

so that $\{\mathbf{x}_{m,j} : j \in \mathbb{N}\}$ is an ordering of those $\mathbf{x} \in \Delta$ for which $\Gamma_{\mathbf{x}}$ has sidelength m. We will iteratively examine the indices (m,j) that label members of Δ', admitting one at each step into a set of *accepted* indices $\mathbf{A}$ while at the same time placing others in a set of *rejected* indices $\mathbf{R}$. We will allow these symbols to denote those indices currently accepted or rejected at each step, without using further labelling. At the start, $\mathbf{A} = \mathbf{R} = \emptyset$. We begin by examining the indices $\{(m_2, j) : j \in \mathbb{N}\}$. At the first step, we put $(m_2, 1)$ in $\mathbf{A}$, and reject (put in $\mathbf{R}$) those (m,i) (except for $(m_2,1)$) for which $B_{\mathbf{x}_{m,i},m}[1] \cap B_{\mathbf{x}_{m_2,1},m_2}[1] \neq \emptyset$. At the generic step for boxes of sidelength m_2, we put (m_2, i) in $\mathbf{A}$, where $i \in \mathbb{N}$ is minimal for which (m_2,i) is not currently in $\mathbf{A} \cup \mathbf{R}$, and put in $\mathbf{R}$,

$$\big\{(m,j) : m \in \{m_1, \ldots, m_2\}, j \in \mathbb{N}, (m,j) \neq (m_2,i) : B_{\mathbf{x}_{m,j},m}[1] \cap B_{\mathbf{x}_{m_2,i},m_2}[1] \neq \emptyset\big\}.$$

(Note that some of these indices may have entered $\mathbf{R}$ at an earlier step). After at most countable many iterations, $(m_2, i) \in \mathbf{A} \cup \mathbf{R}$ for each $i \in \mathbb{N}$. We proceed to deal with those $\{(m,i) : i \in \mathbb{N}\}$ not yet in $\mathbf{A} \cup \mathbf{R}$, for $m = m_2 - 1$, then for each m in descending order until we finish with $m = m_1$. At the generic step when $m \in \{m_1, \ldots, m_2\}$ is some fixed value, (m,i) is admitted to $\mathbf{A}$ for the least i for which it is not already in $\mathbf{A} \cup \mathbf{R}$, while all those other (m',j) for which $B_{\mathbf{x}_{m,i},m}[1]$ intersects $B_{\mathbf{x}_{m',j},m'}[1]$ enter $\mathbf{R}$. At the end of the procedure, each (m,i) lies in $\mathbf{A} \cup \mathbf{R}$. We set $\kappa = \big\{B_{\mathbf{x}_{m,i},m} : (m,i) \in \mathbf{A}\big\}$, with $\mathbf{A}$ now denoting the collection of accepted indices at the end.

The first two properties asserted for the collection κ follow directly by its construction. We claim that

$$\bigcup_{\Gamma \in \Delta'} \Gamma \subseteq \bigcup_{\Gamma \in \kappa} \Gamma[3]. \tag{3.12}$$

To show (3.12), note that each index (m,i) of some box in Δ' is eventually either accepted and so lies in κ (so that the box certainly lies in the set on the right-hand-side of (3.12)), or is rejected by the algorithm. If it is rejected, consider the index (m',j) whose admission to $\mathbf{A}$ resulted in (m,i) joining $\mathbf{R}$. The key point is that this can only happen if $m' \geq m$, because all boxes in Δ' whose sidelength exceeds m' have been dealt with by the time (m',j) is admitted to $\mathbf{A}$. This fact along with the criterion for the rejection of (m,i), namely $B_{\mathbf{x}_{m',j},m'}[1] \cap B_{\mathbf{x}_{m,i},m}[1] \neq \emptyset$ imply that

$B_{\mathbf{x}_{m',j},m'}[3] \supseteq \mathbf{B}_{\mathbf{x}_{m,i},m}$. Thus, (3.12). We may now estimate

$$\begin{aligned} &\Big|\Big(\bigcup_{\Gamma\in\kappa}\Gamma\Big)\cap B_k\Big| \geq \Big|\bigcup\{\Gamma\in\kappa,\Gamma\subseteq B_k\}\Big| = \sum_{\Gamma\in\kappa,\Gamma\subseteq B_k}|\Gamma| = 7^{-d}\sum_{\Gamma\in\kappa,\Gamma\subseteq B_k}|\Gamma[3]| \\ \geq\ & 7^{-d}\Big|\bigcup\big\{\Gamma[3]:\Gamma\in\kappa,\Gamma\subseteq B_k\big\}\Big| \geq 7^{-d}\Big|\Big(\bigcup\big\{\Gamma[3]:\Gamma\in\kappa\big\}\Big)\cap\big\{\mathbf{x}\in B_k:\ell_\infty(x,B_k^c)\geq 4m_2\big\}\Big| \\ \geq\ & 7^{-d}\Big|\Big(\bigcup\big\{\Gamma:\Gamma\in\Delta'\big\}\Big)\cap\big\{\mathbf{x}\in B_k:\ell_\infty(\mathbf{x},B_k^c)\geq 4m_2\big\}\Big| \\ \geq\ & 7^{-d}\Big|\Delta\cap\big\{\mathbf{x}\in B_k:\ell_\infty(\mathbf{x},B_k^c)\geq 4m_2\big\}\Big| \\ \geq\ & 7^{-d}\big|\Delta\cap B_k\big| - 8dm_2 7^{-d}k^{d-1} \geq \big(1/2+o(1)\big)7^{-d}k^d \ - \ O(k^{d-1}), \end{aligned}$$

which implies (3.3). In the first equality, we used the fact that κ is a disjoint collection of sets. In the second equality, we used the fact that $|\gamma[3]| = 7^d|\gamma|$, which is valid for any box γ. In the second inequality, we used that if for any $B_{\mathbf{y},m}\in\kappa$, there exists $\mathbf{x}\in B_k\cap B_{\mathbf{y},m}$ such that $l_\infty(\mathbf{x},B_k^c)\geq 4m_2$, then $B_{\mathbf{y},m}[3]\subseteq B_k$, this following directly from $m\leq m_2$. In the third inequality, (3.12) was used, and, in the fourth, that $\mathbf{x}\in\Delta \implies \mathbf{x}\in\Gamma_x$. In the final inequality, (3.11) was used. The property (3.4) is derived by a similar estimate, that makes use of the disjointness of the collection $\big\{\Gamma[1]:\Gamma\in\kappa\big\}$ and the fact that $|\Gamma[1]| = 3^d|\Gamma|$ for any box Γ. $\square$

In the application, we insist that the value of $m_1\in\mathbb{N}$ satisfy the inequality

$$\frac{\rho\lambda}{m_1^{d-1}} < \frac{\epsilon}{7^{5d}}. \tag{3.13}$$

We will mention conditions stipulating that m_1 must be high relative to ℓ, ρ, λ and c as they arise. We will be joining animals lying in the boxes of κ to a structure of joined lattice animals that lies in ℓ-boxes. To do so, our first step is to make space in the fabric of joined lattice animals for the high boxes lying in κ. We now claim that, for n sufficiently high almost surely,

$$P_{F,C_1}\setminus\bigcup_{B_{\mathbf{x},m}\in\kappa} D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor\big) \tag{3.14}$$

lies in a connected component of $P_{F,\lfloor\rho\rfloor+1}\setminus\cup_{\Gamma\in\kappa}w_\Gamma$. where $C_1 = \lfloor m_2/\ell + 2\log(m_2/\ell)\rfloor + \lfloor\rho\rfloor + 2$. To show this, consider $\mathbf{a}_1,\mathbf{a}_2$ that lie in the set on the left-hand-side of (3.14). Let $\tau = \big(\mathbf{a}_1 = \mathbf{y}_1,\ldots,\mathbf{y}_r = \mathbf{a}_2\big) \subseteq P\cap\{C_1,\ldots,F-1-C_1\}^d$ be a path from $\mathbf{a}_1$ to $\mathbf{a}_2$. We aim to modify the path τ to find a new one from $\mathbf{a}_1$ to $\mathbf{a}_2$ that avoids each of the sets w_Γ for $\Gamma\in\kappa$ while staying in $P\cap\{\lfloor\rho\rfloor+1,\ldots,F-2-\lfloor\rho\rfloor\}^d$. We may assume then that τ does not itself satisfy these requirements, so that $\tau\cap w_{B_{\mathbf{x},m}}\neq\emptyset$ for some $B_{\mathbf{x},m}\in\kappa$. Set

$$r_1 = \inf\big\{i\in\{1,\ldots,r\}:\mathbf{y}_i\in D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor\big)\big\}$$

and

$$r_2 = \inf\big\{i\in\{r_1+1,\ldots,r\}:\mathbf{y}_i\notin D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor\big)\big\} - 1.$$

Note that $r_1 > 1$, because $\mathbf{a}_1\notin D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor\big)$, and that $\ell_\infty(\mathbf{y}_{r_i},w_{B_{\mathbf{x},m}}) = \lfloor\log(m/\ell)\rfloor$ for $i\in\{1,2\}$. The subpath $\big(\mathbf{y}_{r_1},\ldots,\mathbf{y}_{r_2}\big)$ is an excursion of τ inside the set $D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor\big)$. The m-box $B_{\mathbf{x},m}\in\kappa$ being high, we may, by the second requirement in the definition of 'high', find a path $\big(\mathbf{y}_{r_1} = \mathbf{z}_1,\ldots,\mathbf{z}_{r_3} = \mathbf{y}_{r_2}\big)$ such that

$$\mathbf{z}_i\in P\cap\Big(D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor\big)\setminus w_{B_{\mathbf{x},m}}\Big) \tag{3.15}$$

for $i \in \{1, \ldots r_3\}$. Note that

$$
\begin{aligned}
(3.16) \qquad \ell_\infty(\mathbf{z}_i, B_F^c) &\geq \inf\big\{\ell_\infty(\mathbf{z}, B_F^c) : \mathbf{z} \text{ such that } \ell_\infty(\mathbf{z}, w_{B_{\mathbf{x},m}}) = \lfloor \log(m/\ell) \rfloor\big\} \\
&\geq \ell_\infty(\mathbf{y}_{r_1}, B_F^c) - \big(\lfloor m/\ell \rfloor + 1 + 2\lfloor \log(m/\ell) \rfloor\big) \\
&\geq C_1 - \big(\lfloor m/\ell \rfloor + 1 + \lfloor \log(m/\ell) \rfloor\big) \geq \lfloor \rho \rfloor + 1,
\end{aligned}
$$

the third inequality valid by $\mathbf{y}_{r_1} \in \{C_1, \ldots, F - 1 - C_1\}^d$ and the fourth due to $m \leq m_2$. For any $B_{\mathbf{x}',m'} \in \kappa$ for which $(\mathbf{x}', m') \neq (\mathbf{x}, m)$, we have that

$$
(3.17) \qquad \mathbf{z}_i \in D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big) \subseteq D_\infty\big(w_{B_{\mathbf{x}',m'}}, \lfloor \log(m'/\ell) \rfloor\big)^c,
$$

where the containment follows from $B_{\mathbf{x},m}[1] \cap B_{\mathbf{x}',m'}[1] = \emptyset$ and a choice for m_1 that satisfies $m_1 \geq C\ell$. By (3.15),(3.16) and (3.17), we see that the path

$$
\big(\mathbf{a}_1 = \mathbf{y}_1, \ldots, \mathbf{y}_{r_1} = \mathbf{z}_1, \ldots, \mathbf{z}_{r_3} = \mathbf{y}_{r_2}, \ldots, \mathbf{y}_r = \mathbf{a}_2\big) \subseteq P \cap \big\{\lfloor \rho \rfloor + 1, \ldots, F - \lfloor \rho \rfloor - 2\big\}^d
$$

has removed any instance of a visit to $w_{B_{\mathbf{x},m}}$ during the excursion in $D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big)$ from $\mathbf{y}_{r_1}$ to $\mathbf{y}_{r_2}$ without introducing any new visits to

$$
\bigcup_{B_{\mathbf{x}',m'} \in \kappa} D_\infty\big(w_{B_{\mathbf{x}',m'}}, \lfloor \log(m'/\ell) \rfloor\big).
$$

We modify the path in such a way, for each example of an excursion into the set $D_\infty\big(w_{B_{\mathbf{x}',m'}}, \lfloor \log(m'/\ell) \rfloor\big)$ for any $B_{\mathbf{x}',m'} \in \kappa$. After a finite number of such alterations, we obtain a path ϕ from $\mathbf{a}_1$ to $\mathbf{a}_2$ in $P \cap \{\lfloor \rho \rfloor + 1, \ldots, F - \lfloor \rho \rfloor - 2\}^d$ that is disjoint from $\bigcup\big\{w_{B_{\mathbf{x}',m'}} : B_{\mathbf{x}',m'} \in \kappa\big\}$: indeed, any excursion of ϕ in a set $D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big)$ will not visit $w_{B_{\mathbf{x},m}}$, nor $D_\infty\big(w_{B_{\mathbf{x}',m'}}, \lfloor \log(m'/\ell) \rfloor\big) \supseteq w_{B_{\mathbf{x}',m'}}$ by construction. We have proved that the set in (3.14) lies in a connected component of $P_{F,\lfloor \rho \rfloor + 1} \setminus \cup_{\Gamma \in \kappa} w_\Gamma$, as we sought to do. We call this connected component the backdrop and denote it by $BD = BD(n, \ell)$.

We will require the following lower bound on $|BD|$:

$$
(3.18) \qquad \big|BD\big| \geq \big|P_{F,C_1}\big| - \Big(1 + \frac{\epsilon}{10^{10d} y}\Big)\ell^{-d}\Big|\Big(\bigcup_{\Gamma \in \kappa} \Gamma\Big) \cap B_n\Big|.
$$

To obtain this, note that

$$
\begin{aligned}
(3.19) \qquad |BD| &\geq |P_{F,C_1}| - \Big|\Big(\bigcup\Big\{D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big) : B_{\mathbf{x},m} \in \kappa\Big\}\Big) \cap P_{F,C_1}\Big| \\
&\geq |P_{F,C_1}| - \Big|\bigcup\Big\{D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big) : B_{\mathbf{x},m} \in \kappa, B_{\mathbf{x},m} \subseteq B_n\Big\}\Big|,
\end{aligned}
$$

the first inequality being due to the definition of BD. The second is valid because $m \leq m_2$ implies that $C_1 \geq \lfloor m/\ell \rfloor + 2 + 2\lfloor \log(m/\ell) \rfloor$, from which it follows that if $D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big) \cap \big\{C_1, \ldots, F - 1 - C_1\big\}^d \neq \emptyset$, then $D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big) \subseteq B_F$ and thus $B_{\mathbf{x},m} \subseteq B_n$ (for, $B_{\mathbf{x},m} \not\subseteq B_n \implies w_{B_{\mathbf{x},m}} \cap B_F^c \neq \emptyset$). Note that

$$
(3.20) \qquad \big|w_{B_{\mathbf{x},m}}\big| \geq \lfloor m/\ell \rfloor^d
$$

and

$$
(3.21) \qquad \big|w_{B_{\mathbf{x},m}}\big| \leq \big(\lfloor m/\ell \rfloor + 2\big)^d \leq \big(1 + \frac{\epsilon}{3 \cdot 10^{10d} y}\big)\frac{m^d}{\ell^d} = \big(1 + \frac{\epsilon}{3 \cdot 10^{10d} y}\big)\frac{|B_{\mathbf{x},m}|}{\ell^d},
$$

given that $m_1 \geq C(\epsilon, y)\ell$. We find that

$$\begin{aligned} \Big| D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell) \rfloor\big)\Big| &\leq \Big(\lfloor m/\ell \rfloor + 2 + 2\lfloor \log(m/\ell)\rfloor\Big)^d \\ &\leq \Big(1 + \frac{2(1+\log(m/\ell))}{\lfloor m/\ell \rfloor}\Big)^d \big|w_{B_{\mathbf{x},m}}\big| \\ &\leq \Big(1 + \frac{\epsilon}{3 \cdot 10^{10d} y}\Big)\big|w_{B_{\mathbf{x},m}}\big| \\ &\leq \Big(1 + \frac{\epsilon}{3 \cdot 10^{10d} y}\Big)^2 \frac{|B_{\mathbf{x},m}|}{\ell^d} \leq \Big(1 + \frac{\epsilon}{10^{10d} y}\Big)\frac{|B_{\mathbf{x},m}|}{\ell^d}, \end{aligned}$$

the second inequality by (3.20), the third by $m_1 \geq C(\epsilon, y)\ell$, the fourth due to (3.21) and the fifth from $\epsilon < 3 \cdot 10^{10d} y$. Thus,

$$\begin{aligned} &\sum_{B_{\mathbf{x},m} \in \kappa, B_{\mathbf{x},m} \subseteq B_n} \Big| D_\infty\big(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell)\rfloor\big)\Big| \\ \leq\ & \Big(1 + \frac{\epsilon}{10^{10d} y}\Big)\ell^{-d} \sum_{\Gamma \in \kappa, \Gamma \subseteq B_n} \big|\Gamma\big| \leq \Big(1 + \frac{\epsilon}{10^{10d} y}\Big)\ell^{-d}\Big|\Big(\bigcup_{\Gamma \in \kappa} \Gamma\Big) \cap B_n\Big|. \end{aligned} \tag{3.22}$$

By (3.19), (3.22) and the triangle inequality, follows (3.18).

We now define the lattice animal $\hat{\Psi}$ formed from animals in ℓ-boxes corresponding to active sites of the backdrop BD and into which we will stitch animals from high boxes in κ: let

$$\hat{\Psi} = \bigcup_{\mathbf{a} \in BD} \gamma_{\mathbf{a}} \cup \bigcup_{\mathbf{a}_1, \mathbf{a}_2 \in BD : |\mathbf{a}_1 - \mathbf{a}_2| = 1} \phi_{\mathbf{a}_1, \mathbf{a}_2}. \tag{3.23}$$

(Recall from after (2.39) that $\gamma_{\mathbf{a}}$ is the animal γ^* in the condition $\mathbf{A}^\rho_{\lambda,c}$ satisfied by $B_{\ell\mathbf{a},\ell}$, and, from before (2.43), that each λ-white path $\phi_{\mathbf{a}_1,\mathbf{a}_2}$ satisfies $|\phi_{\mathbf{a}_1,\mathbf{a}_2}| \leq \rho\ell$, and intersects each of $\gamma_{\mathbf{a}_1}$ and $\gamma_{\mathbf{a}_2}$.)

There may be a few high boxes of κ and contained in B_n whose greedy lattice animal (or animals) cannot be connected to $\hat{\Psi}$ in the intended way, if it so happens that $\hat{\Psi}$ does not reach into a neighbourhood of these high boxes that would allow the greedy animals therein to attach to $\hat{\Psi}$. In addition, when a path can be formed from inside the high box to $\hat{\Psi}$, we must ensure that the path stays in B_n, which amounts to insisting that the box be at a certain distance from the complement of B_n. We now define the set of high boxes in κ whose greedy lattice animal we will connect to $\hat{\Psi}$, bearing in mind these two requirements.

Definition 3.3. *Let the set of useful high boxes UH be given by*

$$\begin{aligned} UH \ =\ & \Big\{ B_{\mathbf{x},m} \in \kappa : B_{\mathbf{x},m} \subseteq \big\{\lfloor \rho m_2 \rfloor + 1, \ldots, n - \lfloor \rho m_2 \rfloor - 2\big\}^d, \\ & BD \cap \big(D_\infty(w_{B_{\mathbf{x},m}}, \lfloor m/(2\ell)\rfloor) \setminus D_\infty(w_{B_{\mathbf{x},m}}, \lfloor \log(m/\ell)\rfloor)\big) \neq \emptyset \Big\}. \end{aligned} \tag{3.24}$$

We now show that it is only a few boxes in κ contained in B_n that do not make it into UH. Specifically, we prove that, for all sufficiently high n,

$$\Big| \bigcup_{\Gamma \in UH} \Gamma \Big| \geq \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma \Big) \cap B_n \Big| - \frac{\epsilon n^d}{2^{8d-1} y}. \tag{3.25}$$

To this end, note that

$$\Big(\bigcup_{\Gamma\in\kappa}\Gamma\Big)\cap B_n \;\subseteq\; \Big(\bigcup_{\Gamma\in UH}\Gamma\Big)\cup\bigcup\Big\{\Gamma\in\kappa\setminus UH:\Gamma\subseteq\big\{\lfloor\rho m_2\rfloor+1,\ldots,n-\lfloor\rho m_2\rfloor-2\big\}^d\Big\}$$
$$\cup\Big\{\mathbf{x}\in B_n:\ell_\infty(\mathbf{x},B_n^c)\le\lfloor\rho m_2\rfloor+m_2\Big\}. \tag{3.26}$$

To show that the second set on this right-hand-side is small, we adopt a temporary notation, saying that the box $B_{\mathbf{x},m}$ is 'far from BD' if $B_{\mathbf{x},m}\in\kappa\setminus UH$ and $B_{\mathbf{x},m}\subseteq\big\{\lfloor\rho m_2\rfloor+1,\ldots,n-\lfloor\rho m_2\rfloor-2\big\}^d$. For such a box,

$$BD\cap\Big(D_\infty(w_{B_{\mathbf{x},m}},\lfloor m/(2\ell)\rfloor)\setminus D_\infty(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor)\Big)=\emptyset. \tag{3.27}$$

Note that, for $B_{\mathbf{x}',m'}\in\kappa$, $(\mathbf{x}',m')\neq(\mathbf{x},m)$,

$$D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor m/(2\ell)\rfloor\big)\subseteq\big\{\mathbf{a}\in\mathbb{Z}^d:B_{\ell\mathbf{a},\ell}\subseteq B_{\mathbf{x},m}[1]\big\}$$
$$\subseteq\;\big\{\mathbf{a}\in\mathbb{Z}^d:B_{\ell\mathbf{a},\ell}\subseteq B_{\mathbf{x}',m'}[1]\big\}^c\subseteq D_\infty\big(w_{B_{\mathbf{x}',m'}},\lfloor m'/(2\ell)\rfloor\big)^c, \tag{3.28}$$

the first and third containments requiring that $m_1\ge 4\ell$, since this ensures that any $m\ge m_1$ satisfies $\lfloor m/(2\ell)\rfloor+1\le m/\ell-1$. The second containment follows from $B_{\mathbf{x},m}[1]\cap B_{\mathbf{x}',m'}[1]=\emptyset$. We have that

$$D_\infty\big(w_{B_{\mathbf{x},m}},\lfloor m/(2\ell)\rfloor\big)\subseteq B_F. \tag{3.29}$$

Indeed,

$$\rho m_1\le\rho m_2\le\lfloor\rho m_2\rfloor+1\le\ell_\infty(B_{\mathbf{x},m},B_n^c)\le\ell_\infty(w_{B_{\mathbf{x},m}},B_F^c)\ell+2\ell,$$

so that (3.29) follows, given that m_1 may be chosen so that $m_1>2\ell/\rho$. We now show that the collection of sets

$$\Big\{P_{F,C_1},\Big(D_\infty(w_{B_{\mathbf{x},m}},\lfloor m/(2\ell)\rfloor)\setminus D_\infty(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor)\Big):\ B_{\mathbf{x},m}\text{ far from }BD\Big\} \tag{3.30}$$

are disjoint, with union contained in B_F. By (3.27),(3.28),(3.29) and $BD\subseteq B_F$, we find that (3.30) is true with P_{F,C_1} replaced by BD. However, if $\mathbf{y}\in P_{F,C_1}\cap\big(D_\infty(w_{B_{\mathbf{x},m}},\lfloor m/(2\ell)\rfloor)\setminus D_\infty(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor)\big)$ for some $B_{\mathbf{x},m}\in\kappa$, then, using (3.28) along with

$$D_\infty(w_{B_{\mathbf{x}',m'}},\lfloor\log(m'/\ell)\rfloor)\subseteq D_\infty(w_{B_{\mathbf{x}',m'}},\lfloor m'/(2\ell)\rfloor),$$

it follows that $\mathbf{y}$ belongs to the set in (3.14), so that $\mathbf{y}\in BD$. Thus, (3.30). We estimate

$$\Big|\bigcup\big\{\Gamma\text{ far from }BD\big\}\Big|=\sum_{\Gamma\text{ far from }BD}\big|\Gamma\big|\le\ell^d\sum_{\Gamma\text{ far from }BD}\big|w_\Gamma\big|$$
$$\le\;\ell^d\sum_{B_{\mathbf{x},m}\text{ far from }BD}\big|D_\infty(w_{B_{\mathbf{x},m}},\lfloor m/(2\ell)\rfloor)\setminus D_\infty(w_{B_{\mathbf{x},m}},\lfloor\log(m/\ell)\rfloor)\big|$$
$$\le\;\ell^d\big(F^d-|P_{F,C_1}|\big)\le\frac{\epsilon\ell^dF^d}{2^{8d}y}\le\frac{\epsilon n^d}{2^{8d}y}, \tag{3.31}$$

where the second inequality follows from the fact that for any $B_{\mathbf{x},m}\in\kappa$,

$$\big|D_\infty(w_{B_{\mathbf{x},m},\lfloor m/(2\ell)\rfloor})-D_\infty(w_{B_{\mathbf{x},m},\lfloor\log(m/\ell)\rfloor})\big|\ge\big(2m/\ell-3\big)^d-\big(m/\ell+2\log(m/\ell)+2\big)^d,$$

allied with (3.21), the inequality $\epsilon < 3 \cdot 10^{10d} y(2^d - 2)$ and the lower bound $m \geq m_1 \geq C\ell$. The third inequality in (3.31) follows from the claimed property of the collection in (3.30), and the fourth from (3.2). We may now bound

$$\begin{aligned} \Big| \bigcup_{\Gamma \in UH} \Gamma \Big| & \geq \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma \Big) \cap B_n \Big| - \frac{\epsilon n^d}{2^{8d} y} - \Big| \Big\{ \mathbf{x} \in B_n : \ell_\infty(\mathbf{x}, B_n^c) \leq \lfloor \rho m_2 \rfloor + m_2 \Big\} \Big| \\ & \geq \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma \Big) \cap B_n \Big| - \frac{\epsilon n^d}{2^{8d} y} - 2d \big(\lfloor \rho m_2 \rfloor + m_2 \big) n^{d-1} \\ & \geq \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma \Big) \cap B_n \Big| - \frac{\epsilon n^d}{2^{8d-1} y}, \end{aligned}$$

the first inequality valid by (3.26) and (3.31), the final one valid for all high n. We have obtained (3.25).

We need to connect greedy lattice animals in the boxes of UH to the backdrop animal $\hat{\Psi}$. For $\Gamma \in UH$, consider then $\gamma^* = \gamma^*_\Gamma \in \mathcal{A}_\Gamma$, provided by the first of the two conditions that the high box Γ satisfies. As $\Gamma \in UH$, there exists $\mathbf{a} \in BD$ for which $\ell_\infty(\mathbf{a}, w_\Gamma) \leq \lfloor m/(2\ell) \rfloor$ if Γ is of the form $B_{\mathbf{x},m}$. Any $\mathbf{y} \in \gamma_{\mathbf{a}} \subseteq B_{\ell \mathbf{a}, \ell}$ satisfies

$$\ell_\infty(\mathbf{y}, \Gamma) \leq \ell \big(\lfloor m/(2\ell) \rfloor + 1 \big) \leq m/2 + 2\ell \leq 3m/4, \tag{3.32}$$

given that $m \geq m_1 \geq 8\ell$. Note that, by the disjointness of $\{\gamma_{\mathbf{a}} : \mathbf{a} \in BD\}$, (3.18), (3.23), (3.2) and (3.4),

$$\big| \hat{\Psi} \big| \geq |BD| \geq \Big(1 - \frac{\epsilon}{2^{8d} y} \Big) F^d - 2 \cdot 3^{-d} \Big(1 + \frac{\epsilon}{10^{10d} y} \Big) (n/\ell)^d \geq \frac{n^d}{10 \ell^d}, \tag{3.33}$$

for high values of n, given that ϵ may be chosen so that $\epsilon < y \big(2^{-8d} + 2 \cdot 3^{-d} \cdot 10^{-10d} \big)^{-1} \big(9/10 - 2 \cdot 3^{-d} \big)$. Given that m is at most the fixed constant m_2, we may by (3.33) find a lattice animal $\chi \subseteq \hat{\Psi}$ with $|\chi| = \lfloor (\log m)^\rho \rfloor + 1$ and $\mathbf{y} \in \chi$. If $\mathbf{z} \in \chi$, then $\ell_\infty(\mathbf{z}, \Gamma) \leq \lfloor (\log m)^\rho \rfloor + \ell_\infty(\mathbf{y}, \Gamma) \leq m$, by (3.32) and the fact that we may choose m_1 high enough that for each $m \geq m_1$, $(\log m)^\rho \leq m/4$. Thus $\chi \in \Gamma[1]$. By the first condition that the high box Γ satisfies, we may locate a λ-white path $\hat{\phi}_\Gamma$ from a site of γ^* to one of χ, with $|\hat{\phi}_\Gamma| \leq \rho m$. We can now define the lattice animal, modified from $\hat{\Psi}$ in the way that we sought:

$$\Phi = \hat{\Psi} \cup \bigcup_{\Gamma \in UH} \big(\gamma^*_\Gamma \cup \hat{\phi}_\Gamma \big) \tag{3.34}$$

It remains to verify that Φ has the required properties. It is indeed a lattice animal, for each γ^*_Γ is connected to the animal $\hat{\Psi}$ by a path $\hat{\phi}_\Gamma$. We claim that $\Phi \subseteq B_n$. We may show that $\hat{\Psi} \subseteq B_n$, in the same way that we showed that $\Psi \subseteq B_n$ after (2.43). Note also that $\gamma^*_\Gamma \subseteq \Gamma \subseteq B_n$, because $\Gamma \in UH$. If $\mathbf{y} \in \hat{\phi}_{B_{\mathbf{x},m}}$, then

$$\ell_\infty(\mathbf{y}, B_{\mathbf{x},m}) \leq \ell_\infty(\mathbf{y}, \gamma^*_{B_{\mathbf{x},m}}) \leq |\hat{\phi}_{B_{\mathbf{x},m}}| \leq \rho m \leq \rho m_2, \tag{3.35}$$

the second inequality due to $\gamma^*_{B_{\mathbf{x},m}} \cap \hat{\phi}_{B_{\mathbf{x},m}} \neq \emptyset$. However,

$$B_{\mathbf{x},m} \subseteq UH \implies \ell_\infty(B_n^c, B_{\mathbf{x},m}) \geq \lfloor \rho m_2 \rfloor + 1. \tag{3.36}$$

From (3.35) and (3.36), we deduce that $\mathbf{y} \in B_n$. We have shown that $\Phi \subseteq B_n$. Note that

$$S(\Phi) = \sum_{\mathbf{a} \in BD} S(\gamma_{\mathbf{a}}) + \sum_{\Gamma \in UH} S(\gamma^*_\Gamma)$$
$$+ S\Big(\Big(\bigcup_{\Gamma \in UH} \hat{\phi}_\Gamma \cup \bigcup_{\mathbf{a}_1, \mathbf{a}_2 \in BD : |\mathbf{a}_1 - \mathbf{a}_2| = 1} \phi_{\mathbf{a}_1, \mathbf{a}_2}\Big) \setminus \Big(\bigcup_{a \in BD} \gamma_{\mathbf{a}} \cup \bigcup_{\Gamma \in UH} \gamma^*_\Gamma\Big)\Big), \tag{3.37}$$

since, for $\mathbf{a} \in BD$, $\gamma_{\mathbf{a}} \subseteq B_{\ell \mathbf{a}, \ell}$ is disjoint from any ℓ-box intersecting any $\Gamma \in \kappa$, and thus from each $\chi^*_\Gamma \subseteq \Gamma$, by the definition of BD. We bound

$$\sum_{\mathbf{a} \in BD} S(\gamma_{\mathbf{a}}) \geq |BD| c \ell^d \geq \Big(1 - \frac{\epsilon}{2^{8d} y}\Big)\Big(y - \frac{\epsilon}{5^{6d}}\Big) F^d \ell^d - \Big(1 + \frac{\epsilon}{10^{10d} y}\Big) y \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma\Big) \cap B_n \Big|, \tag{3.38}$$

where $S(\gamma_{\mathbf{a}}) \geq c\ell^d$ was used in the first inequality, the second due to (3.18), (3.2) and $y \geq c \geq y - \epsilon/(5^{6d})$. Note also that

$$\sum_{\Gamma \in UH} S(\gamma^*_\Gamma) \geq (y + \epsilon) \Big| \bigcup_{\Gamma \in UH} \Gamma \Big| \geq (y+\epsilon) \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma\Big) \cap B_n \Big| - (y+\epsilon) \frac{\epsilon n^d}{2^{8d-1} y}, \tag{3.39}$$

the first inequality following from the disjointness of the boxes $\Gamma \in UH$ and the definition of the animals γ^*_Γ, the second due to (3.25). Note that

$$\sum_{B_{\mathbf{x},m} \in UH} m \leq \frac{1}{m_1^{d-1}} \sum_{\Gamma \in UH} |\Gamma| \leq \frac{n^d}{m_1^{d-1}}, \tag{3.40}$$

because $m \geq m_1$ for each $B_{\mathbf{x},m} \in UH$, and the collection UH is disjoint, with its union contained in B_n. We find that

$$S\Big(\Big(\bigcup_{\Gamma \in UH} \hat{\phi}_\Gamma \cup \bigcup_{\mathbf{a}_1, \mathbf{a}_2 \in BD : |\mathbf{a}_1 - \mathbf{a}_2| = 1} \phi_{\mathbf{a}_1, \mathbf{a}_2}\Big) \setminus \Big(\bigcup_{a \in BD} \gamma_{\mathbf{a}} \cup \bigcup_{\Gamma \in UH} \gamma^*_\Gamma\Big)\Big)$$
$$\geq -\lambda \Big(\sum_{\Gamma \in UH} |\hat{\phi}_\Gamma| + \sum_{\mathbf{a}_1, \mathbf{a}_2 \in BD : |\mathbf{a}_1 - \mathbf{a}_2| = 1} |\phi_{\mathbf{a}_1, \mathbf{a}_2}|\Big)$$
$$\geq -\lambda \rho \Big(\sum_{B_{\mathbf{x},m} \in UH} m + \ell \Big| \big\{ \{\mathbf{a}_1, \mathbf{a}_2\} : \mathbf{a}_1, \mathbf{a}_2 \in BD, |\mathbf{a}_1 - \mathbf{a}_2| = 1 \big\} \Big| \Big)$$
$$\geq -\lambda \rho \big(n^d / m_1^{d-1} + d \ell F^d \big) \geq - \frac{\epsilon n^d}{7^{5d}} - d \lambda \rho \ell F^d, \tag{3.41}$$

the final inequality by (3.13). Substituting the bounds (3.38), (3.39) and (3.41) into (3.37) yields

$$S(\Phi) \geq \Big[y + \epsilon - \Big(1 + \frac{\epsilon}{10^{10d} y}\Big) y \Big] \Big| \Big(\bigcup_{\Gamma \in \kappa} \Gamma\Big) \cap B_n \Big|$$
$$+ \Big(1 - \frac{\epsilon}{2^{8d} y}\Big)\Big(y - \frac{\epsilon}{5^{6d}}\Big) F^d \ell^d - \frac{\epsilon}{2^{8d-1} y}(y+\epsilon) n^d - \frac{\epsilon}{7^{5d}} n^d - d\lambda\rho\ell F^d$$
$$\geq \Big[y + \epsilon - \Big(1 + \frac{\epsilon}{10^{10d} y}\Big) y \Big] \frac{n^d}{2 \cdot 7^d}$$
$$+ \Big(1 - \frac{\epsilon}{2^{8d} y}\Big)\Big(y - \frac{\epsilon}{5^{6d}}\Big)(n - \ell)^d - \frac{\epsilon}{2^{8d-1} y}(y+\epsilon) n^d - \frac{\epsilon}{7^{5d}} n^d - \frac{\epsilon}{3^{10d}} n^d, \tag{3.42}$$

the second inequality using (3.3) and the inequality $\ell > \big(3^{10d} d\lambda\rho\epsilon^{-1}\big)^{1/(d-1)}$ that we may require that ℓ satisfies. For large values of $n \in \mathbb{N}$, the dominant term in the last expression is the one in n^d, whose coefficient is bounded below by

$$y + \epsilon\Big(\frac{1}{2\cdot 7^d} - \frac{1}{2\cdot 7^d \cdot 10^{10d}} - \frac{1}{2^{8d}} - \frac{1}{2^{8d-1}} - \frac{1}{5^{6d}} - \frac{1}{7^{5d}} - \frac{1}{3^{10d}} - \frac{\epsilon}{2^{8d-1}y}\Big), \tag{3.43}$$

which strictly exceeds y, provided that $\epsilon < y$.

The lattice animal Φ may be formed for all sufficiently large n. We conclude that

$$\liminf_{n\to\infty} \frac{S(\Phi_n)}{n^d} > y,$$

which implies that

$$\liminf_{n\to\infty} \frac{G_n}{n^d} > y,$$

an inconsistency which completes the proof. □

4. Proof of Theorem 1.3

We require a lemma.

Lemma 4.1. *Let P denote a percolation of parameter $p \in (p_c, 1]$. For any $C \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that*

$$\bigcap_{n \geq n_0} P_{n,C} \neq \emptyset,$$

where the sets $P_{n,C}$ were specified in Definition 2.5.

Proof. It follows from [7, Theorem 7.2] and the assumption that $p > p_c$ that there exists almost surely an infinite cluster P_∞^+ of the process $P \cap \mathbb{Z}_+^d$, where $\mathbb{Z}_+^d = \big\{\mathbf{v} \in \mathbb{Z}^d : \mathbf{v}_i \geq 0,\, i \in \{1,\ldots,d\}\big\}$. Let $\mathbf{x} \in P_\infty^+$. Note that the connected component of $P \cap B_n$ in which the site $\mathbf{x}$ lies has radius at least $n - \|\mathbf{x}\|$. Note that, for any $\alpha \in (0,1)$, if the event $Q_n(\alpha)$ defined in (2.34) occurs, and $n > (1-\alpha)^{-1}\|\mathbf{x}\|$, then $\mathbf{x}$ lies in the connected set $C_n(\alpha)$, also defined in (2.34). Recall from the proof of Lemma 2.7 that, if $\alpha \in (0,1)$ is small enough that $\theta(p) > 2\alpha^d + 4d\alpha$, then $C_n(\alpha) = P_{n,0}$ for high values of n. Recalling also that $Q_n(\alpha)$ occurs for all but finitely many n almost surely, we deduce that $\mathbf{x} \in P_{n,0}$ for all high choices of n. The statement of the lemma for a positive value of C is obtained by translating the process P by the vector $(-C,\ldots,-C)$, and applying the result for $C = 0$. □

Proof of Theorem 1.3: Given $\epsilon > 0$, let $C, \ell \in \mathbb{N}$ and the percolation P be those to which the statement of Theorem 1.1 refers. By Lemma 4.1, there exists $F_0 \in \mathbb{N}$ such that we may choose $\mathbf{v} \in \bigcap_{F \geq F_0} P_{F,C}$. Let $\xi_n \in \mathcal{A}_{B_n}$ satisfy $S(\xi_n) = G_n$ and $|\xi_n| = L_n$. Provided that $n \geq F_0\ell$ is also chosen to be so high that we may apply Theorem 1.1 to $\xi_n \subseteq B_n$, we find that $\xi_n \cap B_{\ell\mathbf{v},\ell} \neq \emptyset$. Let $\tau = (\tau_{\mathbf{0}},\ldots,\tau_{\mathbf{r}})$ denote a path in $\mathbb{Z}_+^d$ such that $\mathbf{0} \in \tau$ and $B_{\ell\mathbf{v},\ell} \subseteq \tau$. Let $V = \min\big\{S(\tau^s) : \tau^s = (\tau_{\mathbf{0}},\ldots,\tau_{\mathbf{s}}),\, s \in \{0,\ldots,r\}\big\}$ be equal to the minimal weight of any initial subpath of τ. For each sufficiently high n, we may choose $s(n) \in \{0,\ldots r\}$ such that $\tau^{s(n)} \subseteq B_n \setminus \xi_n$ and $\tau^{s(n)} \cap \partial\xi_n \neq \emptyset$. Note that

$$N_{|\xi_n|+|\tau|} \geq S\big(\xi_n \cup \tau^{s(n)}\big) = S\big(\xi_n\big) + S\big(\tau^{s(n)}\big) \geq G_n + V. \tag{4.1}$$

Given that $|\xi_n| = L_n$, we find from (2.19) and [3, Theorem 2.1] that

$$N_{|\xi_n|+|\tau|} \leq (N+\epsilon)(L_n + |\tau|), \tag{4.2}$$

for all n sufficiently high. From (4.1) and (4.2), we deduce that

$$G_n \leq \big(N+\epsilon\big)L_n + \big(N+\epsilon\big)|\tau| - V. \tag{4.3}$$

Given that $\epsilon > 0$ is arbitrary, and that the path τ is fixed, we obtain, by taking a liminf of the n^{-d}-th multiple of (4.3), the inequality $G \leq NL$ that we sought. □

5. Critical behaviour, proof of Theorem 1.4

We aim to prove that the quantity N is positive under the assumption that, for some $\epsilon > 0$,

$$\limsup_{n\to\infty} n^{-1}(\log n)^{-\frac{d}{d-1}-\epsilon} G_n > 0 \tag{5.1}$$

with positive probability. This limsup is non-random, similarly to $\limsup n^{-d}G_n$, as explained at the beginning of the proof of Theorem 1.2. Hence, the hypothesis (5.1) allows us to fix $\delta > 0$ for which

$$\limsup_{n\to\infty} n^{-1}(\log n)^{-\frac{d}{d-1}-\epsilon} G_n > \delta \ \text{ almost surely.} \tag{5.2}$$

Let $\lambda_0 = \inf\{\lambda \in \mathbb{R} : \mathbb{P}(X_{\mathbf{0}} \geq -\lambda) > p_c\}$. Recall from after (2.21) that, for $\lambda > \lambda_0$, we denote by $\mathcal{W}$ the unique infinite component of λ-white sites in $\mathbb{Z}^d$.

Definition 5.1. *For $\lambda > \lambda_0$, $\rho > 0$, $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{Z}^d$, the event $E(\mathbf{x}, n, \lambda, \rho)$ occurs if there exists $\gamma \in \mathcal{A}_{B_{\mathbf{x},n}}$ such that $S(\gamma) = G_{B_{\mathbf{x},n}} > \delta n(\log n)^{d/(d-1)+\epsilon}$, $|\gamma| = L_{B_{\mathbf{x},n}} > \delta(\log n)^{d/(d-1)+\epsilon}$, and a site $\mathbf{v} \in \gamma \cap \mathcal{W}$ satisfying $D(\mathbf{u},\mathbf{v}) \leq \rho \ell_\infty(\mathbf{u},\mathbf{v})$ for each $\mathbf{u} \in \mathcal{W} \cap B_{\mathbf{x},n}[1]^c$.*

Lemma 5.1. *For λ, ρ sufficiently high,*

$$\begin{aligned} &\big\{G_n > \delta n(\log n)^{\frac{d}{d-1}+\epsilon} \text{ occurs for infinitely many } n\big\} \\ = \ &\big\{E(\mathbf{0}, n, \lambda, \rho) \text{ occurs for infinitely many } n\big\}, \end{aligned}$$

up to a set of measure zero.

Proof: We must show that, for high enough values of n, $G_n > \delta n(\log n)^{d/(d-1)+\epsilon}$ implies the occurrence of $E(\mathbf{0}, n, \lambda, \rho)$ for given choices of λ and ρ. As noted before (2.19), $X_{\mathbf{v}} \geq \|\mathbf{v}\|$ for at most finitely many $\mathbf{v} \in \mathbb{Z}^d$ almost surely, so that $G_n > \delta n(\log n)^{\frac{d}{d-1}+\epsilon}$ implies that

$$L_n > \delta(\log n)^{\frac{d}{d-1}+\epsilon} \tag{5.3}$$

for high n. It follows from (2.25), written with ρ' in place of ρ, and the Borel-Cantelli lemma, that for any $\rho' > d/(d-1)$, and for all n sufficiently high, each $\gamma \in \mathcal{A}_{B_n}$ satisfying $|\gamma| \geq (\log n)^{\rho'}$ intersects $\mathcal{W}$. (Note that we require λ has chosen high enough that (2.25) may be applied). By (5.3), each greedy lattice animal in B_n intersects $\mathcal{W}$ for all high n. Let γ be a greedy lattice animal in B_n, with n chosen to be high enough that we may locate a site $\mathbf{v} \in \mathcal{W} \cap \gamma$. If a site $\mathbf{u} \in B_n[1]^c \cap \mathcal{W}$ satisfies $D(\mathbf{v},\mathbf{u}) \geq \rho \ell_\infty(\mathbf{v},\mathbf{u})$, then $D(\mathbf{v},\mathbf{u}) > (\rho/4)\|\mathbf{u}\|$, since, setting $\mathbf{c} = (\lfloor n/2 \rfloor, \ldots, \lfloor n/2 \rfloor)$,

$$\begin{aligned}\ell_\infty(\mathbf{u},\mathbf{v}) &= \ell_\infty(\mathbf{u}-\mathbf{c},\mathbf{v}-\mathbf{c}) \geq \|\mathbf{u}-\mathbf{c}\| - \|\mathbf{v}-\mathbf{c}\| \geq \|\mathbf{u}-\mathbf{c}\| - \lfloor n/2 \rfloor - 1 \\ &\geq \frac{2}{3}\|\mathbf{u}-\mathbf{c}\| - 1 \geq \frac{2}{3}\big(\|\mathbf{u}\| - \|\mathbf{c}\|\big) - 1 \geq \frac{1}{3}\|\mathbf{u}\| - 1,\end{aligned}$$

the third inequality valid by $\|\mathbf{u}-\mathbf{c}\| \geq 3n/2$, and the fifth by $\|\mathbf{u}\| \geq n \geq 2\|\mathbf{c}\|$. By [3, Lemma 2.14], with $\rho(p,d)$ set equal to 4ρ as it is here, and a union bound, the probability that such a site $\mathbf{u}$ exists is at most $\exp -cn$, for some positive constant c. The Borel-Cantelli lemma implies that each $\mathbf{u} \in B_n[1]^c \cap \mathcal{W}$ satisfies $D(\mathbf{u},\mathbf{v}) \leq \rho \ell_\infty(\mathbf{u},\mathbf{v})$, provided that n is high enough. We have shown that $G_n > \delta n (\log n)^{\frac{d}{d-1}+\epsilon}$ implies the occurence of $E(\mathbf{0}, n, \lambda, \rho)$ for high values of n, as required. □

Defining the event $D(\mathbf{x}, m, \lambda, \rho, C)$, for $\mathbf{x} \in \mathbb{Z}^d$ and $C > 0$, according to

$$\begin{aligned}D(\mathbf{x}, m, \lambda, \rho, C) &= \big\{\exists \gamma \in \mathcal{A}_{B_{\mathbf{x},m}}, \mathbf{v} \in \gamma : \mathbf{v} \text{ is } \lambda\text{-white}, \\ &\qquad S(\gamma) \geq Cm, D(\mathbf{v},\mathbf{u}) \leq \rho m \text{ for each corner } \mathbf{u} \text{ of } B_{\mathbf{x},m}\big\},\end{aligned}$$

we will now show that for any $\epsilon_0 > 0$, we may choose λ, ρ and C sufficiently high that

$$\mathbb{P}\big(D(\mathbf{0}, m, \lambda, \rho, C)\big) \geq 1 - \epsilon_0\,, \tag{5.4}$$

for high values of m. Let $c_1 \in (0,\infty)$ and $N_0 \in \mathbb{N}$ be chosen so that, for $n \geq N_0$,

$$\big(\log n\big)^{\frac{d}{d-1}+\epsilon} > \frac{3\big(2c_1 + (2d-1)\lambda\rho\big)}{\delta\big(1 - \epsilon_0/2\big)}. \tag{5.5}$$

By Lemma 5.1 and (5.2), we may fix $N_1 > N_0$ for which

$$\mathbb{P}\big(E(\mathbf{0}, n, \lambda, \rho) \text{ occurs for some } n \in \{N_0, \ldots, N_1\}\big) > 1 - \epsilon_0^2/4. \tag{5.6}$$

Declare any site $\mathbf{x} \in \mathbb{Z}^d$ to be full if the event $E(\mathbf{x}, n, \lambda, \rho)$ occurs for some $n \in \{N_0, \ldots, N_1\}$. To any full site $\mathbf{x}$, we may associate a lattice animal $\gamma_{\mathbf{x}}$, a site $\mathbf{v}_{\mathbf{x}} \in \gamma_{\mathbf{x}}$ and the box $\Gamma_{\mathbf{x}} = B_{\mathbf{x},n_{\mathbf{x}}}$, these objects arising from the definition of the event $E(\mathbf{x}, n_{\mathbf{x}}, \lambda, \rho)$, for the minimal $n_{\mathbf{x}} \in \{N_0, \ldots, N_1\}$ for which this event occurs. Allowing $\mathbf{e}_1$ to denote the unit vector $(1, 0, \ldots, 0)$, we set $\mathbf{x}_j = j\mathbf{e}_1$, for $j \in \mathbb{N}$. We now form a subsequence $\{\mathbf{y}_j : j \in \mathbb{N}\}$ of the sequence $\{\mathbf{x}_j : j \in \mathbb{N}\}$. The first element $\mathbf{y}_1$ is taken to be $\mathbf{x}_j$, where j is the lowest natural number for which $\mathbf{x}_j$ is a full site. Having constructed an initial segment of the $\mathbf{y}$-sequence, $\{\mathbf{y}_j : j \in \{1, \ldots, K\}\}$, say, we set $\mathbf{y}_{K+1}$ equal to the lowest-labelled site in the $\mathbf{x}$-sequence which is full and has $\mathbf{e}_1$-coordinate exceeding that of any site lying in the box $\Gamma_{\mathbf{y}_K}[1]$.

Noting that $\mathbf{v}_{\mathbf{y}_{i+1}} \notin \Gamma_{\mathbf{y}_i}[1]$, it follows from the definition of the event $E(\mathbf{y}_i, n_{\mathbf{y}_i}, \lambda, \rho)$ that we may join $\mathbf{v}_{\mathbf{y}_i}$ and $\mathbf{v}_{\mathbf{y}_{i+1}}$ by a path τ_i in $\mathcal{W}$ of length at most $\rho \ell_\infty(\mathbf{v}_{\mathbf{y}_i}, \mathbf{v}_{\mathbf{y}_{i+1}})$. For each $J \in \mathbb{N}$, form the animal

$$\kappa_J = \gamma_{\mathbf{y}_1} \cup \tau_1 \cup \gamma_{\mathbf{y}_2} \cup \tau_2 \cup \ldots \cup \tau_{J-1} \cup \gamma_{\mathbf{y}_J}.$$

Let ξ be the collection of sites $\mathbf{x}_i$ that are not full and that lie between $\Gamma_{\mathbf{y}_j}[1]$ and $\mathbf{y}_{j+1}$ for some $j \in \mathbb{N}$, or before $\mathbf{y}_1$. Writing $H = \big\{\frac{|\xi \cap \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}|}{m} > \epsilon_0/2\big\}$, we claim that, for any $m \in \mathbb{N}$,

$$\mathbb{P}\big(H^c\big) > 1 - \epsilon_0/2. \tag{5.7}$$

To see this, we perform an experiment in which we sample $z \in \{1,\ldots,m\}$ uniformly at random, and ask whether the site $\mathbf{x}_z$ is full. If $\mathbb{P}(H^c) \leq 1-\epsilon_0/2$, then

$$\mathbb{P}\big(\mathbf{x}_z \text{ is not full}\big) \geq \mathbb{P}\big(\mathbf{x}_z \text{ is not full}\big|H\big)\mathbb{P}(H) \geq \epsilon_0^2/4;$$

however, $\mathbb{P}(\mathbf{x}_z$ is not full) is the probability that a given site is not full, contradicting (5.6) and establishing (5.7).

For $m \in \mathbb{N}$, let $J(=J(m))$ be maximal such that $\Gamma_{\mathbf{y}_J}[1]$ has maximum $\mathbf{e}_1$-co-ordinate at most $m-1$. Let us estimate the weight $S(\kappa_J)$ of the animal κ_J, for fixed m. The animals γ_j are disjoint for distinct j, and the paths τ lie in $\mathcal{W}$. Thus,

$$S(\kappa_J) \geq \sum_{j=1}^{J} S(\gamma_{\mathbf{y}_j}) - \lambda \sum_{j=1}^{J-1} |\tau_j|. \tag{5.8}$$

In bounding the first term on the right-hand-side of (5.8), note that, for $j \in \{1,\ldots,J\}$, we have that

$$S\big(\gamma_{\mathbf{y}_j}\big) \geq \delta n_{\mathbf{y}_j} (\log n_{\mathbf{y}_j})^{\frac{d}{d-1}+\epsilon}.$$

Since $n_{\mathbf{y}_i} \geq N_0$ for any such j, from (5.5), it follows that

$$\sum_{j=1}^{J} S\big(\gamma_{\mathbf{y}_j}\big) \geq \frac{3\big(2c_1 + (2d-1)\rho\lambda\big)}{1-\epsilon_0/2} \sum_{j=1}^{J} n_{\mathbf{y}_j}. \tag{5.9}$$

To bound from below the quantity $\sum_{j=1}^{J} n_{\mathbf{y}_J}$, note the following inclusion:

$$\{\mathbf{x}_0,\ldots,\mathbf{x}_{m-1}\} \subseteq \Big(\bigcup_{j=1}^{J} \Gamma_{\mathbf{y}_j}[1]\Big) \cup \xi \cup R, \tag{5.10}$$

where the set R denotes the final $2N_1 - 1$ sites of the interval $\{\mathbf{x}_1,\ldots,\mathbf{x}_{m-1}\}$, and appears because of the possibility that the site $\mathbf{y}_{J+1}$ lies in this interval. From (5.10), it follows that, on the event H^c,

$$\sum_{j=1}^{J} n_{\mathbf{y}_j} \geq \frac{m(1-\epsilon_0/2)}{3} - \frac{2N_1 - 1}{3}. \tag{5.11}$$

We must also bound from above the quantity $\sum_{j=1}^{J-1} \big|\tau_j\big|$. Note that

$$\begin{aligned}
\sum_{i=1}^{J-1} \ell_\infty\big(\mathbf{v}_{\mathbf{y}_i}, \mathbf{v}_{\mathbf{y}_{i+1}}\big) &\leq \sum_{i=1}^{J-1}\sum_{l=1}^{d} \big|\mathbf{v}^l_{\mathbf{y}_i} - \mathbf{v}^l_{\mathbf{y}_{i+1}}\big| \\
&\leq \sum_{i=1}^{J-1} \big|\mathbf{v}^1_{\mathbf{y}_i} - \mathbf{v}^1_{\mathbf{y}_{i+1}}\big| + \big(d-1\big) \sum_{i=1}^{J-1} \max\big\{n_{\mathbf{y}_i}, n_{\mathbf{y}_{i+1}}\big\} \\
&\leq \sum_{i=1}^{J-1} \big|\mathbf{v}^1_{\mathbf{y}_i} - \mathbf{v}^1_{\mathbf{y}_{i+1}}\big| + 2\big(d-1\big) \sum_{i=1}^{J} n_{\mathbf{y}_i} \leq \big(2d-1\big)m,
\end{aligned} \tag{5.12}$$

where, in the second inequality, we used the fact that, for each $i \in \mathbb{N}$ and $l \in \big\{2,\ldots,d\big\}$,

$$\big|\mathbf{v}^l_{\mathbf{y}_i} - \mathbf{v}^l_{\mathbf{y}_{i+1}}\big| \leq \max\big\{n_{\mathbf{y}_i}, n_{\mathbf{y}_{i+1}}\big\},$$

while in the fourth, we used the bounds

$$\sum_{i=1}^{J-1} \big| \mathbf{v}^1_{\mathbf{y}_i} - \mathbf{v}^1_{\mathbf{y}_{i+1}} \big| \leq m, \text{ and } \sum_{i=1}^{J} n_{\mathbf{y}_i} \leq m.$$

(In the first of these, we used the fact that $\mathbf{v}^1_{\mathbf{y}_i}$ is increasing, which is true because the boxes $\Gamma_{\mathbf{y}_i}$ are disjoint. The second also uses this disjointness). From (5.12) and $|\tau_i| \leq \rho \ell_\infty(\mathbf{v}_{\mathbf{y}_i}, \mathbf{v}_{\mathbf{y}_{i+1}})$, it follows that

$$\sum_{j=1}^{J-1} \big| \tau_j \big| \leq \big(2d-1\big)\rho m. \tag{5.13}$$

Substituting the bounds (5.9) and (5.13) into (5.8) yields

$$S(\kappa_J) \geq \frac{3\big(2c_1 + (2d-1)\rho\lambda\big)}{1-\epsilon_0/2} \sum_{j=1}^{J} n_{\mathbf{y}_j} \, - \, \lambda\rho(2d-1)m.$$

Substituting (5.11) into this inequality, we see that, for high values of m,

$$H^c \subseteq \Big\{ S(\kappa_J) \geq 2c_1 m - \frac{\big(2N_1 - 1\big)\big(2c_1 + (2d-1)\rho\lambda\big)}{1-\epsilon_0/2} \Big\} \subseteq \big\{ S(\kappa_J) \geq c_1 m \big\}. \tag{5.14}$$

We now claim that

$$H^c \subseteq \Big\{ \kappa_J \subseteq \big\{ \mathbf{v} \in \mathbb{Z}^d : \ell_\infty(\mathbf{v}, B_m) \leq \rho\big(m\epsilon_0/2 + (d+2)N_1\big) \big\} \Big\}. \tag{5.15}$$

To show (5.15), note that $\gamma_{\mathbf{y}_j} \subseteq \Gamma_{\mathbf{y}_j} \subseteq B_m$ for each $j \in \{1, \ldots, J\}$, the latter inclusion valid by the definition of $J = J(m)$. For $j \in \{1, \ldots, J-1\}$ and $\mathbf{v} \in \tau_j$,

$$\ell_\infty(\mathbf{v}, B_m) \leq |\tau_j| \leq \rho \ell_\infty(\mathbf{v}_{y_j}, \mathbf{v}_{y_{j+1}}) \leq \rho\big(|\mathbf{v}^1_{\mathbf{y}_j} - \mathbf{v}^1_{\mathbf{y}_{j+1}}| + (d-1)\max\{n_{\mathbf{y}_j}, n_{\mathbf{y}_{j+1}}\} \big), \tag{5.16}$$

the third inequality following similarly to (5.12). Note that

$$|\mathbf{v}^1_{\mathbf{y}_j} - \mathbf{v}^1_{\mathbf{y}_{j+1}}| \leq |\xi| + 2n_{\mathbf{y}_j} + n_{\mathbf{y}_{j+1}}, \tag{5.17}$$

because $\mathbf{v}^1_{\mathbf{y}_j}$ is at most $2n_{\mathbf{y}_j}$ less than the maximum $\mathbf{e}_1$-coordinate of the box $\Gamma_{\mathbf{y}_j}[1]$, $\mathbf{v}^1_{\mathbf{y}_j}$ is at most $n_{\mathbf{y}_{j+1}}$ more than $\mathbf{y}_j$, the minimum $\mathbf{e}_1$-coordinate of the box $\Gamma_{\mathbf{y}_j}$, while each site $\mathbf{x}_j = j\mathbf{e}_1$ for which j lies strictly between this maximum and this minimum belongs to ξ. Given that $|\xi| \leq m\epsilon_0/2$ on the event H^c, and that $\max\{n_{\mathbf{y}_j}, n_{\mathbf{y}_{j+1}}\} \leq N_1$, we find that (5.16) and (5.17) imply (5.15).

By (5.14), (5.15) and the bound $\mathbb{P}(H^c) > 1 - \epsilon_0/2$, we find, provided that ϵ_0 has been chosen so that $\epsilon_0 < 2\rho^{-1}$, that, for m sufficiently high,

$$\begin{aligned} &\mathbb{P}\Big(\exists \gamma \in \mathcal{A}_{B_m[1]}, \mathbf{v} \in \gamma \cap \mathcal{W} : S(\gamma) \geq c_1 m, \\ &\quad \mathbf{u} \in \mathcal{W} \cap \big\{ \mathbf{x} \in \mathbb{Z}^d : \ell_\infty(\mathbf{x}, B_m) > m/2 \big\} \implies D(\mathbf{v}, \mathbf{u}) \leq \rho \ell_\infty(\mathbf{v}, \mathbf{u}) \Big) > 1 - \epsilon_0/2, \end{aligned} \tag{5.18}$$

the role of $\mathbf{v}$ in (5.18) being played by any $\mathbf{v}_{\mathbf{y}_i}$ for $i \in \{1, \ldots, J\}$. (We are using the fact that $B_{\mathbf{y}_i, n_{\mathbf{y}_i}}[1] \subseteq \big\{ \mathbf{x} \in \mathbb{Z}^d : \ell_\infty(\mathbf{x}, B_m) \leq m/2 \big\}$, which is implied, provided that $m \geq 2N_1$, by $B_{\mathbf{y}_i, n_{\mathbf{y}_i}} \subseteq B_m$ and $N_1 \geq n_{\mathbf{y}_i}$.) Note also that

$$\mathbb{P}\Big(\text{each corner of } B_m[1] \text{ lies in } \mathcal{W} \Big) \geq 1 - 2^d \mathbb{P}(\mathbf{0} \notin \mathcal{W}) \geq 1 - \epsilon_0/2, \tag{5.19}$$

since (2.31) permits us to choose $\lambda \in \mathbb{R}$ so that the second inequality is valid. By (5.18), (5.19) and the translation invariance of the process $\{X_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^d\}$, we find that, for m large and divisible by three,

$$\mathbb{P}\Big(\exists \gamma \in \mathcal{A}_{B_m}, \mathbf{v} \in \gamma \cap \mathcal{W} : \ S(\gamma) \geq (c_1/3)m, \ D(\mathbf{v},\mathbf{u}) \leq \rho m \, \text{for each corner } \mathbf{u} \text{ of } B_m\Big) > 1 - \epsilon_0,$$

The condition that m is divisible by three occurs because the sidelength of the box $B_m[1]$ must satisfy this. It may be dropped by replacing $B_m[1]$ in (5.19) by a box that extends by one or two sites further on one half of its faces. Writing $c_1 = 3C$, we have shown (5.4).

By the proof of Lemma 2.5, the process

$$\Big\{\mathbf{a} \in \mathbb{Z}^d : D(m\mathbf{a}, m, \lambda, \rho, C) \text{ occurs}\Big\} \tag{5.20}$$

is a $(2\rho+1)$-near percolation, for any given $m \in \mathbb{N}$. By (5.4), Lemma 2.6 and (2.31), we may fix $\lambda, \rho, C > 0$ and a high value of m so that there exists a subset P of (5.20) that is a percolation of supercritical parameter. Let $\{\mathbf{a}_i : i \in \mathbb{N}\}$ denote an infinite self-avoiding path in P. Note that, for each $i \in \mathbb{N}$, $D(\gamma_{\mathbf{a}_i}, \gamma_{\mathbf{a}_{i+1}}) \leq 2\rho m + 1$, where $\gamma_{\mathbf{a}} \subseteq B_{m\mathbf{a},m}$ and $\mathbf{v}_{\mathbf{a}} \in \gamma_{\mathbf{a}}$ denote the lattice animal and the site therein resulting from the occurence of $D(m\mathbf{a}, m, \lambda, \rho, C)$: the inequality is due to $D(\gamma_{\mathbf{a}_i}, \gamma_{\mathbf{a}_{i+1}}) \leq D(\mathbf{v}_{a_i}, \mathbf{w}) + D(\mathbf{w}', \mathbf{v}_{a_{i+1}}) + 1$, where $\mathbf{w}$ $\mathbf{w}'$ are adjacent corners of $B_{m\mathbf{a}_i,m}$ and $B_{m\mathbf{a}_{i+1},m}$. Let ϕ_i denote a white path of length at most $2\rho m + 1$ from $\gamma_{\mathbf{a}_i}$ to $\gamma_{\mathbf{a}_{i+1}}$. Let ϕ_0 denote an arbitrary path from $\mathbf{0}$ to $\gamma_{\mathbf{a}_1}$. We form an increasing sequence of lattice animals $\{R_i : i \in \mathbb{N}\}$, satisfying $|R_i| = i$, with $R_0 = \{\mathbf{0}\}$, which successively collect the sites of the path ϕ_0, then the animal $\gamma_{\mathbf{a}_i}$ and the path ϕ_i, for each $i \in \{1, 2, \ldots\}$ in turn.

Let $n_i = \inf\{j \in \mathbb{N} : R_j \supset \gamma_{\mathbf{a}_i}\}$. Then

$$R_{n_i} = \phi_0 \cup \bigcup_{j=1}^{i-1} \big(\gamma_{\mathbf{a}_j} \cup \phi_j\big) \cup \gamma_{\mathbf{a}_i}.$$

We find that

$$S(R_{n_i}) \geq \sum_{j=1}^{i} S(\gamma_{\mathbf{a}_j}) \, - \, S(\phi_0) \, - \, \lambda \sum_{j=1}^{i-1} \big(|\phi_j| - 2\big) \geq Cmi - \lambda(2\rho m - 1)(i-1) - S(\phi_0), \tag{5.21}$$

where the first inequality follows from the animals $\gamma_{\mathbf{a}_j} \subseteq B_{m\mathbf{a}_j,m}$ being disjoint and the paths ϕ_j being white, their endpoints lying in $\gamma_{\mathbf{a}_j}$ or $\gamma_{\mathbf{a}_{j+1}}$. Note also that

$$n_i \leq |\phi_0| + m^d i + (2\rho m - 1)(i-1), \tag{5.22}$$

since $|\gamma_{\mathbf{a}_j}| \leq m^d$ and $|\phi_{\mathbf{a}_j}| - 2 \leq 2\rho m - 1$. By (5.21), (5.22) and $|R_{n_i}| = n_i$, we obtain

$$\liminf_{i \to \infty} \frac{S(R_{n_i})}{n_i} \geq \frac{Cm - \lambda(2\rho m - 1)}{m^d + 2\rho m - 1} > 0 \, ,$$

provided that the constant C is chosen so that $C > 2\lambda\rho$. Thus, on our hypothesis (5.1), $N > 0$, by the definition of N. This completes the proof. □

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