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Random Growth Models, Phase Separation
and Hamilton–Jacobi PDE

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Brownian Gibbs resampling, phase separation and Kardar–Parisi–Zhang universality

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Lecture notes

Abstract

These lectures offer a probabilistic prospect on Kardar–Parisi–Zhang universality through resampling arguments, phase-separation geometry, last passage percolation, and Brownian Gibbs line ensembles. The central objects are random paths, random interfaces, and random collections of ordered curves. We will study how their laws may be analysed by erasing and resampling parts of a configuration, or by embedding a visible object into a richer ensemble whose conditional structure is tractable.

A first example is classical. For Brownian motion, Lévy’s construction and related resampling arguments lead to estimates for twin peaks, namely the event that two well-separated locations nearly compete for the maximum. The same point of view is then brought to two-dimensional supercritical percolation. There we study the rare event that the origin lies in a large finite open cluster. This cluster may be viewed as a trapped region cut from the infinite open cluster, and its boundary exhibits the one-third and two-third fluctuation scales that foreshadow KPZ geometry.

The lectures then turn to last passage percolation and its line ensembles. Poissonian and Brownian models provide a route from maximizing paths to ordered systems of curves. In Brownian last passage percolation, multi-path energies form an ordered ensemble with the Brownian Gibbs property: erased curve segments are resampled as mutually avoiding Brownian bridges, conditional on neighbouring curves. This property gives a probabilistic route to the Airy line ensemble, the universal ordered ensemble of curves whose top curve encodes multi-path KPZ fluctuations. From there one is led to the Airy sheet, which records the limiting energy for travel between two spatial endpoints across one time interval, and to the directed landscape, the full space-time random metric governing KPZ point-to-point energies. The final lectures describe a simplified construction of the directed landscape from ongoing work of Fardin Syed with Duncan Dauvergne, and then turn to fractal geometry and noise sensitivity of KPZ polymers, including connections with joint work with Shirshendu Ganguly.

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CHAPTER 1

Introduction

These lectures are about probability measures on structured random objects. The object may be a Brownian path, a percolation cluster, a polymer, a random interface, or an ordered ensemble of curves. In each case, there is a law of interest, and the problem is to understand what a typical sample from that law looks like. Often the questions are geometric. Where is the maximizer? How does a boundary fluctuate? When do two polymers coalesce? How stable is an optimizer under resampling of the environment?

The first guiding idea is to study a law by means of a transformation that preserves it. One starts with an input sample and produces an output sample with the same distribution:

$$\text{input sample} \longmapsto \text{output sample}.$$

The map may be random. It may erase a portion of the configuration and resample that portion from the correct conditional law. It may rotate part of the object, or splice together pieces in a way that respects the distribution. The point is not that the output has a new law. The point is that the procedure which constructs the output may make visible something that was hard to see in the input. Since the two samples have the same distribution, whatever is learned from the output is information about the original law.

The second guiding idea is embedding. A one-dimensional object of interest may be the shadow or marginal of a richer system. A random interface may be the top curve of an ordered line ensemble. A point-to-point energy may be one entry in a larger array of multi-path energies. The richer system may have symmetries, Gibbs properties, or algebraic structure which are not visible in the marginal alone. In these lectures, resampling and embedding are complementary methods: we enlarge the view until a useful resampling rule appears, and then we use that rule to return information to the object we first cared about.

Chapter 2 begins with Brownian motion. The aim is partly to refresh some measure-theoretic probability, but in a form that already contains the course's main idea. Lévy's construction builds Brownian motion from countably many independent Gaussian inputs. This representation allows us to hold most of the randomness fixed and vary one component. Even in this elementary setting, such a local variation can reveal geometric information. We use it to study a twin-peak event: the event that two well-separated locations nearly compete for the maximum of a Brownian-like function. This theme will return later when we study noise sensitivity of last passage geodesics.

Chapter 3 is a case study inspired by two-dimensional supercritical percolation. Above the critical point there is almost surely a unique infinite open cluster, which one may think of as a well-woven fabric extending through the plane. Conditioning the origin to lie in a large finite open cluster is a rare event: it is as though a macroscopic piece has been cut from this fabric. The chapter uses resampling to study the geometry of a toy model that acts as a surrogate for this trapped region in the percolation problem. Its boundary fluctuates inside its convex hull on scales governed

by the exponents $1/3$ and $2/3$. This is not yet KPZ in its full dynamical form, but it is a clean phase-separation model in which KPZ-type geometry is already visible.

Chapter 4 turns to Poissonian last passage percolation. Here random growth is encoded by energy-maximizing paths through a random environment. As the endpoint varies, the maximal energy gives a growth interface; when several disjoint maximizing paths are considered at once, their bouquet energies form a line ensemble. This ensemble records multi-path information lying beneath the top interface and has a natural Gibbs resampling property. The chapter explains how the Poissonian model produces this ordered system of curves, and how it serves as a pre-limiting model for the Brownian Gibbs structure used in the later chapters.

Chapter 5 introduces Brownian last passage percolation and the Airy line ensemble. Brownian last passage is a particularly clean microscopic model because its line ensemble has an exact Brownian Gibbs property: if a curve, or a block of curves, is erased on an interval, then conditional on the remaining data it is resampled as a system of mutually avoiding Brownian bridges. The usual KPZ exponents are present: vertical fluctuations are of order $1/3$, the gaps between the top curves are of order $1/3$, and the horizontal correlation scale is $2/3$. After parabolic scaling, the limiting object is the Airy line ensemble. The chapter's purpose is to explain how Brownian Gibbs resampling gives robust probabilistic control of this object.

The final chapter, Chapter 6, moves from the Airy line ensemble to the Airy sheet and the directed landscape. The Airy line ensemble describes scaled energies from a fixed starting point. Geometrically, this is a rooted picture: as the endpoint varies, geodesics coalesce into a random tree. If the starting point is also allowed to vary, one must understand how many such rooted pictures are coupled. The Airy sheet records the resulting two-endpoint energy field across one time strip, and the directed landscape records the full space-time field. We describe a streamlined construction, based on ongoing work of Fardin Syed with Duncan Dauvergne, and indicate how the same objects support questions about fractal geometry and noise sensitivity.

Noise sensitivity is a dynamical version of the same geometric problem. If the random environment is resampled slowly in time, how long does it take before a geodesic substantially changes its route? Such a change is possible when competing routes have nearly tied energies, so the question connects back to the twin-peak estimates from Chapter 2. This circle of ideas, pursued in joint work with Shirshendu Ganguly, is one of the reasons that the course begins with Brownian resampling and ends with KPZ geodesics.

Comments on the literature

There is no single reference which develops exactly the version of RSK used in these notes, with all of the accompanying last passage, isometry, and Gibbs property viewpoints. The references below should therefore be read as complementary.

The paper of Noumi and Yamada [NY04] is perhaps the closest single source to the full algebraic picture. It discusses the tropical RSK correspondence, the birational RSK correspondence, and the relationship between the two through the change of algebra from $(\max, +)$ to $(+, \times)$. It also contains versions of the RSK isometry and Greene-type statements. Its approach is quite algebraic, using matrix factorizations and discrete Toda equations, so it is not always the most direct probabilistic route into the subject. Still, the first few sections give a useful structural overview of why RSK, last passage values, and tropical algebra naturally belong together.

For a more probabilistic discussion of last passage symmetries, see Dauvergne [Dau22]. This paper formulates RSK-type invariance statements for last passage percolation and directed polymer models. In particular, it is a useful reference for the idea that RSK is not just a combinatorial bijection, but also a mechanism which preserves a family of last passage values. This is close in spirit to the isometry point of view used in these notes.

The paper of Dauvergne, Nica, and Virág [DNV22] gives a geometric formulation of RSK using Pitman transforms. This perspective unifies several versions of RSK, including ordinary, dual, and continuous RSK. It is also especially relevant for the line ensemble viewpoint: the map pushes optimizers as far left as possible while preserving last passage values, and the bijective part of the theory leads naturally to Gibbs properties for pre-limiting line ensembles. The proof of bijectivity is one of the main points of that paper. In the Poissonian last passage setting, where the model is tied closely to the PNG droplet, the corresponding invertibility is more immediate, which is one reason the Gibbs property is easier to see there.

Finally, Romik's book [Rom15] is a good entry point for the classical Robinson-Schensted and Robinson-Schensted-Knuth correspondences via insertion algorithms and Young tableaux. It is not tailored to the last passage isometry or Gibbs property viewpoint, but it is useful background for the classical combinatorial objects which underlie the constructions used here.

CHAPTER 2

The Lévy construction of Brownian motion and the Brownian twin peaks estimate

The first two lectures have a double purpose. They offer a brief refreshment of measure-theoretic probability, and they do so in a way that already speaks to one of the main themes of the course. That theme is probabilistic resampling. We will repeatedly study random objects by finding a representation of them in terms of simpler random ingredients, and then asking what happens if one or a few of these ingredients are varied while the others are held fixed. In later chapters, this idea will take a sophisticated form in Brownian Gibbs resampling. In the present chapter, it already appears in an elementary and very concrete way.

We begin from a simple probability experiment. A dart lands uniformly at random in the unit interval. The mathematical model for this experiment is familiar: the probability space is $[0, 1]$ equipped with its Borel σ -field and Lebesgue measure. What deserves emphasis is how much randomness is already sitting inside this model. By reading the binary expansion of the landing point, we recover an infinite sequence of independent Bernoulli random variables. By reorganising this sequence, we may build a countable family of independent uniform random variables. By applying inverse distribution functions, we may transform these into a countable family of independent standard Gaussian random variables. If one likes, these Gaussian variables may then be thought of as sitting on the shelf, ready to be picked off one by one.

This point of view leads naturally to the Lévy construction of Brownian motion. Brownian motion is built by taking these Gaussian variables off the shelf one at a time and using them to perturb linear interpolation at successive dyadic scales. At each stage, a tent is inserted whose height is Gaussian and whose support is a dyadic interval. The result is a random continuous function. The continuity is not put in by hand at the end; it emerges from the rapid decay of the dyadic perturbations. This construction is valuable not merely because it builds Brownian motion. It displays Brownian motion as a random object assembled from countably many independent Gaussian inputs. That representation is already suggestive for later purposes. If one wants to understand some event involving the geometry of the graph of Brownian motion, or of Brownian motion plus a deterministic function, one may try to condition on almost all of the Gaussian inputs and vary one of them. This sort of manoeuvre already lies behind simple resampling arguments, and later it will take a much more powerful form.

2.1. The unit interval as a probability model

We begin with the probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda),$$

where $\mathcal{B}([0, 1])$ denotes the Borel σ -field and λ denotes Lebesgue measure on $[0, 1]$. The identity map

$$U : \Omega \rightarrow [0, 1], \quad U(\omega) = \omega,$$

is then a random variable with the uniform distribution on $[0, 1]$. In this way, the Borel space on the unit interval is the mathematical model for a dart that lands uniformly at random in $[0, 1]$.

There is nothing deep in this first step, but it is worth pausing over what it says. A probability space is not merely a passive stage on which random variables happen to live. A well chosen probability space can act as a reservoir of randomness from which other random variables may be made. Here the reservoir is especially simple: it is just the unit interval with its usual measurable structure and its usual notion of length. Yet it will prove rich enough for all of the countable randomness that we need in this chapter.

This use of the unit interval is also a good moment to recall the role of the Borel σ -field. Since the audience has measure theory in hand, there is no need to redevelop the general theory here. The point is simply that the Borel subsets of $[0, 1]$ form a large and natural class of events to which Lebesgue measure may be assigned. This permits us to interpret events such as $\{U \leq x\}$ or $\{U \in I\}$ for intervals $I \subseteq [0, 1]$ in the familiar probabilistic way. If $0 \leq a \leq b \leq 1$, then

$$\mathbb{P}(U \in [a, b]) = b - a.$$

Thus the probability that the dart lands in a subinterval is just the length of that interval.

One very basic but very useful principle is that a random variable may be built by transporting this uniform randomness through a measurable map. For example, if we set

$$D(\omega) = \lceil 6\omega \rceil$$

for $\omega \in (0, 1]$, and define $D(0) = 1$, then D has the law of a fair die. More generally, given a real-valued distribution function, we may build a random variable with that law by using the quantile map. We record this now, because it will soon allow us to pass from uniform random variables to Gaussian ones.

Let $F : \mathbb{R} \rightarrow [0, 1]$ be a distribution function: that is, F is non-decreasing and right-continuous, with

$$\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow \infty} F(x) = 1.$$

Define the generalised inverse

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad u \in (0, 1).$$

LEMMA 2.1. If U is uniform on $[0, 1]$, then $F^{-1}(U)$ has distribution function F .

PROOF. For any $x \in \mathbb{R}$,

$$\{F^{-1}(U) \leq x\} = \{U \leq F(x)\}.$$

Therefore

$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x).$$

□

In particular, if Φ denotes the standard normal distribution function, then $\Phi^{-1}(U)$ is a standard Gaussian random variable.

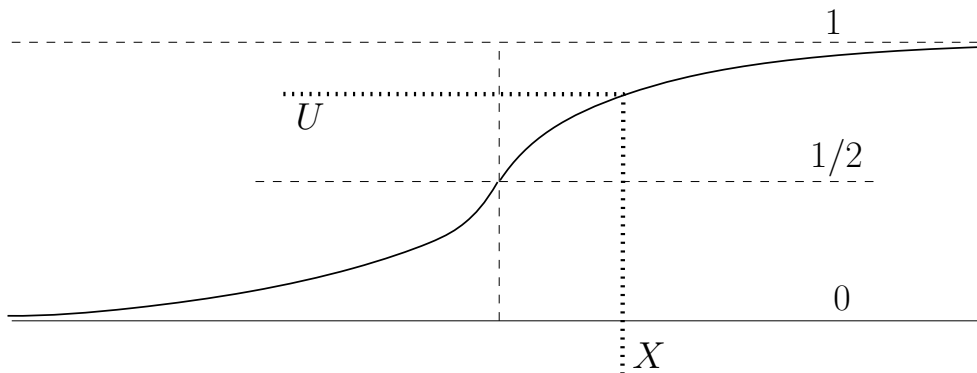


FIGURE 2.1. The quantile transform. A uniform random height $U \in [0, 1]$ is sent horizontally to the graph of a distribution function F and then dropped to the horizontal axis. The resulting random point has distribution function F . In particular, taking $F = \Phi$ yields a standard Gaussian random variable.

2.2. Binary expansion and the extraction of Bernoulli randomness

The first important idea is that the single uniform random variable U contains within it a countable sequence of independent fair coin flips.

Every point $x \in [0, 1]$ has a binary expansion

$$x = \sum_{i=1}^{\infty} 2^{-i} B_i(x), \quad B_i(x) \in \{0, 1\}.$$

There is the familiar ambiguity at dyadic rationals: for instance,

$$\frac{1}{2} = 0.1000 \dots = 0.0111 \dots.$$

To avoid this nuisance, we make once and for all the convention that dyadic rationals are represented by the expansion that is eventually zero. Since the set of dyadic rationals is countable, this choice affects only a null set.

Thus, for each $i \geq 1$, the map

$$B_i : [0, 1] \rightarrow \{0, 1\}$$

is well defined and Borel measurable. We may regard $B_i(U)$ simply as the i^{th} binary digit of the landing point of the dart.

The first few of these functions are very concrete. The first is

$$B_1 = \mathbf{1}_{[1/2, 1]},$$

since the first binary digit records whether the point lies in the left or right half of the interval. The second is

$$B_2 = \mathbf{1}_{[1/4, 1/2) \cup [3/4, 1]},$$

since it records the second stage of the left-right dyadic descent. In general, B_i oscillates between 0 and 1 on dyadic intervals of length 2^{-i} .

The crucial fact is that these digits form an independent Bernoulli sequence.

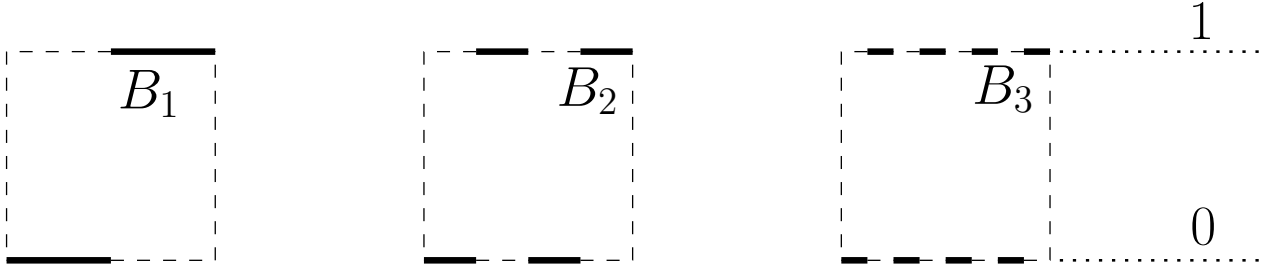


FIGURE 2.2. Three copies of the unit square $[0, 1]^2$ are depicted. In the respective squares are B_1 , B_2 and B_3 , the values of the first three digits in the binary expansion of the variable in the domain.

PROPOSITION 2.2. The random variables $(B_i)_{i \geq 1}$ are independent and each has the Bernoulli(1/2) law.

PROOF. Fix $n \in \mathbb{N}$, and fix digits $a_1, \dots, a_n \in \{0, 1\}$. The event

$$\{B_1 = a_1, \dots, B_n = a_n\}$$

specifies the first n stages in the dyadic descent. It is therefore a dyadic interval of length 2^{-n} , up to the harmless issue of endpoints. Hence

$$\mathbb{P}(B_1 = a_1, \dots, B_n = a_n) = 2^{-n}.$$

Since each B_i takes the values 0 and 1 with probability 1/2, this is exactly

$$\prod_{i=1}^n \mathbb{P}(B_i = a_i).$$

Thus $B_1, \dots, B_n$ are independent, and as n was arbitrary, the whole sequence is independent. $\square$

This is a very useful point to keep in mind. The randomness of a single uniform dart has been unpacked into a countable sequence of independent coin flips. One may think of the dart as a diffuse continuum object, and the coin-flip sequence as a discrete store of randomness extracted from it. The two descriptions are equivalent. Which one is more useful depends on what one wants to do.

2.3. A countable family of independent darts

We now take the Bernoulli sequence that has just been extracted and reorganise it into a two-indexed family. The purpose is to manufacture not one but countably many independent uniform random variables.

Choose a bijection

$$\psi : \mathbb{N} \rightarrow \mathbb{N}^2.$$

One may picture this bijection by picturing the positive integers draped along a snaking path through the first quadrant lattice, as Figure 2.3 depicts. If $\psi(m) = (i, j)$, define

$$\eta_{i,j} = B_m.$$

The family $(\eta_{i,j})_{i,j \geq 1}$ is again a family of independent Bernoulli(1/2) random variables, because it is simply a relabelling of the original independent sequence.

Now for each fixed $i \geq 1$, define

$$U_i = \sum_{j=1}^{\infty} 2^{-j} \eta_{i,j}.$$

This is the real number in $[0, 1]$ whose binary digits are read from the i^{th} row of the array $(\eta_{i,j})$.

PROPOSITION 2.3. The sequence $(U_i)_{i \geq 1}$ is independent, and each U_i is uniformly distributed on $[0, 1]$.

PROOF. For each fixed i , the binary digits of U_i are independent Bernoulli(1/2) variables, so the same dyadic-interval argument as before shows that U_i is uniform on $[0, 1]$.

To see independence, let $n \in \mathbb{N}$, and let $I_1, \dots, I_n$ be dyadic intervals. Membership of U_i in I_i is determined by finitely many of the digits $\eta_{i,j}$. For different values of i , these sets of digits are disjoint, so the relevant events are independent. Hence

$$\mathbb{P}(U_1 \in I_1, \dots, U_n \in I_n) = \prod_{i=1}^n \mathbb{P}(U_i \in I_i).$$

Since dyadic intervals generate the Borel σ -field on $[0, 1]$, a standard monotone class argument yields independence of the U_i . $\square$

Thus one uniformly random point in the unit interval may be reorganised into a countable family of independent uniformly random points in the unit interval.

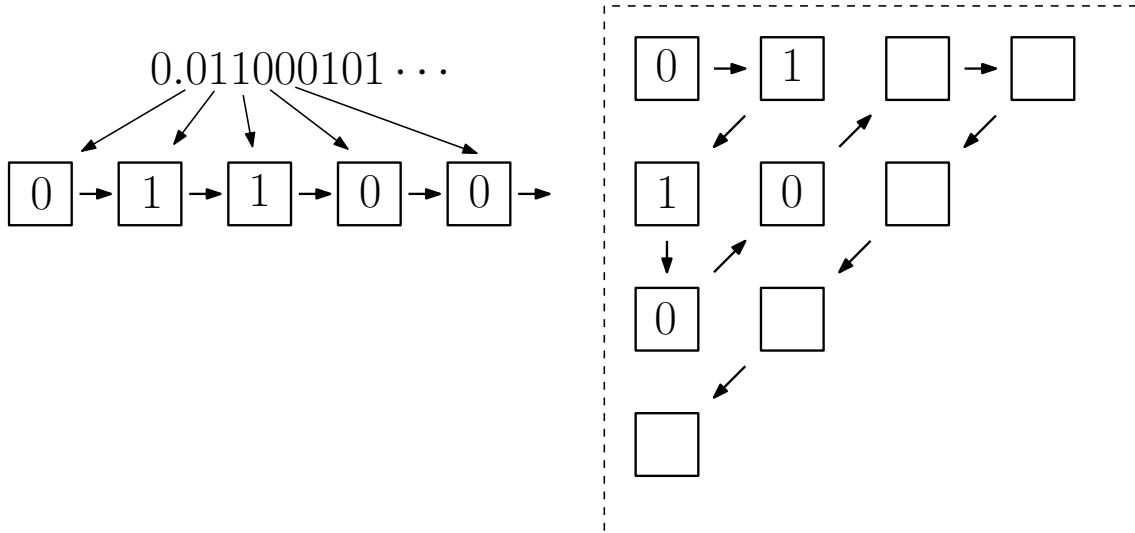


FIGURE 2.3. The successive binary digits of a value in $[0, 1]$ are assigned to integer sites in the lower-right quadrant by placing them along a path that snakes its way through all of these sites. The first five digits have been placed in the picture. When all the digits are placed, each row will offer an independent binary expansion of a uniform random number in $[0, 1]$. Here we are using a trick associated to countability to gain access to a countably infinite collection of uniform random darts using as input just one such dart.

There is something conceptually useful in this step. We have not added any randomness. We have merely reorganised the randomness already present. Countability makes this possible. A countable sequence may be repackaged as a doubly indexed countable family, or in many other ways. This modest observation is often a source of rather useful probabilistic constructions.

2.4. A countable Gaussian shelf

Let Φ denote the standard normal distribution function,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz.$$

Define

$$Z_i = \Phi^{-1}(U_i), \quad i \geq 1.$$

By the quantile transform, the sequence $(Z_i)_{i \geq 1}$ is independent and each Z_i has the $N(0, 1)$ distribution.

This countable family of independent standard Gaussians is the stock of randomness from which we will now build Brownian motion. Since they are already available before the construction begins, it is convenient to think of them as sitting on the shelf. The Lévy construction proceeds by taking them off the shelf one by one and using them to determine successive dyadic refinements of the path.

It is worth emphasising the shift in viewpoint here. Brownian motion is often introduced abstractly, by specifying its finite-dimensional distributions and then appealing to a construction theorem. That is not our present route. We want to see it being assembled. The reason is not merely pedagogical. The assembly itself matters. Later, when we study events involving local geometry, it will be helpful to know which Gaussian input influences which part of the path.

2.5. Brownian motion and the midpoint law

We recall the definition.

DEFINITION 2.4. A process $(B(t) : 0 \leq t \leq 1)$ is called standard Brownian motion on $[0, 1]$ if:

- (1) $B(0) = 0$ almost surely;
- (2) for any $0 \leq t_0 < t_1 < \dots < t_n \leq 1$, the increments
$$B(t_1) - B(t_0), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})$$
are independent;
- (3) for $0 \leq s < t \leq 1$, the increment $B(t) - B(s)$ has the normal law with mean 0 and variance $t - s$;
- (4) with probability one, the map $t \mapsto B(t)$ is continuous on $[0, 1]$.

The dyadic construction rests on a simple Gaussian computation. Suppose for the moment that Brownian motion on an interval $[a, b]$ is already in hand, and that we condition on the endpoint values $B(a)$ and $B(b)$. What then is the law of the midpoint value $B((a+b)/2)$? It is enough first to consider the unit interval. Write

$$X = B(1/2) - B(0), \quad Y = B(1) - B(1/2).$$

Then X and Y are independent centred Gaussian random variables with variance $1/2$. If we condition on $X + Y = z$, the conditional density of X is proportional to

$$e^{-x^2} e^{-(z-x)^2}.$$

Completing the square,

$$-x^2 - (z-x)^2 = -2\left(x - \frac{z}{2}\right)^2 - \frac{z^2}{2},$$

so the x -dependence is that of a Gaussian density with mean $z/2$ and variance $1/4$. Thus, conditional on the endpoint values $B(0) = u$ and $B(1) = v$, the midpoint value $B(1/2)$ is Gaussian with mean

$$\frac{u+v}{2}$$

and variance $1/4$. After translation and scaling, it follows that for a general interval $[a, b]$, conditional on $B(a)$ and $B(b)$, the midpoint value $B((a+b)/2)$ is Gaussian with mean

$$\frac{B(a) + B(b)}{2}$$

and variance

$$\frac{((a+b)/2 - a)(b - (a+b)/2)}{b - a} = \frac{b - a}{4}.$$

This is the bridge rule that drives the entire construction.

We record the corresponding general Gaussian fact in a form that we will use repeatedly.

LEMMA 2.5. Let X and Y be independent centred Gaussian random variables with variances σ_1^2 and σ_2^2 . Conditional on the value of $X + Y$, the variable X is Gaussian with mean

$$\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(X + Y)$$

and conditional variance

$$\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

PROOF. Since $(X, X+Y)$ is jointly Gaussian, the conditional law of X given $X+Y$ is Gaussian. Its mean is the linear regression of X on $X+Y$,

$$\frac{\text{Cov}(X, X+Y)}{\text{Var}(X+Y)}(X+Y) = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(X+Y),$$

and its conditional variance is therefore

$$\text{Var}(X) - \frac{\text{Cov}(X, X+Y)^2}{\text{Var}(X+Y)} = \sigma_1^2 - \frac{\sigma_1^4}{\sigma_1^2 + \sigma_2^2} = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}. \quad \square$$

This is precisely the law that Lévy's construction will impose at each dyadic midpoint.

2.6. The dyadic construction

We now build Brownian motion on $[0, 1]$.

Write

$$\mathcal{D}_n = \{k2^{-n} : 0 \leq k \leq 2^n\}$$

for the dyadic mesh of level n , and let

$$\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$$

denote the set of dyadic rationals in $[0, 1]$.

We will define a sequence $(B_n)_{n \geq 0}$ of continuous random functions on $[0, 1]$, where B_n is piecewise linear on the dyadic mesh $\mathcal{D}_n$. The guiding idea is simple. At stage n , the values at the mesh points of $\mathcal{D}_n$ have already been fixed. To pass to stage $n + 1$, one inserts new values at the midpoints of the dyadic intervals of level n , using the conditional Gaussian law just described, and then linearly interpolates.

Set first

$$B_0(0) = 0.$$

Take the first Gaussian Z_1 off the shelf and set

$$B_1(1) = Z_1.$$

We may regard B_1 as the straight line joining $(0, 0)$ to $(1, Z_1)$.

At the next stage, take the next Gaussian Z_2 and define

$$B_2(1/2) = \frac{B_1(0) + B_1(1)}{2} + 2^{-1}Z_2 = \frac{Z_1}{2} + \frac{Z_2}{2}.$$

This is exactly the rule suggested by the midpoint law, since the variance to be inserted at the midpoint of $[0, 1]$ is $1/4$. We then linearly interpolate on $[0, 1/2]$ and on $[1/2, 1]$.

At the next stage, we use two new Gaussian variables, say Z_3 and Z_4 , and define

$$B_3(1/4) = \frac{B_2(0) + B_2(1/2)}{2} + 2^{-3/2}Z_3,$$

$$B_3(3/4) = \frac{B_2(1/2) + B_2(1)}{2} + 2^{-3/2}Z_4,$$

again followed by linear interpolation on each dyadic subinterval of length $1/4$.

And so the procedure continues. In general, when a midpoint of a dyadic interval of length 2^{-n} is inserted, the Gaussian perturbation has variance $2^{-(n+2)}$, and so its standard deviation is $2^{-(n+2)/2}$.

It is convenient to formalise the recursion. Let

$$\Gamma_{n,k}, \quad n \geq 0, \quad k \in \{1, 3, \dots, 2^n - 1\},$$

be an independent family of standard Gaussian random variables. One may take these from the shelf by relabelling the sequence (Z_i) ; the precise indexing does not matter. Define B_0 by $B_0 \equiv 0$. Suppose that B_n has been defined and is piecewise linear on the mesh $\mathcal{D}_n$. For each odd $k \in \{1, 3, \dots, 2^n - 1\}$, set

$$B_{n+1}\left(\frac{k}{2^n}\right) = \frac{1}{2}\left(B_n\left(\frac{k-1}{2^n}\right) + B_n\left(\frac{k+1}{2^n}\right)\right) + 2^{-(n+2)/2}\Gamma_{n,k},$$

and retain the old values at the old mesh points:

$$B_{n+1}(t) = B_n(t), \quad t \in \mathcal{D}_n.$$

Then extend B_{n+1} linearly between adjacent points of $\mathcal{D}_{n+1}$.

Thus $B_{n+1} - B_n$ is a collection of tents, one on each dyadic interval of level n , each vanishing at the endpoints and peaking at the midpoint.

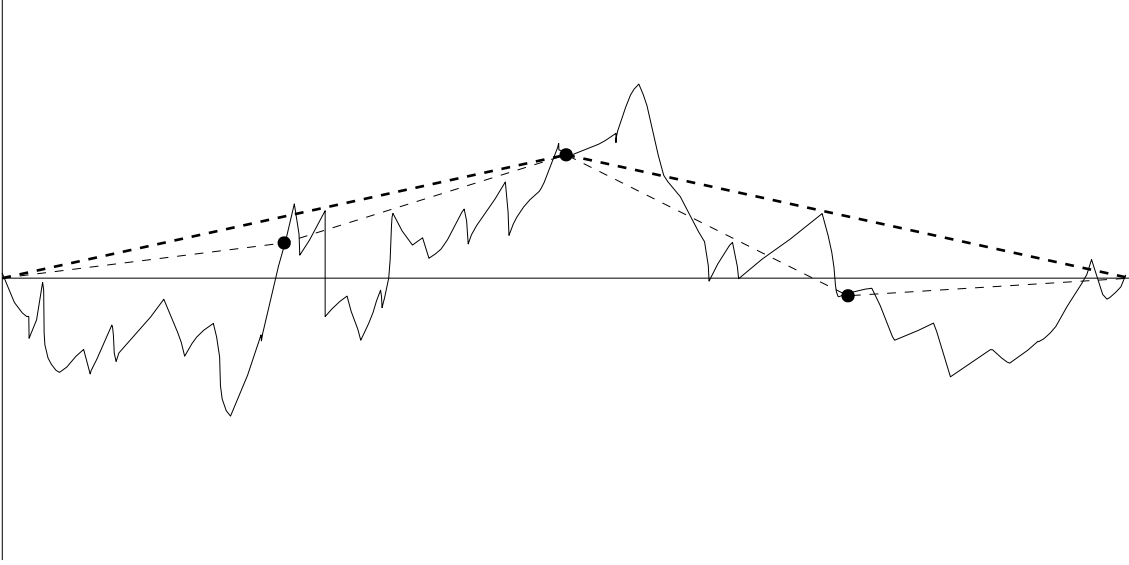


FIGURE 2.4. One stage of the Lévy construction. Starting from the piecewise linear interpolation on a dyadic mesh, one inserts Gaussian midpoint perturbations on each dyadic interval and then linearly interpolates again. The added function is a collection of tents supported on dyadic intervals.

This is the construction in its bare form. Its appeal is partly visual. One sees the path being formed by a succession of ever finer random wrinklings. But the picture is not only suggestive. It is also honest. The continuity of the limiting path will follow from the fact that the tents inserted at high scales are uniformly tiny.

2.7. Uniform convergence of the dyadic approximants

We now show that the sequence (B_n) converges uniformly almost surely on $[0, 1]$. This gives a continuous limit.

Let

$$\Delta_n = B_{n+1} - B_n.$$

The function Δ_n is piecewise linear, supported on the dyadic intervals of level n , and on each such interval it is a single tent. Its maximum absolute value on that interval is attained at the midpoint, where it equals the absolute value of the inserted Gaussian height. Thus

$$\|\Delta_n\|_\infty = 2^{-(n+2)/2} \max_{1 \leq j \leq 2^n} |G_{n,j}|,$$

where $G_{n,1}, \dots, G_{n,2^n}$ denote the 2^n independent standard Gaussian variables used at level n .

To prove summability, it is enough to show that the maximum on the right-hand side grows at most linearly in n , eventually almost surely. This is a very generous allowance, since the maximum of 2^n standard Gaussian variables is in fact of order $\sqrt{n}$, but the weaker bound is enough for our present purpose and keeps the argument short.

Fix a constant $C > 0$, and consider the event

$$A_n = \left\{ \max_{1 \leq j \leq 2^n} |G_{n,j}| > Cn \right\}.$$

By the union bound and the standard Gaussian tail estimate,

$$\mathbb{P}(A_n) \leq 2^n \mathbb{P}(|N(0, 1)| > Cn) \leq 2^{n+1} \exp \left\{ -\frac{C^2 n^2}{2} \right\}.$$

The series $\sum_n \mathbb{P}(A_n)$ converges. Hence, by the Borel–Cantelli lemma, with probability one only finitely many of the events A_n occur.

On that almost sure event, for all sufficiently large n ,

$$\max_{1 \leq j \leq 2^n} |G_{n,j}| \leq Cn,$$

and therefore

$$\|\Delta_n\|_\infty \leq 2^{-(n+2)/2} Cn.$$

Since

$$\sum_{n=0}^{\infty} n 2^{-n/2} < \infty,$$

we conclude that

$$\sum_{n=0}^{\infty} \|\Delta_n\|_\infty < \infty$$

almost surely.

This implies that (B_n) is almost surely Cauchy in the supremum norm, and so converges almost surely uniformly on $[0, 1]$ to a continuous limit B .

THEOREM 2.6. The dyadic approximants (B_n) converge almost surely uniformly on $[0, 1]$ to a continuous random function $B : [0, 1] \rightarrow \mathbb{R}$.

The argument is short, but it is worth noting what it already contains. It gives continuity because the dyadic perturbations are summable in the supremum norm. In fact, it gives more. Since

$$\|\Delta_n\|_\infty \leq Cn 2^{-n/2}$$

eventually almost surely, one may sum over scales and deduce that the limiting path is almost surely Hölder continuous of every exponent strictly less than $1/2$.

It is also easy to see, just by eyeballing the proof, how far this may be pushed. If one replaces the threshold Cn by $C\sqrt{n}$, with C chosen large enough, then the same union bound gives

$$\mathbb{P} \left(\max_{1 \leq j \leq 2^n} |G_{n,j}| > C\sqrt{n} \right) \leq 2^{n+1} e^{-C^2 n/2},$$

which is summable as soon as $C^2/2 > \log 2$. On that basis one obtains

$$\|\Delta_n\|_\infty \leq C\sqrt{n} 2^{-n/2}$$

eventually almost surely. Summing over dyadic scales, this points to a modulus of continuity of order

$$x^{1/2}(\log x^{-1})^{1/2}$$

up to multiplicative constants.

2.8. Identification of the limit as Brownian motion

It remains to show that the continuous limit B has the Brownian law.

The easiest way to do this is first to understand the values of B on the dyadic rationals. The recursive rule was chosen precisely so that the dyadic skeleton has the correct Brownian bridge structure from one level to the next. This may be proved inductively.

PROPOSITION 2.7. For each $n \geq 0$, the random vector

$$(B(t) : t \in \mathcal{D}_n)$$

has the same law as standard Brownian motion sampled at the times in $\mathcal{D}_n$.

PROOF. The claim is clear at level 0, and at level 1 it says only that $B(1)$ is standard Gaussian. Assume the claim at level n . Consider any dyadic interval

$$I = [(m-1)2^{-n}, m2^{-n}].$$

Conditional on the values at the endpoints of I , the value inserted at the midpoint at the next stage is Gaussian with mean equal to the average of the endpoint values and variance equal to one quarter of the interval length, that is, $2^{-(n+2)}$. Moreover, these midpoint insertions are conditionally independent from one dyadic interval of level n to another, because they are made using independent Gaussian inputs.

By Lemma 2.5, this is exactly the conditional law of Brownian motion at the dyadic midpoints, given the values at the level- n dyadic points. Hence the joint law on $\mathcal{D}_{n+1}$ is the correct Brownian law. $\square$

From this, the conclusion is straightforward. Fix times

$$0 \leq t_1 < \cdots < t_r \leq 1.$$

Choose dyadic approximations $t_i^{(n)} \in \mathcal{D}_n$ with $t_i^{(n)} \rightarrow t_i$. Since $B_n \rightarrow B$ uniformly almost surely, we have

$$B(t_i^{(n)}) \rightarrow B(t_i)$$

almost surely for each i . But for each n , the vector

$$(B(t_1^{(n)}), \dots, B(t_r^{(n)}))$$

has the same law as Brownian motion sampled at the times $t_1^{(n)}, \dots, t_r^{(n)}$. Passing to the limit, we find that

$$(B(t_1), \dots, B(t_r))$$

is a centred Gaussian vector with covariance

$$\text{Cov}(B(s), B(t)) = \min\{s, t\}.$$

This is exactly the finite-dimensional law of Brownian motion. Since continuity has already been proved, we obtain the desired result.

THEOREM 2.8. The limit process B constructed above is standard Brownian motion on $[0, 1]$.

PROOF. By Theorem 2.6, the sample paths are almost surely continuous and $B(0) = 0$. The preceding discussion shows that the finite-dimensional distributions are those of standard Brownian motion. Therefore B is standard Brownian motion on $[0, 1]$. $\square$

To construct Brownian motion on all of $[0, \infty)$, one may repeat the same construction independently on successive unit intervals and concatenate. Since our later discussion will mainly concern local behaviour on compact intervals, the construction on $[0, 1]$ is the essential part.

2.9. A first quantitative resampling estimate: twin peaks

The Lévy construction does more than build Brownian motion. It places the law of interest in a useful coordinate system. The process is sampled, the random data that specify it are written as a list, and a line is drawn through that list. The data above the line are retained and frozen. The data below the line are forgotten. One then redraws the forgotten part from its correct conditional law given what has been kept, and reconstructs the process. The reconstructed object is the output. By construction, the output has the same law as the input. This input-to-output update is then a tool for studying the law itself.

We now illustrate this principle for the process

$$X(x) = B(x) - x^2, \quad x \in \mathbb{R},$$

where B is standard two-sided Brownian motion with $B(0) = 0$. This process is a natural toy model for several later themes. It is locally Brownian, globally parabolic, and it displays in a particularly transparent way the competition between fluctuation and curvature.

For $\epsilon > 0$, let $\text{TP}(\epsilon)$ denote the event that there exist points $x_1, x_2 \in \mathbb{R}$ such that

$$|x_1 - x_2| \geq 1$$

and

$$X(x_1) \geq \sup_{x \in \mathbb{R}} X(x) - \epsilon, \quad X(x_2) \geq \sup_{x \in \mathbb{R}} X(x) - \epsilon.$$

Thus $\text{TP}(\epsilon)$ is the event that the global maximum is accompanied by a second competitor at distance at least one whose value lies within ϵ of it. We will show that this event has probability of order ϵ .

Before doing so, it is useful to isolate the conditional law on a compact interval. The parabolic term alters the bridge law slightly, and it is worth making this explicit before any resampling is attempted.

LEMMA 2.9. Let $a < b$, and condition on the values of X outside $[a, b]$. Then the conditional law of X on $[a, b]$ is that of

$$t \mapsto \frac{b-t}{b-a}X(a) + \frac{t-a}{b-a}X(b)$$

plus a standard Brownian bridge on $[a, b]$, plus the deterministic correction

$$t \mapsto (t-a)(b-t).$$

PROOF. Write

$$X(t) = B(t) - t^2.$$

Conditional on $B(a)$ and $B(b)$, the process B on $[a, b]$ is the affine interpolation between its endpoint values plus a standard Brownian bridge on $[a, b]$. Thus the conditional law of X on $[a, b]$ is that of

$$\frac{b-t}{b-a}B(a) + \frac{t-a}{b-a}B(b) + \beta_{[a,b]}(t) - t^2,$$

where $\beta_{[a,b]}$ is a standard Brownian bridge on $[a, b]$. Since

$$B(a) = X(a) + a^2, \quad B(b) = X(b) + b^2,$$

this becomes

$$\frac{b-t}{b-a}X(a) + \frac{t-a}{b-a}X(b) + \beta_{[a,b]}(t) + \frac{b-t}{b-a}a^2 + \frac{t-a}{b-a}b^2 - t^2.$$

A direct calculation shows that

$$\frac{b-t}{b-a}a^2 + \frac{t-a}{b-a}b^2 - t^2 = (t-a)(b-t),$$

which proves the claim. $\square$

The preceding lemma shows that the conditional law is not literally that of a Brownian bridge. Rather, once the endpoint values are fixed, the process on the interval is obtained by taking the usual bridge representation and adding a deterministic correction. In particular, the random part is still a standard Brownian bridge, so the on-the-shelf Gaussian coordinates from its Lévy description remain available exactly as before. The deterministic correction may simply be absorbed into the frozen part of the reconstruction.

We begin with a simpler consequence of the same idea.

PROPOSITION 2.10. Almost surely, the process $X(x) = B(x) - x^2$ has a unique point of global maximum.

PROOF. Since $x^2 \rightarrow \infty$ as $|x| \rightarrow \infty$ while Brownian motion grows only on the order of $|x|^{1/2+o(1)}$, we have

$$X(x) \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty$$

almost surely. Thus the global maximum is almost surely attained on some random compact interval.

It is therefore enough to prove that, on any fixed compact interval, the maximiser is almost surely unique. Fix such an interval I , and suppose that the maximum on I were attained at two distinct points. Then there would exist two disjoint closed subintervals $J_1, J_2 \subseteq I$ with rational endpoints such that

$$\sup_{x \in J_1} X(x) = \sup_{x \in J_2} X(x).$$

Since there are only countably many such ordered pairs, it is enough to show that the event displayed above has probability zero for each fixed pair.

So let J_1 and J_2 be disjoint compact intervals. We sample X , erase the portion of the graph over J_2 , and ask for the conditional law of the erased piece given what remains. By Lemma 2.9, conditional on the retained data, the process on J_2 is the sum of a deterministic function and a Brownian bridge.

The quantity $\sup_{x \in J_1} X(x)$ is then fixed, while the maximum on J_2 has a continuous distribution. It therefore hits the fixed target level $\sup_{x \in J_1} X(x)$ with conditional probability zero. Hence

$$\mathbb{P}\left(\sup_{x \in J_1} X(x) = \sup_{x \in J_2} X(x)\right) = 0.$$

Taking the union over rational pairs of disjoint subintervals completes the proof. $\square$

The same general picture leads to the twin peaks estimate. The point is to draw the line through the data in such a way that, after conditioning, only one Gaussian coordinate remains to be redrawn. The event of a second distant near-maximiser is then reduced to the event that this one-dimensional Gaussian falls in a short interval.

PROPOSITION 2.11. There exists $C < \infty$ such that, for all $\epsilon \in (0, 1]$,

$$\mathbb{P}(\text{TP}(\epsilon)) \leq C\epsilon.$$

PROOF. The first step is localisation. Since $X(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$ almost surely, there exist positive constants c and C_0 such that

$$\mathbb{P}\left(\sup_{|x| \geq R} X(x) \geq \sup_{|x| \leq 3} X(x) - 1\right) \leq C_0 e^{-cR^3}$$

for all large R . Thus, after choosing R large and absorbing the tail event into the final constant, it is enough to work on the event that every $(1 + \epsilon)$ -near maximiser lies in $[-R, R]$.

Fix such an R . Let $\mathcal{K}_R$ be the finite collection of unit intervals

$$K_m = \left[\frac{m}{2}, \frac{m}{2} + 1\right]$$

whose middle halves meet $[-R, R]$. For such an interval $K = [u, u + 1]$, write

$$K^{\text{mid}} = \left[u + \frac{1}{4}, u + \frac{3}{4}\right].$$

The middle halves of the intervals in $\mathcal{K}_R$ cover $[-R, R]$. Accordingly, on the localisation event, if $\text{TP}(\epsilon)$ occurs, then there exists some $K \in \mathcal{K}_R$ such that one of the two near-maximisers, say x_2 , lies in K^{mid} . Since the other near-maximiser x_1 lies at distance at least one from x_2 , it cannot belong to K .

Now let $\mathcal{I}_R$ be any finite collection of unit intervals with rational endpoints that covers $[-R, R]$. On the event just described, there is some interval $I \in \mathcal{I}_R$ disjoint from K such that $x_1 \in I$. Hence it is enough to prove that, for each fixed ordered pair (I, K) with $I \cap K = \emptyset$,

$$\mathbb{P}\left(\sup_{x \in I} X(x) \geq \sup_{x \in \mathbb{R}} X(x) - \epsilon, \sup_{x \in K^{\text{mid}}} X(x) \geq \sup_{x \in \mathbb{R}} X(x) - \epsilon\right) \leq C\epsilon, \quad (1)$$

with C independent of the pair. Summing over the finitely many such pairs then yields the proposition.

Fix such a pair (I, K) . Write

$$M_I = \sup_{x \in I} X(x).$$

We now present the data that determine the process over K in the form of a list. The first item is the path outside K ; this is retained, and in particular it fixes the endpoint values $X(u)$ and $X(u + 1)$. The next items are the Gaussian coordinates in the Lévy description of the Brownian bridge part on K . We draw the line just below the first of these coordinates. Everything above

the line is retained. Everything below it is forgotten. Thus, after conditioning, only one Gaussian variable remains to be redrawn.

By Lemma 2.9, conditional on the retained data, the process on K takes the form

$$X(t) = F(t) + \xi \varphi_K(t), \quad t \in K,$$

where F is a continuous function determined by the retained data, ξ is a centred Gaussian random variable of variance $1/4$, and φ_K is the basic tent function on K , equal to 0 at the endpoints and 1 at the midpoint. Here F absorbs the affine interpolation between the endpoint values, the deterministic parabolic correction, and all frozen higher bridge coordinates. On the middle half K^{mid} , the tent function satisfies

$$\varphi_K(t) \geq \frac{1}{2}.$$

Now fix the retained data. The quantity M_I is then fixed, because I lies outside K . Define, for $z \in \mathbb{R}$,

$$S(z) = \sup_{t \in K^{\text{mid}}} (F(t) + z \varphi_K(t)).$$

This is the maximum over the candidate interval for the second peak when the single resampled Gaussian takes the value z . The function S is increasing, and more is true: if $z_2 > z_1$, then

$$S(z_2) - S(z_1) \geq \frac{z_2 - z_1}{2}.$$

Indeed, for any $t \in K^{\text{mid}}$,

$$F(t) + z_2 \varphi_K(t) - (F(t) + z_1 \varphi_K(t)) = (z_2 - z_1) \varphi_K(t) \geq \frac{z_2 - z_1}{2}.$$

Taking suprema over $t \in K^{\text{mid}}$ yields the displayed bound.

It follows that the set of z for which

$$S(z) \in [M_I - \epsilon, M_I + \epsilon]$$

is an interval of length at most 4ϵ . Since ξ is Gaussian with bounded density, the conditional probability that

$$\sup_{x \in K^{\text{mid}}} X(x) \in [M_I - \epsilon, M_I + \epsilon]$$

is at most $C\epsilon$.

Finally, the event in (1) can only occur if both

$$M_I \geq \sup_{x \in \mathbb{R}} X(x) - \epsilon \quad \text{and} \quad \sup_{x \in K^{\text{mid}}} X(x) \geq \sup_{x \in \mathbb{R}} X(x) - \epsilon,$$

which implies

$$\left| \sup_{x \in K^{\text{mid}}} X(x) - M_I \right| \leq \epsilon.$$

Thus the conditional probability of the event in (1) is at most $C\epsilon$, and therefore so is its unconditional probability. This proves (1), and hence the proposition. $\square$

A matching lower bound may also be obtained by a local construction, so that the probability of twin peaks is in fact of order ϵ . The upper bound is the part that is most revealing here. The law of interest is sampled; its data are presented in a useful form; a line is drawn through that data; what lies above the line is frozen and what lies below it is forgotten and resampled; the output is reconstructed and has the same law as the input. By drawing the line so that only one well

chosen Gaussian variable remains to be redrawn, the event of two well separated competing peaks is reduced to the event that a one-dimensional Gaussian land in a short interval. This is a very simple instance of a general method that will later reappear in richer and much more powerful forms.

2.10. Remarks

The presentation here has been deliberately concrete. We could have taken Kolmogorov's extension theorem as the starting point and obtained Brownian motion more abstractly. But that would have hidden the countable Gaussian structure that will later matter. Here we have preferred to see the path being built.

It is also worth noting that the Borel space on $[0, 1]$ is playing a rather universal role. From the single random dart one extracts coin flips, from coin flips one extracts a countable family of uniform random variables, and from these one extracts a countable Gaussian family. This is one very simple way of expressing the fact that a standard Borel probability space without atoms is already rich enough for an enormous range of probabilistic constructions.

Finally, the figures in this chapter carry mathematical content. The binary expansion picture, the arrangement of digits into rows, the midpoint rule, and the dyadic tents in the Lévy construction each make visible part of the structure by which the randomness is organised. The geometric viewpoint introduced here continues into the later discussion of random curves and resampling.

CHAPTER 3

A case study in random resampling: phase separation and area trap

Important stochastic processes such as Brownian motion and the Airy_2 process describe scaled universal behaviour in random physical systems. One way to analyse such processes is to embed them in richer random objects whose larger structure carries a useful probabilistic rule. The rule may take the form of a *Gibbs resampling* principle: if a bounded part of the larger object is removed, then, conditional on what remains, the missing piece may be redrawn according to an explicit conditional law. The replacement is random, but it is random in a controlled way. In favourable cases, useful information may be read from this conditional law, and then carried back to the original object of interest.

A central example for the later lectures is the Airy_2 process. This random continuous function may be embedded as the uppermost curve $\mathcal{A}(1, \cdot)$ in the Airy line ensemble, a random $\mathbb{N}$ -indexed collection of continuous curves. With suitable interpretation of the boundary conditions in this infinite system, the Airy line ensemble is a system of Brownian motions conditioned on mutual avoidance. It follows that, if one removes the uppermost few curves on a given compact interval, then their conditional law is that of a finite system of Brownian bridges with the appropriate endpoint data, conditioned on mutual avoidance and on avoidance of the next lower curve. This Brownian Gibbs property will be specified later, and will form a central tool in what follows.

Before turning to that setting, it is useful to examine two simpler random systems. They will exhibit four characteristic features of the Kardar–Parisi–Zhang scaling picture:

- a horizontal exponent of two-thirds;
- a vertical exponent of one-third;
- local Gaussian fluctuation;
- and global curvature.

The second of the two examples will also show, in a fairly direct setting, how resampling may be used as a tool of proof.

3.1. A Brownian toy model

Let $B : \mathbb{R} \rightarrow \mathbb{R}$ be standard two-sided Brownian motion with $B(0) = 0$. Fix a large parameter $t > 0$, and define

$$X(x) = B(x) - \frac{x^2}{t}, \quad x \in \mathbb{R}.$$

Let $\mathcal{C} = \mathcal{C}_X$ denote the least concave majorant of X .

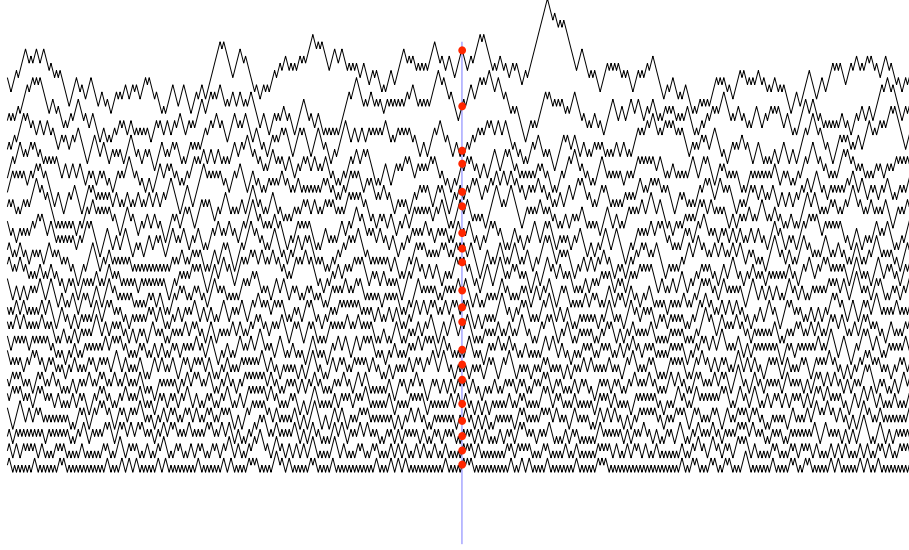


FIGURE 3.1. A discrete cyclic variant of the Airy line ensemble. In the scaling limit, the walkers converge to the Airy line ensemble, while the indicated point process converges to the scaled upper eigenvalue process in the Gaussian unitary ensemble. Simulation by Judit Zádor.

This is a baby version of the Airy_2 process. The process X is locally Brownian and globally parabolic. In fact, it also displays the characteristic KPZ powers of $2/3$ and $1/3$. One way to see these powers is to measure how closely the graph of X follows the global contour given by its concave majorant.

The graph of $\mathcal{C}$ is a closed subset of $\mathbb{R}^2$ formed from a countable union of closed planar line segments. The segment that crosses the vertical axis through the origin will be called the *facet through zero*. The *facet length* is the horizontal extent of this facet. A second quantity is the *local roughness at zero*, namely

$$\mathcal{C}(0) - X(0).$$

This is the inward deviation of the random interface from its concave hull at a typical location.

The claims are that, in mean value, the facet length has scale $t^{2/3}$ as $t \rightarrow \infty$, while the local roughness has scale $t^{1/3}$. The facet length is the horizontal scale above which parabolic curvature dominates and below which Brownian fluctuation dominates. To identify the transition, one equates the two effects:

$$\text{Brownian fluctuation} \approx \text{parabolic curvature}.$$

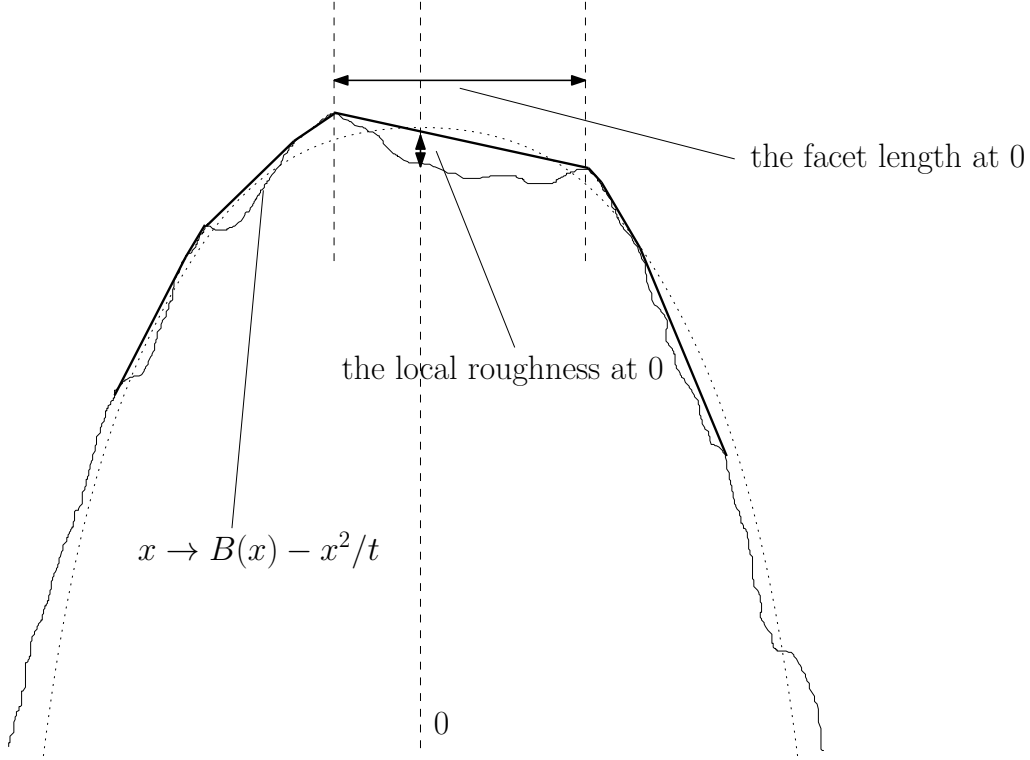


FIGURE 3.2. A toy model for the Airy_2 process: Brownian motion with a parabola of small positive curvature subtracted. The graph is locally Brownian, globally parabolic, and its concave majorant records the scale at which the two effects meet.

On a horizontal scale ℓ , the Brownian fluctuation has size $\ell^{1/2}$, while the sag contributed by the parabola has size ℓ^2/t . Thus

$$\ell^{1/2} \approx \frac{\ell^2}{t},$$

which yields

$$\ell \approx t^{2/3}.$$

This identifies the horizontal scale of the facet. Once that scale is known, the vertical scale follows from Gaussian fluctuation: the inward deviation made by X as it travels from one extreme point of the graph of its concave majorant to the next should have order equal to the square root of the horizontal extent of that journey. Thus the local roughness has scale

$$(t^{2/3})^{1/2} = t^{1/3}.$$

This example is intentionally simple, but it already isolates the mechanism that will persist later: local randomness and global curvature coexist, and the balance point between them is the source of the $2/3$ and $1/3$ powers.

3.2. Oriented random walk under an area trap

We now turn to a discrete model of phase separation.

Let Λ denote the state space of oriented, that is, downright, paths in the first quadrant of $\mathbb{Z}^2$ that begin on the y -axis and end on the x -axis. For $\gamma \in \Lambda$, let $L(\gamma)$ denote the number of edges in the

path. This is also the sum of the vertical coordinate at which γ begins and the horizontal coordinate at which it ends. Let $A(\gamma)$ denote the area of the finite region enclosed by the graph of γ and the two coordinate axes.

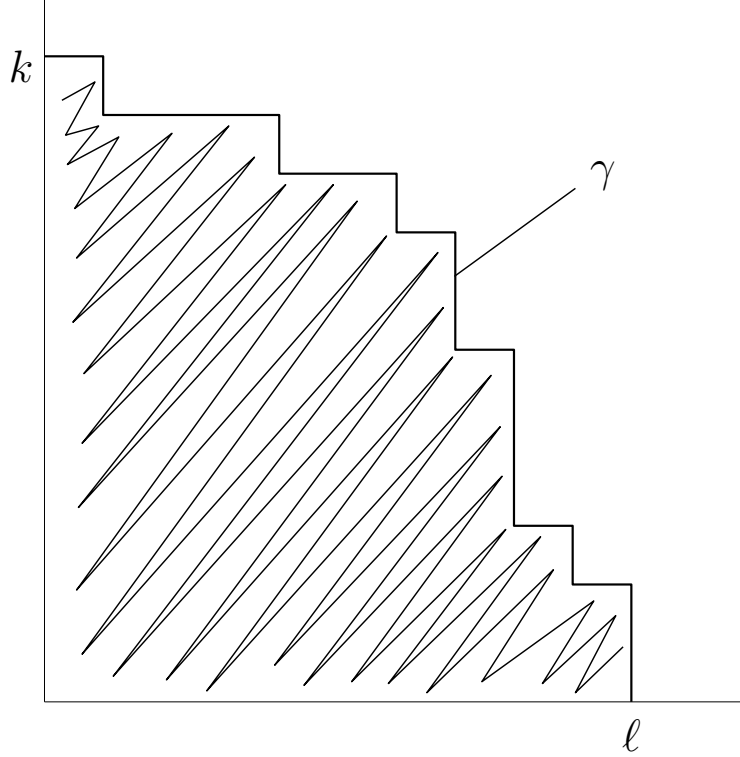


FIGURE 3.3. An element $\gamma \in \Lambda$ with $L(\gamma) = k + \ell$ and with area $A(\gamma)$ equal to the hatched region enclosed by the path and the coordinate axes.

For $n \in \mathbb{N}$, define

$$\Lambda_n = \{\gamma \in \Lambda : L(\gamma) = n\}.$$

Then $|\Lambda_n| = 2^n$. Indeed, the number of downright paths of length n started at the origin is 2^n , and each such path may be translated directly upward to a unique element of Λ_n .

Now fix $\lambda \in (0, 1/2)$, and define a probability measure $\mathbb{P}_\lambda$ on Λ by

$$\mathbb{P}_\lambda(\gamma) = Z^{-1} \lambda^{L(\gamma)},$$

where $Z \in (0, \infty)$ is the normalising constant. Since $\lambda < 1/2$, the partition function is finite. If Γ denotes a sample from $\mathbb{P}_\lambda$, then

$$\mathbb{P}_\lambda(L(\Gamma) = n) = Z^{-1} (2\lambda)^n,$$

so the random length has exponential tail decay.

This law will serve as the background measure on which an area constraint is now imposed. For $N \in \mathbb{N}$, define

$$\mathbb{P}_\lambda^{N^2} = \mathbb{P}_\lambda(\cdot \mid A(\Gamma) \geq N^2).$$

This is the area-trap model.

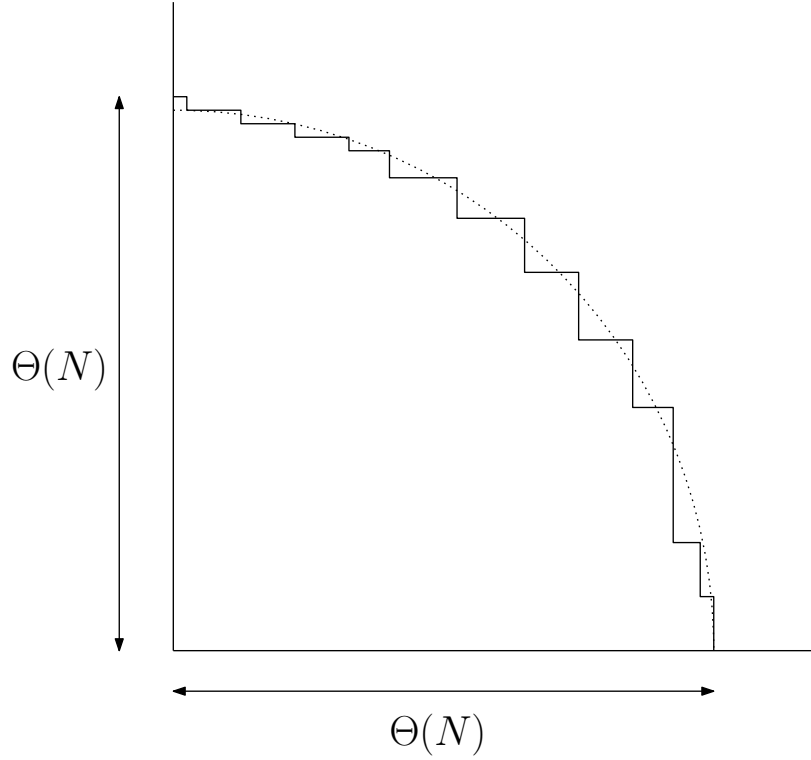


FIGURE 3.4. A caricature of a typical sample of $\mathbb{P}_\lambda^{N^2}$. The area constraint enforces a macroscopic arch, while local fluctuation remains present along the interface.

Two quantities are natural to examine at once, because they help to set the scale of the model: the length $L(\Gamma)$ of the sampled path, and the *excess area*

$$\text{ExcessArea}(\Gamma) = A(\Gamma) - N^2.$$

The typical length under $\mathbb{P}_\lambda^{N^2}$ has order N . Indeed, the event $A(\Gamma) \geq N^2$ is already typical among paths whose length is a large constant multiple of N , and under the reference law the length has exponential tail. This identifies the correct scale.

The excess area also has order N . More precisely, there exist positive constants C and c such that

$$\mathbb{P}_\lambda^{N^2}(\text{ExcessArea}(\Gamma) \geq tN) \leq Ce^{-ct} \quad (2)$$

for $t \geq C \log N$.

Why should these be the right answers? For the excess area, consider the following simple operation. Given a typical sample of $\mathbb{P}_\lambda^{N^2}$, shift the path one unit to the right, and add a horizontal edge at the starting point so that a new element of the state space is obtained. Since under the conditioned law the sampled path has vertical extent of order N , a gain in area of order N typically results from this manoeuvre. But the new path and the old one have probabilities under $\mathbb{P}_\lambda$ that differ only by a constant factor, because only one edge has been added. In this way, one sees that excess area of order N has positive probability.

The upper tail bound is obtained, roughly speaking, by reversing the same idea. If the excess area is at least tN , then one may shift the path left by order t units, cut off the part pushed into the second

quadrant, and be left with a new path that still traps area at least N^2 but has order t fewer edges than the original. The reference law therefore prefers the shortened path by a factor exponential in t , and this is the source of the estimate (2).

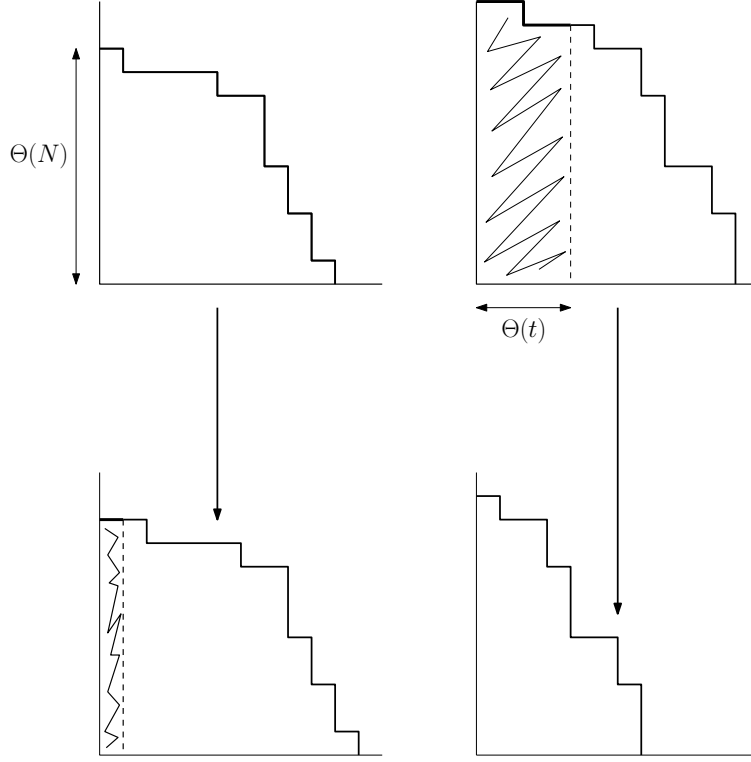


FIGURE 3.5. *Left:* a unit rightward displacement produces an area gain of order N at the cost of one additional edge. *Right:* if the excess area is of order tN , a leftward displacement of order t reduces the length by order t while preserving the area constraint.

These preliminary facts will now be brought to bear on the geometry of the conditioned interface.

The area trap model is an idealization of the interface formed in supercritical $p > p_c = 1/2$ bond percolation in two dimensions $(\mathbb{Z}^2, \sim)$ conditioned on the origin lying in an open cluster that is finite but of cardinality at least n^2 . Under this conditioning, a subcritical dual path surrounds the origin, isolating it from the unique (and generically pervasive) infinite open cluster. The dual path makes its way through the first quadrant from a location at order n above the origin on the y -axis to such a location to the right of the origin on the x -axis, following an overall downright trajectory, with local anomalies or backtracks. The path in the area trap is a surrogate for this first quadrant dual path crossing in the percolation model.

The KPZ exponent pair $(2/3, 1/3)$ for the geometry of such conditioned interfaces was established rigorously for the planar Fortuin–Kasteleyn random cluster model—a close relative of supercritical percolation—in a series of three papers [Ham12a, Ham12b, Ham11]. Those papers develop the surgical resampling technique for conditioned circuits in the random cluster setting, from which the analogous percolation results follow by the FK representation. The model studied in the present chapter is a more direct analogue: downright paths in the first quadrant conditioned to enclose

a prescribed area, with no integrability requirement. Exact cube-root fluctuations for this area-constrained walk model, including the polylogarithmic corrections, were subsequently established by D'Alimonte and Panis [DP23] using the same resampling strategy.

3.3. Facet length and local roughness

Let $\mathcal{C}_\Gamma$ denote the least concave majorant of the graph of Γ . The counterpart in this model of the facet through zero is the facet that crosses the diagonal line $y = x$. Its horizontal extent will again be called the *facet length*. The *local roughness* at a given vertex of Γ is its distance to the graph of $\mathcal{C}_\Gamma$.

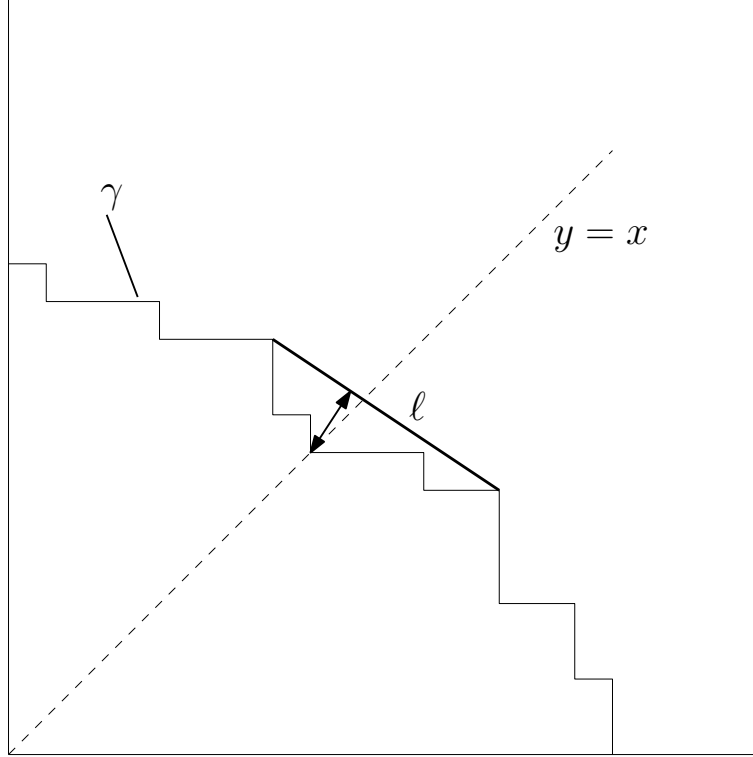


FIGURE 3.6. *Left:* the facet through the diagonal and its length. *Right:* local roughness measured by the distance from a vertex of γ to the graph of its concave majorant.

The model exhibits the same two powers as the Brownian toy model.

THEOREM 3.1. Under $\mathbb{P}_\lambda^{N^2}$, the mean facet length has order $N^{2/3}$, and the mean local roughness has order $N^{1/3}$.

We will explain the substantive elements of the proof by means of resampling. The random path will be sampled, a suitable part of it will be erased and redrawn, and the law of the output will again be $\mathbb{P}_\lambda^{N^2}$. The key point is that the resampling may, with positive probability, force an atypically large gain in area whenever the input possesses too long a facet. The excess area estimate will then imply that such facets are rare.

3.4. The resampling rule

To prepare for the argument, we first note the resampling property of the reference law $\mathbb{P}_\lambda$.

Take a sample Γ of $\mathbb{P}_\lambda$, and select two points a and b on Γ , with b more clockwise than a . These points may be chosen at fixed distances along the path from the starting point, or by a search procedure that locates them without revealing the subpath between them. What matters is that the randomness of the middle section remains untouched when the points are chosen.

The path may then be decomposed into three pieces: an initial segment from the y -axis to a , a middle segment from a to b , and a final segment from b to the x -axis. Remove the middle section. Conditional on what remains, the missing piece has the uniform law on downright paths from a to b . This is because, under $\mathbb{P}_\lambda$, all such middle pieces have the same length and thus the same weight.

Under the conditioned law $\mathbb{P}_\lambda^{N^2}$, the same statement remains true subject to the inherited area constraint. If the segment between a and b is removed, then the missing piece is uniformly distributed among all downright paths from a to b such that, when inserted back into the surrounding path, the resulting full path traps total area at least N^2 . Equivalently, within the rectangle with northwest corner a and southeast corner b , the resampled path is uniform subject to a lower bound on the area it traps against the lower and left edges of that rectangle.

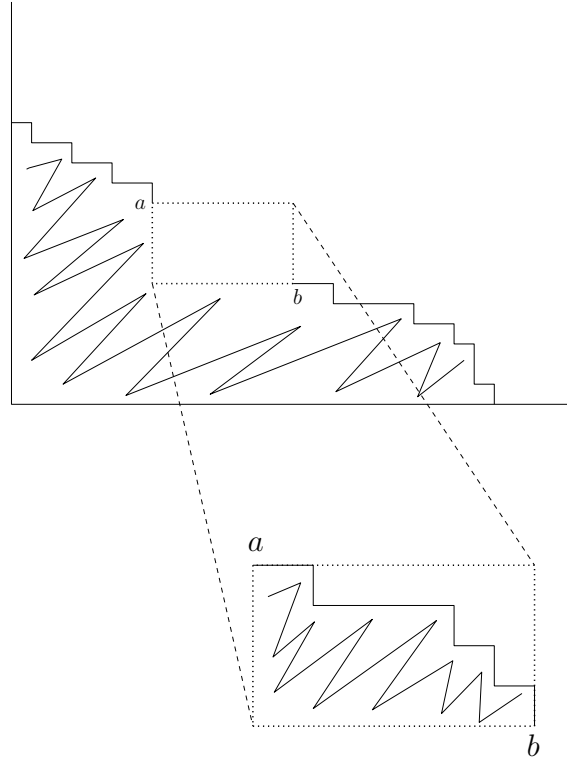


FIGURE 3.7. Resampling under $\mathbb{P}_\lambda^{N^2}$. The removed segment between a and b is redrawn uniformly among all downright paths compatible with the required total trapped area.

This is the random surgery that will drive the proof.

3.5. The resampling argument for the facet-length upper bound

Fix $\varepsilon > 0$. We will show that

$$\mathbb{P}_\lambda^{N^2}(\text{the facet length of } \Gamma \text{ is at least } N^{2/3+\varepsilon}) \rightarrow 0 \quad (3)$$

as $N \rightarrow \infty$. This gives the upper bound on the horizontal exponent.

Let $\mathbb{P}$ denote the law of the following experiment. First sample Γ^{in} from $\mathbb{P}_\lambda^{N^2}$. This is the input path. Then pick two vertices a and b independently and uniformly along Γ^{in} , subject to the condition that b is clockwise of a . Remove the segment of the path from a to b , and replace it by a new downright path connecting a to b , sampled from the conditional resampling law described above. The resulting path will be denoted Γ^{out} . By construction, if the input has law $\mathbb{P}_\lambda^{N^2}$, then so does the output.

Define the event

$$\text{BigFacet} = \{\text{the facet length of } \Gamma^{\text{in}} \text{ is at least } N^{2/3+\varepsilon}\}.$$

We next isolate the event that the resampling happens to strike exactly this facet. Let

$$\text{GoodHit} = \{a \text{ and } b \text{ are the endpoints of the facet through the diagonal of } \Gamma^{\text{in}}\}.$$

Since the typical length of Γ^{in} has order N , the chance of selecting the correct ordered pair of vertices is of order N^{-2} :

$$\mathbb{P}(\text{GoodHit} \mid \text{BigFacet}) = O(1)N^{-2}.$$

There is one further favourable event. Let Rect be the rectangle whose northwest corner is a and whose southeast corner is b . The resampled path from a to b divides Rect into a lower-left region and an upper-right region. Define

$$\text{MoreOutThanIn}$$

to be the event that the lower-left region has area at least one half of the area of Rect together with an additional amount of order $\|a - b\|^{3/2}$. A uniform random path across a rectangle has Gaussian fluctuation, and the area enclosed below it fluctuates on precisely this scale. Consequently there exists a constant $c > 0$ such that, for every N ,

$$\mathbb{P}(\text{MoreOutThanIn} \mid \text{BigFacet} \cap \text{GoodHit}) \geq c.$$

The central claim is that

$$\text{BigFacet} \cap \text{GoodHit} \cap \text{MoreOutThanIn} \subseteq \left\{A(\Gamma^{\text{out}}) \geq N^2 + cN^{1+3\varepsilon/2}\right\}. \quad (4)$$

To verify this, let Γ' denote the path obtained from Γ^{in} by replacing the segment between a and b with the straight line segment joining these endpoints. On GoodHit , the points a and b are consecutive extreme points of the concave majorant, and so

$$A(\Gamma^{\text{in}}) \leq A(\Gamma').$$

On the other hand, the event $\text{BigFacet} \cap \text{GoodHit} \cap \text{MoreOutThanIn}$ ensures that the resampled path encloses below the chord an additional area of at least $c\|a - b\|^{3/2}$. Hence

$$A(\Gamma^{\text{out}}) \geq A(\Gamma') + c\|a - b\|^{3/2}.$$

But on $\text{BigFacet} \cap \text{GoodHit}$ we have

$$\|a - b\| \geq N^{2/3+\varepsilon},$$

so

$$c\|a - b\|^{3/2} \geq cN^{1+3\varepsilon/2}.$$

Combining the last three inequalities yields (4).

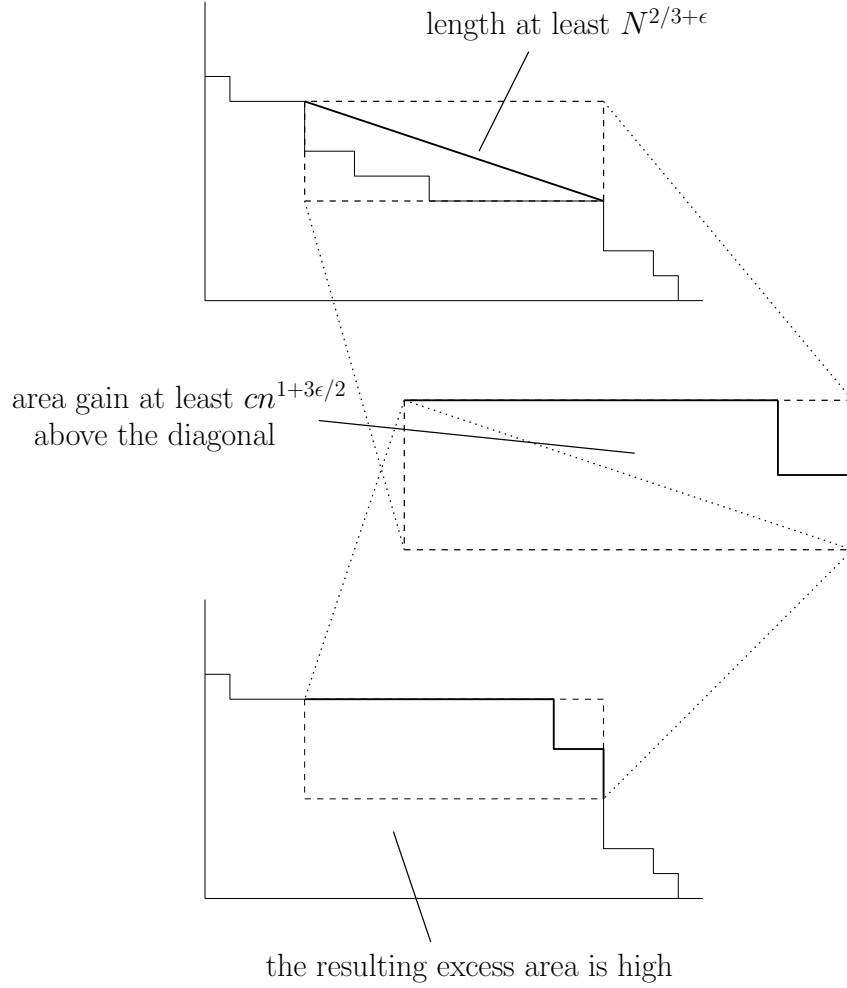


FIGURE 3.8. If the input path has a long facet and the resampling strikes exactly that facet, then a favourable fluctuation of the resampled middle piece creates a macroscopic gain in enclosed area.

We now compare probabilities. Since

$$\mathbb{P}(\text{GoodHit} \mid \text{BigFacet}) = O(1)N^{-2} \quad \text{and} \quad \mathbb{P}(\text{MoreOutThanIn} \mid \text{BigFacet} \cap \text{GoodHit}) \geq c,$$

we obtain

$$\begin{aligned} \mathbb{P}_\lambda^{N^2}(\text{the facet length of } \Gamma \text{ is at least } N^{2/3+\epsilon}) &\times O(1)N^{-2} \times c \\ &\leq \mathbb{P}(\text{BigFacet} \cap \text{GoodHit} \cap \text{MoreOutThanIn}). \end{aligned}$$

By the central claim, this is at most

$$\mathbb{P}(A(\Gamma^{\text{out}}) \geq N^2 + cN^{1+3\epsilon/2}).$$

Since Γ^{out} under $\mathbb{P}$ has law $\mathbb{P}_\lambda^{N^2}$, the last quantity is

$$\mathbb{P}_\lambda^{N^2}(\text{ExcessArea}(\Gamma) \geq cN^{1+3\epsilon/2}).$$

Applying the excess area estimate (2) with $t = cN^{3\varepsilon/2}$ yields

$$\mathbb{P}_\lambda^{N^2}(\text{ExcessArea}(\Gamma) \geq cN^{1+3\varepsilon/2}) \leq Ce^{-c'N^{3\varepsilon/2}}$$

for some $c' > 0$. Therefore

$$\mathbb{P}_\lambda^{N^2}(\text{the facet length of } \Gamma \text{ is at least } N^{2/3+\varepsilon}) \leq O(1)N^2e^{-c'N^{3\varepsilon/2}},$$

which proves (3).

The argument is simple in outline but quite telling. A facet that is too long creates, after a lucky hit by the resampling and a further favourable but constant-probability fluctuation, an output path whose excess area is much too large to be typical. Since the output law is the same as the input law, the long facet must itself have been atypical.

3.6. The remaining bounds and the KPZ picture

Theorem 3.1 makes two assertions: that facet length has order $N^{2/3}$, and that local roughness has order $N^{1/3}$. We have explained the reason for the upper bound on facet length. We will not pursue in detail the accompanying lower bound, nor the corresponding bounds for local roughness. Their source is the same broad mechanism. Below the scale $N^{2/3}$, the path is largely free from the demands of the area constraint and retains Gaussian fluctuation inherited from the reference law; above that scale, the global arch forced by trapped area becomes dominant. The vertical scale of order $N^{1/3}$ is then the Gaussian fluctuation associated to horizontal scale $N^{2/3}$.

Thus the area-trap model exhibits the same broad KPZ features as the Brownian toy model: the exponents $2/3$ and $1/3$, local Gaussianity, and global curvature. In that sense it belongs to a baby KPZ universality class. It does not display the full rich scaled distributional structure of the genuine KPZ class, but it does organise fluctuation and shape in the same way, and it does so in a setting where resampling may be seen at work very concretely.

The basic resampling method also leads to stronger conclusions. Let $\text{MFL}(\Gamma)$ denote the maximum facet length among all facets of the concave majorant of Γ , and let $\text{MLR}(\Gamma)$ denote the maximum local roughness, that is, the maximum over vertices of Γ of the distance from the vertex to the graph of the concave majorant. The same line of argument, pursued more finely, yields the polylogarithmic corrections to the one-third and two-thirds powers.

THEOREM 3.2. There exist positive constants c and C such that

$$\mathbb{P}_\lambda^{N^2}\left(c \leq \frac{\text{MLR}(\Gamma)}{N^{1/3}(\log N)^{2/3}} \leq C\right) \rightarrow 1$$

and

$$\mathbb{P}_\lambda^{N^2}\left(c \leq \frac{\text{MFL}(\Gamma)}{N^{2/3}(\log N)^{1/3}} \leq C\right) \rightarrow 1$$

as $N \rightarrow \infty$.

These logarithmic corrections have close analogues in genuinely KPZ settings, for example in problems concerning the maximal fluctuation and maximal energy of short geodesics. In this respect, too, the area-trap model is a useful testing ground. It permits the resampling method to be seen in a comparatively direct form, while already leading to the characteristic exponents, curvature, local Gaussian fluctuation, and logarithmic refinements that later reappear in richer models.

CHAPTER 4

Poissonian last passage percolation and its line ensemble

This chapter has two complementary aims. On the geometric side, we explain how Poissonian last passage percolation gives rise to a multiline object whose lower lines encode the energies of disjoint bouquets. On the probabilistic side, we show that the same object may be described through conditioned systems of continuous-time simple random walks, and hence satisfies a natural Gibbs property. This pre-limiting picture is the model for the Airy line ensemble and its Brownian Gibbs property, to which we shall turn later in the course.

4.1. Poissonian last passage percolation and bouquets

In Poissonian last passage percolation, a Poisson point process $\mathcal{P}$ of unit intensity is sampled in the cone

$$C = \{(x, t) \in \mathbb{R} \times [0, \infty) : |x| \leq t\}.$$

Let $\mathbb{Q}$ denote the associated probability measure. The apex $(0, 0)$ of the cone will play the role of the common starting point for the paths that we consider.

Fix $(x, t) \in \mathbb{R} \times [0, \infty)$. An *upgoing* path from $(0, 0)$ to (x, t) is formed of a finite collection of planar line segments, each of which makes an angle with the vertical of at most one half of a right angle. Thus, in travelling from $(0, 0)$ to (x, t) along such a path, one always moves north-east or north-west, or else vertically, changing direction only finitely many times. An upgoing path from $(0, 0)$ to (x, t) exists precisely when $|x| \leq t$, that is, precisely when (x, t) lies in the cone C .

For $x \in \mathbb{R}$, let $h(x, t)$ denote the maximum number of elements of the Poisson cloud $\mathcal{P}$ that lie on an upgoing path from $(0, 0)$ to (x, t) . If no such path exists, because $|x| > t$, we set $h(x, t) = 0$. The profile $x \mapsto h(x, t)$ is the basic last passage object associated to the cloud at time t .

To pass from a single maximizing path to a line ensemble, it is natural to consider several disjoint paths at once. For $j \in \mathbb{N}$, define

$$D_j(x, t) = \left\{ (\phi_1, \dots, \phi_j) : \begin{array}{l} \phi_i \text{ is an upgoing path from } (0, 0) \text{ to } (x, t) \text{ for each } i \in \llbracket 1, j \rrbracket, \\ \phi_1, \dots, \phi_j \text{ are pairwise disjoint except at their common endpoints} \end{array} \right\}.$$

An element of $D_j(x, t)$ is thus a collection of j upgoing paths, each starting at $(0, 0)$ and ending at (x, t) , whose only points of mutual contact are these endpoints. The generic element is a family of disjoint paths tied together at both ends, and we shall call it a j -bouquet.

For $\phi \in D_1(x, t)$, let $\ell(\phi)$ denote the number of Poisson points visited by ϕ . For a bouquet $(\phi_1, \dots, \phi_j) \in D_j(x, t)$, define its *energy* by

$$\ell((\phi_1, \dots, \phi_j)) = \sum_{i=1}^j \ell(\phi_i).$$

We then write

$$M_j(x, t) = \max \{ \ell(\phi) : \phi \in D_j(x, t) \}$$

for the maximum energy of a j -bouquet from $(0, 0)$ to (x, t) .

These quantities will later be encoded by the multiline polynuclear growth (PNG) construction: a dynamical procedure, developed in detail in the next section, that grows from the Poisson cloud an ordered family of height lines $L_1, L_2, \dots$, the first of which is the last passage profile $h(\cdot, t)$. We preview the geometric punchline here, anticipating the terminology to be introduced shortly. The lower lines carry genuine geometric content: they record the optimal energies of disjoint bouquets in the Poisson cloud. To state this cleanly, let L_i denote the i^{th} line in this multiline evolution. Since the line L_i begins at height $1 - i$, the quantity $L_i(x, t) + i - 1$ measures how far that line has risen above its initial level at the space-time point (x, t) . The first j lines together then recover the maximum energy of a j -bouquet.

PROPOSITION 4.1. For $j \in \mathbb{N}$ and $(x, t) \in \mathbb{R} \times [0, \infty)$,

$$\sum_{i=1}^j (L_i(x, t) + i - 1) = M_j(x, t).$$

We shall return to this identity after developing the multiline PNG construction. At present, its role is to indicate the geometric meaning of the line ensemble that we are about to build.

4.2. PNG dynamics and the multiline construction

We now explain how the Poisson cloud generates a system of height lines. The construction is dynamical. One should imagine watching the horizontal time-line rise through the cone and, as it does so, seeing a family of interfaces evolve in response.

4.2.1. The PNG interface. The PNG dynamics was introduced by Prähofer and Spohn [PS02]. The dynamics begins at time zero with a single flat interface,

$$L_1(\cdot, 0) \equiv 0.$$

As time runs, the Poisson cloud $\mathcal{P}$ is revealed by the rising horizontal line $\mathbb{R} \times \{t\}$. When this rising line first encounters a point $(x, t) \in \mathcal{P}$, a nucleation occurs on the interface L_1 : at time t , a small island of height one is created at the spatial location x . Equivalently, one may think of a $+1$ jump and a -1 jump being born at the same point.

After nucleation, the two jump discontinuities move apart deterministically. The left jump travels with velocity -1 , and the right jump with velocity $+1$. Thus, if a nucleation occurs at (x, t) , then at a later time $t' > t$ the region created by this event occupies the interval

$$(x - (t' - t), x + (t' - t)),$$

provided that no collision has intervened.

When a second Poisson point is encountered by the rising time-line, a second nucleation occurs, again on the uppermost interface L_1 . This may happen over a portion of the interface that is presently at height zero or at height one; in either case, the new island is built on top of the current profile. Continuing in this way, every Poisson point creates a nucleation on L_1 at the moment when it is encountered by the rising time-line.

The evolution is deterministic between nucleations. The only further events are collisions: a right-moving down-step may meet a left-moving up-step. When this happens, the two jumps annihilate. In the single-layer picture, that is the end of the story: the interface simply closes up across the

collision point, and the resulting height profile continues to evolve until the next nucleation or collision.

For each $t \geq 0$, the random function $x \mapsto L_1(x, t)$ is the PNG interface at time t . Its connection to last passage percolation is fundamental.

PROPOSITION 4.2. For every $(x, t) \in \mathbb{R} \times [0, \infty)$,

$$L_1(x, t) = h(x, t).$$

This identity expresses the fact that the top PNG line records the ordinary last passage profile. The multiline construction extends this picture by retaining information that is lost when collisions occur.

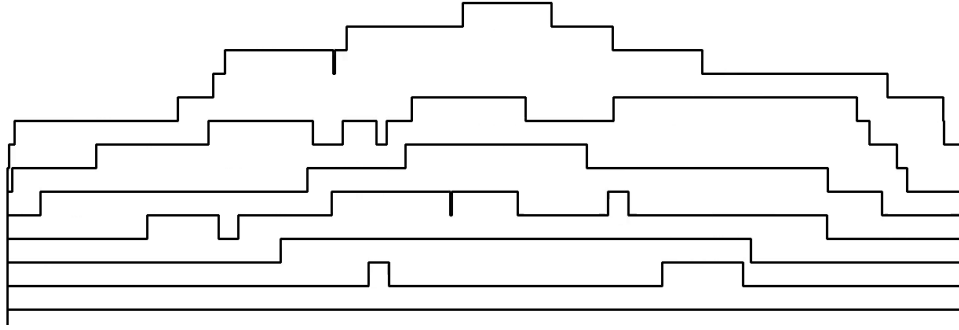


FIGURE 4.1. A snapshot of the multi-line PNG dynamics. Simulation: Derek Wu.

4.2.2. The multiline construction. To define the lower lines, one modifies only the collision rule. The first line still evolves exactly as before: every Poisson point creates a nucleation on L_1 , and between such events the jumps on L_1 move laterally with velocities -1 and $+1$.

Now consider a collision on L_1 . In the single-line picture, the two colliding jumps would simply annihilate. In the multilayer picture, they still annihilate on L_1 , but the information carried by the collision is not discarded. Instead, at that same space-time point a new nucleation is created on the next line below, L_2 .

The line L_2 evolves by exactly the same local rules: its nucleations occur at the collision points of L_1 ; after each nucleation, the two resulting jumps move apart with velocities -1 and $+1$; and when a right-moving down-step and a left-moving up-step collide on L_2 , they annihilate there and simultaneously create a nucleation on L_3 .

Proceeding inductively, one obtains a countable family of lines

$$L_i(\cdot, t), \quad i \in \mathbb{N},$$

with initial conditions

$$L_i(\cdot, 0) \equiv 1 - i.$$

so that successive lines are initially separated by exactly one unit, and all lines are initially flat. This unit gap between consecutive lines is preserved by the dynamics, as we verify in Lemma 4.3 below. The first line is driven directly by the Poisson cloud, while each lower line is created from the collision history of the line above.

It is also important to note that the multilayer system remains finite at each fixed time. Every time the rising time-line encounters a Poisson point, a single nucleation is created on the top line, and this contributes exactly two new fronts to the full system. By contrast, a collision on any line removes two fronts from that line and simultaneously creates two fronts on the next line down. Thus collisions preserve the total number of fronts in the multilayer system. After the first m Poisson points have been encountered, the total number of fronts present in all lines together is therefore exactly $2m$. Since the number of Poisson points in $C \cap (\mathbb{R} \times [0, t])$ is almost surely finite for each fixed t , it follows that only finitely many lines are non-flat at time t .

The lower lines arise canonically from the collision cascade, and they record deeper combinatorial information about the cloud. In particular, as indicated in Proposition 4.1, the first j lines together encode the maximum energy of a disjoint j -bouquet.

We shall not prove Proposition 4.1 at this point. For the moment, the important point is that the multiline construction is both geometrically natural and information-preserving. In the next section, we make this more precise by fixing a time T and analysing the deterministic map that sends a finite cloud in the cone up to height T to the corresponding multiline configuration at that time.

4.3. The cloud-line map at fixed time

We now fix a time horizon $T > 0$ and make precise the deterministic map that sends a finite cloud in the cone up to height T to the corresponding multiline PNG configuration at time T . This map is already implicit in the film description of the previous section, but it is worth isolating it carefully. It is the fixed-time backbone of the later probabilistic comparison with conditioned random walks.

Write

$$C_T = C \cap (\mathbb{R} \times [0, T])$$

for the cone up to height T . A *cloud configuration* in C_T is a finite subset of C_T .

A convenient way to understand the map from clouds to lines is again to follow the PNG evolution up to time T as a film, exactly as described in the previous section. Starting from a cloud configuration $\omega \subseteq C_T$, the rising time-line creates a zero-width nucleation on the first line at each point of ω ; thereafter the two fronts of each nucleation move apart with velocities -1 and $+1$; and whenever a rightgoing front and a leftgoing front meet on a line, they annihilate there and create a new nucleation at the same space-time point on the line below. Running this evolution until time T produces a multiline configuration

$$\{L_i(\cdot, T) : i \in \mathbb{N}\},$$

and only finitely many lines are non-flat, because the cloud in C_T is finite.

At this point it is worth making one convention explicit. We regard the time evolution as càdlàg. Thus a nucleation at time t is already present in the configuration at time t , though still of zero width, and similarly a collision at time t is recorded in its post-collision form at time t : the two colliding jumps have disappeared from the upper line, and the new nucleation on the next line below has already been created. This convention makes the evolution right-continuous and gives each event a definite status at its own time.

To describe the correspondence cleanly, we restrict attention to cloud configurations for which no two points lie on a common characteristic line of slope ± 1 , for which no two nucleation or collision events in the induced PNG evolution occur simultaneously, and for which the terminal time T is not itself an event time. This is the generic situation. The excluded set is negligible under the Poisson

law, and the restriction serves only to ensure that the event structure is strictly ordered and that the fixed-time profile at time T is an ordinary one, with all jump locations distinct.

A basic structural feature of the multilayer PNG construction is that the lines remain strictly ordered throughout the evolution: each line stays at least one unit below the line above it. Since the proof is a little separate from the main thread of the section, we postpone it to the end.

LEMMA 4.3. Assume that the terminal time T is not an event time of the multilayer PNG evolution induced by the cloud ω . Then, for every $i \in \mathbb{N}$, every time $t \in [0, T]$ that is not an event time, and every $x \in \mathbb{R}$,

$$L_{i+1}(x, t) \leq L_i(x, t) - 1.$$

In particular, the fixed-time configuration at time T is strictly ordered, and hence non-intersecting.

Correspondingly, we call a multiline configuration at time T *admissible* if it is a sequence of integer-valued càdlàg functions on $[-T, T]$ such that only finitely many differ from their original constant heights, such that the jumps are all of size ± 1 , such that the jump locations are all distinct, and such that the lines are strictly ordered. By Lemma 4.3, the forward PNG film associates an admissible configuration to every generic cloud in C_T .

The same film may be run backwards. Start from an admissible multiline configuration at time T and let time decrease from T to 0. Since the event times are distinct, there is a well-defined last event. If that event is a nucleation on the first line, then in the reversed film it disappears, and we record the corresponding space-time point in the cloud. If the last event is a nucleation on a lower line, then in the forward dynamics it could only have been created by a collision on the line above, so in the reversed film we restore that collision. Iterating this procedure removes the events one by one and reconstructs a unique cloud configuration in C_T .

This discussion leads to the following proposition.

PROPOSITION 4.4. Fix $T > 0$. The PNG evolution up to time T induces a bijection between generic finite cloud configurations in C_T and admissible multiline PNG configurations at time T .

PROOF. The forward map is given by the PNG evolution. Since the cloud in C_T is finite, only finitely many nucleations occur before time T , and hence only finitely many lines are affected. By Lemma 4.3, the resulting multiline configuration is admissible.

Conversely, let an admissible multiline configuration at time T be given. Running the film backwards, one removes the last event first. If this last event is a nucleation on the first line, it yields a point of the cloud. If it is a nucleation on a lower line, it is reversed by reinstating the collision on the line above that created it. Since event times are distinct and the lines are ordered, this backward reconstruction is unambiguous. Repeating it finitely many times recovers a unique cloud configuration in C_T , and the two constructions are inverse. $\square$

We now return to the deferred proof of the ordering lemma.

PROOF OF LEMMA 4.3. Let

$$0 < \tau_1 < \tau_2 < \cdots < \tau_M < T$$

be the global event times of the multilayer PNG evolution induced by the cloud ω , listed in increasing order. Here an event means either a nucleation or a collision on one of the lines. By genericity,

these times are distinct. Write $\tau_0 = 0$, and set $\tau_{M+1} = +\infty$. Since T is not an event time and τ_M is the last event time before T , no event occurs in the half-open interval $(\tau_M, T]$.

We shall prove that for every $r \in \{0, 1, \dots, M\}$, every $t \in (\tau_r, \tau_{r+1}) \cap [0, T]$, every $i \in \mathbb{N}$, and every $x \in \mathbb{R}$,

$$L_{i+1}(x, t) \leq L_i(x, t) - 1.$$

Since T is not an event time, T lies in (τ_M, τ_{M+1}) , and the case $r = M$ of this claim then gives the lemma.

The proof has two parts. First, we record the deterministic evolution on an interval containing no event on either of two adjacent lines. Secondly, we check that each possible event preserves the one-level gap immediately after the event.

No-event evolution. Fix $i \in \mathbb{N}$, and suppose that on a time interval $[s, u]$ no event occurs on either of the two lines L_i and L_{i+1} . For an integer-valued profile f and $m \in \mathbb{Z}$, write

$$E_m(f) = \{x \in \mathbb{R} : f(x) \geq m\}.$$

Since no event occurs on line L_i during $[s, u]$, the combinatorial jump pattern of L_i is frozen there: every up-step moves left with speed 1 and every down-step moves right with speed 1. Consequently, for each $m \in \mathbb{Z}$, every connected component of $E_m(L_i(\cdot, s))$ simply expands linearly at speed 1 on both sides until time u . The same is true of L_{i+1} . In particular, if for some $t_0 \in [s, u]$ one has

$$E_m(L_{i+1}(\cdot, t_0)) \subseteq E_{m+1}(L_i(\cdot, t_0)) \quad \text{for all } m \in \mathbb{Z},$$

then the same inclusion holds for every $t \in [t_0, u]$, because corresponding interval components on the two sides are subjected to the same symmetric expansion. Equivalently, if

$$L_{i+1}(x, t_0) \leq L_i(x, t_0) - 1 \quad \text{for all } x \in \mathbb{R},$$

then the same inequality holds for every $t \in [t_0, u]$.

Initial interval. On the interval $(\tau_0, \tau_1) = (0, \tau_1)$, no event occurs anywhere. Since the initial conditions are

$$L_i(\cdot, 0) \equiv 1 - i, \quad L_{i+1}(\cdot, 0) \equiv -i,$$

we have

$$L_{i+1}(x, 0) = L_i(x, 0) - 1 \quad \text{for all } x \in \mathbb{R}.$$

By the no-event evolution just described, the same equality, and hence the desired inequality, holds throughout $(0, \tau_1)$.

Inductive step. Assume now that for some $r \in \{1, \dots, M\}$ the desired inequality holds on the whole interval (τ_{r-1}, τ_r) . We shall show that it also holds on (τ_r, τ_{r+1}) .

Fix an index $i \in \mathbb{N}$. We inspect what may happen at the event time τ_r for the pair L_i, L_{i+1} . There are four possibilities.

Case 1: no event occurs on either L_i or L_{i+1} at time τ_r . Then both lines are unchanged at τ_r , and the no-event evolution carries the inequality from (τ_{r-1}, τ_r) across the whole interval (τ_r, τ_{r+1}) .

Case 2: a nucleation occurs on L_i at time τ_r . For positive times immediately after τ_r , the line L_i has acquired a new small terrace, one unit above its former height at the nucleation point, while L_{i+1} is unchanged. Thus the inequality

$$L_{i+1} \leq L_i - 1$$

still holds for all spatial points sufficiently soon after τ_r . Since no further event occurs before τ_{r+1} , the no-event evolution then extends this to the whole interval (τ_r, τ_{r+1}) .

Case 3: a collision occurs on L_i at time τ_r . Let x_0 be the collision point, and let h be the retained height on L_i after the collision. Just before the event, L_i has a valley of height $h - 1$ collapsing at x_0 . By the càdlàg convention, at time τ_r that valley has already disappeared, so L_i is locally flat at height h near x_0 . Simultaneously, a zero-width nucleation is created on L_{i+1} , raising it there to height at most $h - 1$, since just before the event $L_{i+1}(x_0) \leq L_i(x_0) - 1 = h - 2$ and the nucleation adds one unit. Hence, for all sufficiently small positive times after τ_r , the lower line has a tiny terrace at height at most $h - 1$ while the upper line remains locally at height h . Thus the one-level gap is preserved immediately after the event, and the no-event evolution propagates it through (τ_r, τ_{r+1}) .

Case 4: a collision occurs on L_{i+1} at time τ_r . Let x_0 be the collision point, and let h be the retained height on L_{i+1} after the collision. Just before the event, there is a disappearing valley on L_{i+1} of height $h - 1$, bounded by a rightgoing front on the left and a leftgoing front on the right, and the line takes the value h immediately to either side of this valley.

We claim that L_i is locally flat at height at least $h + 1$ across the spatial span of this collapsing valley just before time τ_r . Indeed, by the induction hypothesis $L_{i+1} \leq L_i - 1$ just before τ_r , so at the two shoulders of the valley, where $L_{i+1} = h$, we have $L_i \geq h + 1$. Suppose L_i had a jump strictly inside the valley span at this time. As the valley floor shrinks to the single point x_0 at time τ_r , such a jump would bring a level set of L_i to closure at x_0 at that same instant, that is, a collision on L_i at x_0 at time τ_r . This would be a second global event occurring simultaneously with the collision on L_{i+1} , contradicting the genericity assumption that event times are distinct. Hence L_i carries no jump inside the span and is locally flat there at height at [wrong description, could be disappearing a moment earlier] least $h + 1$. When the lower collision occurs, the valley on L_{i+1} disappears and the line is raised to the retained level h , which is still at least one unit below the unchanged upper height. Hence the inequality again holds for all sufficiently small positive times after τ_r , and the no-event evolution propagates it through (τ_r, τ_{r+1}) .

This completes the induction on r . Therefore, for every r and every $t \in (\tau_r, \tau_{r+1}) \cap [0, T]$, one has

$$L_{i+1}(x, t) \leq L_i(x, t) - 1 \quad \text{for all } x \in \mathbb{R}.$$

Since T is not an event time, $T \in (\tau_M, \tau_{M+1})$, and so the same inequality holds at time T . Thus the configuration at time T is strictly ordered, and in particular non-intersecting. $\square$

The significance of Proposition 4.4 goes beyond providing a convenient coding of the PNG evolution. It says that, at a fixed time, the cloud picture and the line picture carry exactly the same finite information. In the next section we place natural reference measures on the two sides and compare them locally. That comparison leads to the finite-volume bridge identity, Proposition 4.5.

4.4. Finite walk ensembles and a finite-volume bridge identity

We now turn from the deterministic correspondence of the previous section to the probabilistic comparison that will identify the finite-volume law of the PNG lines. The key point is that the same local line configurations admit two natural ambient descriptions. On the one hand, they arise from a Poisson cloud in the cone. On the other hand, they may be generated by a finite system of independent continuous-time simple random walks. Proposition 4.5 will say that, after the appropriate conditioning and normalisation, these two pictures agree exactly.

The walk picture is introduced for a very concrete reason. Before any conditioning is imposed, its infinitesimal weights are especially simple: each active line is driven by independent Poisson clocks, so the probability of realising a prescribed local jump pattern can be read off directly. This is the line-space analogue of the elementary Poisson calculation on the cloud side. It is this local tractability that makes the walk ensemble useful.

One must, however, work first with only finitely many active lines. If one were to begin immediately with an infinite system, the corresponding reference measure would have infinite total jump intensity on any time interval, and the basic local calculation would no longer be literal. This is the reason for introducing the finite ensembles $\mathbb{P}_{n,T}$. They capture the top n lines, while all lower lines remain frozen at their initial heights.

Of course, under this unconditioned walk law the configurations that resemble the PNG picture are rare. One must require that each active walk return to its assigned endpoint at time T , and one must also require that the walks remain mutually non-intersecting. These conditions are restrictive, but that causes no difficulty. They are precisely the constraints needed to isolate from the ambient independent-walk system the same class of admissible line configurations that arises from the PNG dynamics. Once one conditions on these events, the walk law is simply renormalised onto the relevant finite configuration space.

The cloud side has a parallel finite-volume restriction. A cloud in C_T almost surely generates only finitely many active lines by time T , and the event $\{D(T) \leq n\}$ says precisely that no activity has reached below the n^{th} line. Thus the finite walk ensemble and the finite-depth cloud picture meet on the same portion of the configuration space. As n increases, this portion captures more and more of the full picture. As already noted, the depth at fixed time is almost surely finite; this is recorded formally in Lemma 4.7 below.

4.4.1. The finite walk ensemble. Fix $n \in \mathbb{N}$ and $T > 0$. For $i \in \llbracket 1, n \rrbracket$, let

$$X_i : [-T, T] \rightarrow \mathbb{Z}$$

be independent continuous-time simple random walks of jump rate two, started at time $-T$ from

$$X_i(-T) = 1 - i.$$

Equivalently, each walk is driven by a rate-two Poisson clock, and at each ring makes a jump of size ± 1 with equal probability. Let $\mathbb{P}_{n,T}$ denote the joint law of the n -tuple $(X_1, \dots, X_n)$.

Write R for the event that each walk returns to its initial height at time T :

$$R = \bigcap_{i=1}^n \{X_i(T) = 1 - i\}.$$

Write also NI for the event that the walks remain strictly ordered throughout the time interval and stay strictly above the level $-n$:

$$\text{NI} = \left\{ X_1(s) > X_2(s) > \dots > X_n(s) > -n \text{ for all } s \in [-T, T] \right\}.$$

The lower bound $X_n(s) > -n$ records the presence of a frozen bottom line at height $-n$, which plays the role of an $(n+1)^{\text{th}}$ curve held identically at $-n$; on the cloud side it corresponds to the constraint that the n^{th} line cannot descend below its initial level when $\{D(T) \leq n\}$ holds. On the event $R \cap \text{NI}$, the n walks form a system of mutually avoiding bridges from the levels $0, -1, \dots, 1-n$ back to themselves, all remaining strictly above $-n$. This is the finite line ensemble that will be compared with the PNG picture.

PROPOSITION 4.5. Fix $n \in \mathbb{N}$ and $T \in [0, \infty)$, and let Φ be the cloud–line bijection of Proposition 4.4.

- (i) The two conditioning events have masses in a fixed ratio,

$$\mathbb{P}_{n,T}(R \cap \text{NI}) = \exp\{T(T - 4n)\} \mathbb{Q}(D(T) \leq n).$$

- (ii) Once normalised, the two pictures are the same law: the conditioned walk law $\mathbb{P}_{n,T}(\cdot \mid R \cap \text{NI})$ is the image under Φ of the conditioned cloud law $\mathbb{Q}(\cdot \mid D(T) \leq n)$.

Before proving this, we compare small neighbourhoods on the cloud side and on the line side.

4.4.2. Local neighbourhoods in cloud space and line space. Fix $n \in \mathbb{N}$ and $T > 0$, and let L be a proper admissible multiline configuration at time T with depth at most n . Let ω denote the corresponding cloud in C_T under Proposition 4.4, and let m be the number of points of ω . Equivalently, m is the number of nucleations on the first line in the forward film.

Because we are in the generic situation, the relevant event locations are separated from one another and from the boundary of the time interval. Accordingly, there exists $\epsilon_0 > 0$ such that, whenever $\epsilon \in (0, \epsilon_0)$, perturbing each point of ω by at most ϵ in space and time does not alter the combinatorial structure of the film: no two events exchange their order, no new collision is created, and no existing collision disappears.

Fix such an ϵ . On the cloud side, for each point $\omega_i = (x_i, t_i)$ of ω , let $D(\omega_i, \epsilon)$ denote the diagonally oriented square of side-length 2ϵ centred at ω_i . For $\epsilon < \epsilon_0$, these squares are pairwise disjoint. Let $\mathbf{B}(\omega, \epsilon)$ be the event that the Poisson cloud in C_T contains exactly one point in each square $D(\omega_i, \epsilon)$ and no other points.

On the line side, let $B(L, \epsilon)$ denote the set of proper n -line configurations whose up-jumps and down-jumps are in natural correspondence with those of L , with each jump occurring on the same line and displaced by at most ϵ from its original location. The choice of ϵ_0 ensures that these two local neighbourhoods correspond exactly.

LEMMA 4.6. For sufficiently small $\epsilon > 0$, the cloud–line bijection of Proposition 4.4 restricts to a bijection between $\mathbf{B}(\omega, \epsilon)$ and $B(L, \epsilon)$.

PROOF. Take $\epsilon \in (0, \epsilon_0)$. Recall that each square $D(\omega_i, \epsilon)$ is oriented along the characteristic directions of slope ± 1 , so that displacing a cloud point within its square moves the two fronts it generates, and hence every jump descended from it, by at most ϵ . Every cloud in $\mathbf{B}(\omega, \epsilon)$ is obtained by moving each point of ω inside its own square and adding no further points. By the definition of ϵ_0 , such perturbations leave the event structure of the film unchanged. The same local moves therefore occur in the same order as for ω , and the only effect is that the jump locations in the resulting line configuration are shifted by at most ϵ . Thus the image lies in $B(L, \epsilon)$.

Conversely, let $L' \in B(L, \epsilon)$. Since L' has the same combinatorial jump pattern as L , the backward film unwinds the same sequence of events. The resulting top-line nucleations occur in the same order as before, and each recovered cloud point lies in the corresponding square $D(\omega_i, \epsilon)$. No additional point can appear, because the total number of top-line nucleations is unchanged. Thus the reconstructed cloud lies in $\mathbf{B}(\omega, \epsilon)$.

The two restricted maps are inverse to one another, because the full cloud–line correspondence is a bijection. $\square$

The local weights of these neighbourhoods under the two ambient laws are now easy to compute. On the walk side, the law $\mathbb{P}_{n,T}$ is generated by independent rate-two Poisson jump clocks on the n active lines, together with independent fair choices of jump direction. The event $B(L, \epsilon)$ therefore asks for one jump of the prescribed sign in each of $2m$ disjoint intervals of length 2ϵ , and no further jumps anywhere on the n lines during the time interval $[-T, T]$. Hence, as $\epsilon \downarrow 0$,

$$\mathbb{P}_{n,T}(B(L, \epsilon)) = e^{-4nT}(2\epsilon)^{2m}(1 + o(1)).$$

On the cloud side, the event $\mathbf{B}(\omega, \epsilon)$ requires exactly one Poisson point in each square $D(\omega_i, \epsilon)$ and no other points in C_T . Since the Poisson process in the cone has unit intensity and the area of C_T is T^2 , we likewise have

$$\mathbb{Q}(\mathbf{B}(\omega, \epsilon)) = e^{-T^2}(2\epsilon)^{2m}(1 + o(1))$$

as $\epsilon \downarrow 0$.

4.4.3. Proof of Proposition 4.5. Both parts come from one comparison of weights, worth seeing first in its uncluttered form. Classify an admissible configuration by the integer m : on the cloud side the number of Poisson points, on the line side the number of upstep–downstep pairs, the two agreeing under Φ . For fixed m the configurations form a $2m$ -dimensional family, coordinatised by the locations of the m cloud points, equivalently by the $2m$ jump times. Divide this family into the small boxes of the previous subsection. Lemma 4.6 matches each line box $B(L, \epsilon)$ with a cloud box $\mathbf{B}(\omega, \epsilon)$, and the two ambient laws weight the matched pair as

$$\mathbb{P}_{n,T}(B(L, \epsilon)) = e^{-4nT}(2\epsilon)^{2m}(1 + o(1)), \quad \mathbb{Q}(\mathbf{B}(\omega, \epsilon)) = e^{-T^2}(2\epsilon)^{2m}(1 + o(1)).$$

The box volume $(2\epsilon)^{2m}$ is shared, so the ratio of the two weights is $e^{T(T-4n)}$: one constant, the same for every box and for every m . A weight standing in a fixed ratio box by box, the ratio blind to m , already displays the proposition. Summing the matched boxes over a fixed m places the mass of that stratum on the two sides in the ratio $e^{T(T-4n)}$; summing over m does the same for $R \cap \mathbf{NI}$ and $\{D(T) \leq n\}$, which is part (i), while the box-by-box equality of the normalised weights is part (ii).

Two points separate this from a proof. At a fixed scale ϵ the boxes of a stratum overlap and leave a thin boundary layer uncovered, so they do not partition it; and the corrections $o(1)$ would have to be controlled uniformly over infinitely many boxes. Both are met at once by reading the displayed weights not as a near-partition but as densities, the box ratio becoming an identity of Radon–Nikodym derivatives.

PROOF. Fix $n \in \mathbb{N}$ and $T > 0$. Let μ be the measure on the configuration space that, on the stratum of each combinatorial type, is Lebesgue measure in the $2m$ jump-time coordinates. The configurations carrying a coincident pair of events, an event at time $\pm T$, or a tangency between consecutive lines form a μ -null set, null also for the two laws below; off this set the box $B(L, \epsilon)$ is, for ϵ small, the product of the $2m$ coordinate windows, of μ -volume $(2\epsilon)^{2m}$. By Lemma 4.6 the bijection Φ carries the cloud-area Lebesgue measure to μ , the two box volumes agreeing at $(2\epsilon)^{2m}$.

Read against μ , the box weights of the previous subsection are Lebesgue derivatives. Using $B(L, \epsilon) \subseteq R \cap \mathbf{NI}$ and $\mathbf{B}(\omega, \epsilon) \subseteq \{D(T) \leq n\}$, which hold for ϵ below the threshold of Lemma 4.6, at μ -almost every configuration

$$\frac{d\mathbb{P}_{n,T}(\cdot \cap R \cap \mathbf{NI})}{d\mu} = e^{-4nT}, \quad \frac{d\Phi_*\mathbb{Q}(\cdot \cap \{D(T) \leq n\})}{d\mu} = e^{-T^2}.$$

Both densities are constant, and their ratio is the single factor $e^{T(T-4n)}$, independent of the stratum: the box ratio above, now exact rather than to leading order. Hence, as measures on the configuration space,

$$\mathbb{P}_{n,T}(\cdot \cap R \cap \mathbf{NI}) = e^{T(T-4n)} \Phi_* \mathbb{Q}(\cdot \cap \{D(T) \leq n\}).$$

Taking total mass gives part (i). Normalising both sides to probability measures cancels the constant and gives part (ii). $\square$

Part (i) of Proposition 4.5 is a finite-volume normalisation identity, and part (ii) says that, once this factor is accounted for, the cloud picture and the conditioned n -line walk picture are the same law. The two describe the same local configurations under different ambient reference measures: Poisson area measure in the cone on one side, independent Poisson jump clocks on finitely many lines on the other. Thus the finite walk systems cease to be merely auxiliary and become a genuine probabilistic description of the PNG line ensemble. In the next section we explain how, by letting n grow, one arrives at the infinite line ensemble at fixed time.

4.5. The infinite line ensemble at fixed time

The finite walk ensembles of the previous section serve a very specific purpose. They provide a concrete reference measure on line space whose local weights may be compared directly with those of the Poisson cloud. They are not, however, the final object. The genuine line ensemble is obtained from the Poissonian model itself, and the role of finite n is to make this object accessible through finite-volume approximations.

Before defining the infinite ensemble, it is worth pausing over a natural concern. Suppose we tried to build the infinite ensemble directly on the walk side, as a limit of the conditioned non-intersecting systems $\mathbb{P}_{n,T}(\cdot \mid R \cap \mathbf{NI})$ as $n \rightarrow \infty$. One might fear a kind of upward congestion. A low curve that rises must push the curve above it, which must push the next, and so on up the stack; if that pressure accumulated, the lower curves would be forced steadily higher as n grew, the finite systems would fail to be tight, and no genuinely finite configuration would survive in the limit at a fixed time.

The resolution is that the very same coupling works in our favour. Precisely because a move by a deep curve forces a matching move on every curve above it, a deep curve sits under great pressure from above to stay where it is. Far down in the stack the configuration is therefore overwhelmingly flat: what looked like a threat to tightness is the mechanism that secures it. The multiline PNG construction lets us make this precise.

The Poissonian construction sidesteps this difficulty. The multiline PNG configuration is built directly from the cloud, and we have already seen that at any fixed time only finitely many lines are active. There is therefore no infinite conditioning to control and no tightness argument to make: at a fixed time the depth is finite as a structural consequence of the dynamics, since every deformation descends through the collision cascade from a Poisson point of the cloud, and the cloud in C_T is almost surely finite. We record this formally for later use.

LEMMA 4.7. For every $T > 0$, the random variable $D(T)$ is almost surely finite.

PROOF. Let $N(T)$ denote the number of Poisson points in C_T . Since C_T has finite area, $N(T) < \infty$ almost surely. Every nucleation on the first line is created by a point of the cloud, and every nucleation on a lower line is created by a collision on the line above. Thus each active line at time T must ultimately be descended from some point of the cloud in C_T . In particular, the depth of

deformation cannot exceed the number of points in the cloud, so $D(T) \leq N(T)$ almost surely. Hence $D(T)$ is almost surely finite. $\square$

This observation allows one to define the full line ensemble at fixed time without any appeal to an infinite conditioning procedure.

Let $\mathcal{L}_T$ denote the space of admissible multiline configurations at time T having finite depth, that is, the space of sequences $\ell = (\ell_i)_{i \geq 1}$ such that each $\ell_i : [-T, T] \rightarrow \mathbb{Z}$ is càdlàg, the lines are strictly ordered, the jumps all have size ± 1 , only finitely many lines differ from their original constant heights $1 - i$, and all jump locations are distinct. By Proposition 4.4, each generic cloud in C_T determines a unique element of $\mathcal{L}_T$, and by Lemma 4.7 this is the generic situation under $\mathbb{Q}$.

DEFINITION 4.8. For $T > 0$, let $\mathbb{P}_{\infty, T}$ denote the law on $\mathcal{L}_T$ obtained by pushing forward the Poisson law $\mathbb{Q}$ under the fixed-time cloud–line map of Proposition 4.4.

Thus $\mathbb{P}_{\infty, T}$ is the law of the full multiline PNG configuration at time T . It is an infinite ordered line ensemble only in the indexing sense: almost surely, only finitely many lines are active in any given realisation. This is why the object is well defined. One does not begin with infinitely many independent walks and then condition them never to intersect. Rather, one begins with the Poissonian growth model, whose fixed-time output is already meaningful, and only afterwards recognises that its finite-depth truncations admit a random-walk description.

The preceding section shows precisely how this finite-volume description works. For each n , the event $\{D(T) \leq n\}$ is exactly the event that the fixed time configuration lies in the n -line sector of $\mathcal{L}_T$, meaning that all lines below the n^{th} remain frozen at their initial heights. On that event, the law of the top n lines is described by the conditioned walk ensemble.

PROPOSITION 4.9. Fix $n \in \mathbb{N}$ and $T > 0$. Under the law $\mathbb{P}_{\infty, T}$, conditioned on the event $\{D(T) \leq n\}$, the top n lines have the same distribution as the n -line walk ensemble under the conditional law $\mathbb{P}_{n, T}(\cdot \mid R \cap \text{NI})$.

PROOF. This is exactly the content of the cloud–line correspondence together with the normalisation identity in Proposition 4.5. The event $\{D(T) \leq n\}$ is the deterministic restriction that all lines below the n^{th} remain undeformed. On this restricted state space, Proposition 4.5 identifies the cloud law with the conditioned n -line walk law up to the normalising factor, and hence the two conditional distributions coincide. $\square$

In this way, the finite walk ensembles are seen not as separate models, but as the finite-depth shadows of $\mathbb{P}_{\infty, T}$. As n increases, the events $\{D(T) \leq n\}$ exhaust the full space $\mathcal{L}_T$, and so the finite conditioned walk descriptions capture more and more of the genuine line ensemble. The finite-volume bridge identity is therefore the mechanism by which the walk picture is connected to the Poissonian one, while $\mathbb{P}_{\infty, T}$ is the object that remains once the finite cut-off is removed.

The next step is to express the probabilistic structure of $\mathbb{P}_{\infty, T}$ in a more intrinsic way. The conditioned finite walk systems already suggest the correct resampling rule: conditional on suitable boundary data, a block of consecutive curves should behave like mutually avoiding continuous-time simple random-walk bridges. This is the continuous-time simple random walk Gibbs property, to which we now turn.

4.6. The continuous-time simple random walk Gibbs property

We have now reached the point at which the probabilistic structure of the line ensemble may be stated in its natural form. The finite walk ensembles of Section 4 already suggest what the answer should be. If one looks only at a finite block of consecutive lines, and if one conditions on the neighbouring curves that surround this block, then the remaining randomness should come from allowing the inner curves to evolve as independent continuous-time simple random walks, subject to the endpoint data and to the requirement that they remain strictly ordered between the two surrounding boundaries. This is the Gibbs property in the present setting.

It is helpful to formulate this first in finite volume, where it is almost a restatement of the conditioned walk construction, and only afterwards to pass to the infinite ensemble $\mathbb{P}_{\infty, T}$.

4.6.1. Bridge laws between two boundary curves. Fix $T > 0$. Let $f, g : [-T, T] \rightarrow \mathbb{Z}$ be càdlàg functions such that

$$f(s) > g(s) \quad \text{for all } s \in [-T, T].$$

Fix also $k \in \mathbb{N}$, and let

$$\vec{x} = (x_1, \dots, x_k), \quad \vec{y} = (y_1, \dots, y_k)$$

be integer vectors satisfying

$$f(-T) > x_1 > \dots > x_k > g(-T) \quad \text{and} \quad f(T) > y_1 > \dots > y_k > g(T).$$

We define a reference law on k curves between these entrance and exit data as follows. Let $Y_1, \dots, Y_k$ be independent continuous-time simple random walks of jump rate two on the time interval $[-T, T]$, started from

$$Y_i(-T) = x_i, \quad i \in \llbracket 1, k \rrbracket.$$

Let $\mathbb{P}_{T, \vec{x}, \vec{y}}^{\text{free}}$ denote the conditional law of $(Y_1, \dots, Y_k)$ given the endpoint event

$$Y_i(T) = y_i, \quad i \in \llbracket 1, k \rrbracket.$$

Thus $\mathbb{P}_{T, \vec{x}, \vec{y}}^{\text{free}}$ is simply the law of k independent continuous-time simple random walk bridges.

We then write

$$\text{NI}(f, g)$$

for the event that these bridges remain strictly ordered between the two boundary curves, in the sense that

$$f(s) > Y_1(s) > \dots > Y_k(s) > g(s) \quad \text{for all } s \in [-T, T].$$

Whenever this event has positive probability, we define

$$\mathbb{P}_{T, \vec{x}, \vec{y}}^{f, g} = \mathbb{P}_{T, \vec{x}, \vec{y}}^{\text{free}}(\cdot \mid \text{NI}(f, g)).$$

This is the law of k mutually avoiding continuous-time simple random walk bridges between the boundary curves f and g , with entrance data $\vec{x}$ and exit data $\vec{y}$.

This notation isolates the local random-walk object that appears in the Gibbs property. The point of the property is that, once the surrounding curves are held fixed, the inner block of curves carries no further memory of the global Poissonian construction.

4.6.2. The finite-volume Gibbs property. We first record the resampling statement for the conditioned finite walk ensemble. Fix $n \in \mathbb{N}$ and $T > 0$, and consider the law

$$\mathbb{P}_{n,T}(\cdot | R \cap \mathbf{NI})$$

on ordered n -tuples of bridges. Let $a, b \in \mathbb{N}$ satisfy

$$1 \leq a \leq b \leq n.$$

We think of the curves $X_a, \dots, X_b$ as a consecutive block inside the full ensemble. Their upper boundary is the curve X_{a-1} , with the convention that this boundary is identically $+\infty$ when $a = 1$. Their lower boundary is the curve X_{b+1} , with the convention that this boundary is the flat curve identically equal to $-n$ when $b = n$, matching the lower wall $X_n > -n$ imposed in the event $\mathbf{NI}$.

PROPOSITION 4.10. Fix $n \in \mathbb{N}$, $T > 0$, and integers $1 \leq a \leq b \leq n$. Under the law $\mathbb{P}_{n,T}(\cdot | R \cap \mathbf{NI})$, condition on the curves outside the block $(X_a, \dots, X_b)$. Then the conditional law of the block $(X_a, \dots, X_b)$ is

$$\mathbb{P}_{T, \vec{x}, \vec{y}}^{X_{a-1}, X_{b+1}},$$

where $\vec{x}$ and $\vec{y}$ are the entrance and exit values of the block at times $-T$ and T , and where the conventions $X_0 \equiv +\infty$ and $X_{n+1} \equiv -n$ are used when the block touches the top or bottom of the ensemble.

PROOF. By its definition, $\mathbb{P}_{n,T}$ is the unconditioned law of the n independent rate-two simple random walks $X_1, \dots, X_n$; the lower wall at height $-n$ and the strict ordering are not part of this law but enter only through the conditioning event $\mathbf{NI}$. Conditioning first on the outer curves and on the endpoint values of the block, the inner block therefore has the law of independent continuous-time simple random walk bridges between its entrance and exit data. Imposing the global event $\mathbf{NI}$ then amounts, inside the block, exactly to the requirement that these bridges remain strictly ordered between the two fixed boundary curves X_{a-1} and X_{b+1} , the lower of which is the flat curve at $-n$ when $b = n$. Thus the conditional law is the bridge law $\mathbb{P}_{T, \vec{x}, \vec{y}}^{X_{a-1}, X_{b+1}}$. $\square$

This proposition is the finite-volume Gibbs property in its clearest form. It says that the conditioned walk ensemble may be resampled locally by forgetting the inner block and redrawing it from the natural bridge law with the induced boundary data. Nothing more global is needed.

4.6.3. The Gibbs property for $\mathbb{P}_{\infty,T}$. We now pass to the infinite line ensemble $\mathbb{P}_{\infty,T}$. Since every configuration under $\mathbb{P}_{\infty,T}$ has finite depth almost surely, any fixed finite block of curves is meaningful. Moreover, on the event that all curves below the n^{th} are frozen, the law of the top n curves is exactly the finite conditioned walk law of Proposition 4.9. Thus the finite-volume resampling rule carries over directly to the infinite ensemble.

To state the result, let

$$\mathcal{L} = (L_1, L_2, \dots)$$

denote the line ensemble under $\mathbb{P}_{\infty,T}$. Fix integers $1 \leq a \leq b$, and let $A_{a,b}$ denote the block of curves

$$(L_a, \dots, L_b).$$

When $a > 1$, the upper boundary of this block is L_{a-1} ; when $a = 1$, we again take the upper boundary to be $+\infty$. Since the ensemble has finite depth almost surely, there exists almost surely some index $m \geq b$ such that all lines below L_m remain at their initial heights. In particular, the lower boundary of the block is always well defined.

THEOREM 4.11. Fix $T > 0$ and integers $1 \leq a \leq b$. Under the law $\mathbb{P}_{\infty, T}$, condition on all curves outside the block $A_{a,b}$. Then the conditional law of the block $A_{a,b}$ is that of mutually avoiding continuous-time simple random walk bridges between the induced entrance and exit data and between the two neighbouring boundary curves. More precisely, this conditional law is

$$\mathbb{P}_{T, \vec{x}, \vec{y}}^{L_{a-1}, L_{b+1}},$$

with the convention $L_0 \equiv +\infty$, and where $\vec{x}$ and $\vec{y}$ are the values of the block at times $-T$ and T .

PROOF. Fix $a \leq b$. On the event $\{D(T) \leq n\}$ for $n \geq b$, the law of the top n curves under $\mathbb{P}_{\infty, T}$ agrees, by Proposition 4.9, with the conditioned finite walk law $\mathbb{P}_{n, T}(\cdot | R \cap \mathbf{NI})$. Under this law, the conditional distribution of the block $A_{a,b}$ given the curves outside the block is described by Proposition 4.10. Since the events $\{D(T) \leq n\}$ increase almost surely to the full space as $n \rightarrow \infty$, the same conditional resampling rule holds under $\mathbb{P}_{\infty, T}$. This gives the stated bridge law for the block. $\square$

The content of Theorem 4.11 is worth dwelling on. The line ensemble was constructed geometrically from a Poisson cloud by the PNG dynamics, and yet its local probabilistic behaviour is described entirely in terms of conditioned random walks. If one fixes the neighbouring curves and the entrance and exit data of a finite block, then the block itself forgets its global origin and is resampled from the most natural law available: mutually avoiding continuous-time simple random walk bridges.

This is the probabilistic heart of the chapter. The Poissonian growth model has produced an infinite line ensemble, and that ensemble satisfies a Gibbs property whose local specification is as simple and canonical as one could hope for. In the final section, we return to the geometric side of the story and emphasise that this same line ensemble also encodes the bouquet energies of Poissonian last passage percolation.

4.7. Geometric meaning of the lines and outlook

We have now reached the point at which the two themes of the chapter may be brought together. On the one hand, the multiline PNG construction produces a well-defined infinite line ensemble at each fixed time T , and the previous section showed that this ensemble satisfies a natural continuous-time simple random walk Gibbs property. On the other hand, the line ensemble was introduced in the first place because it retains geometric information about Poissonian last passage percolation that is not visible from the top line alone. We now return to this geometric side of the story.

The basic point is that the lower lines encode the optimal energies of disjoint bouquets. The top line L_1 records only the ordinary last passage value. When a collision occurs on the top line, the corresponding local feature is removed from that line and passed to the next line down as a nucleation. Iterating this mechanism through the layers, the multiline construction retains exactly the information needed to recover not just one optimizing path, but whole families of disjoint optimizing paths. This is the content of Proposition 4.1. We prove it now as an addendum: the conceptual weight of the chapter lies in the embedding established above, of the profile $L_1 = h$ as the top curve of an ordered ensemble carrying a continuous-time random-walk Gibbs property, and the identity below is a second, combinatorial reading of that same ensemble, tying it to the classical theory of last passage; a reader chiefly interested in the Gibbs side may pass over the argument.

PROOF OF PROPOSITION 4.1. Write $(a, s) \preceq (b, u)$ when $u \geq s$ and $|b - a| \leq u - s$, that is, when an upgoing path runs from (a, s) to (b, u) ; this is a partial order, and the points met by a

single upgoing path form a $\preceq$ -chain. Conversely, the points of any chain inside the *lens*

$$\Lambda = \Lambda(x, t) = \{p : (0, 0) \preceq p \preceq (x, t)\}$$

may be threaded, in $\preceq$ -order, by one upgoing path from $(0, 0)$ to (x, t) . We take $(x, t) \in C$; off the cone no path exists and both sides vanish. The cloud being generic in the standing sense, $\mathcal{P} \cap \Lambda$ is almost surely finite and in general position.

Step 1: locality. Both sides depend on $\mathcal{P}$ only through $\mathcal{P} \cap \Lambda$. Every point met by an upgoing path to (x, t) lies in Λ , so $M_j(x, t)$ ignores the remainder. The multiline dynamics, in turn, carries all influence forward at unit speed: each front and each collision lies in the forward cone of the nucleation that spawned it, hence of some point of $\mathcal{P}$, and a point with a descendant event $\preceq (x, t)$ is itself $\preceq (x, t)$. Points outside Λ thus leave $L_i(x, t)$ untouched, and we may run the whole evolution on the finite cloud $\mathcal{P} \cap \Lambda$.

Step 2: bouquets are chain covers. For finite $A \subseteq \Lambda$ set

$$c_j(A) = \max\{|S| : S \subseteq A \text{ is a union of } j \text{ chains}\}.$$

We claim $M_j(x, t) = c_j(\mathcal{P} \cap \Lambda)$. The points met by the r^{th} path of a j -bouquet form a chain S_r ; the paths being disjoint off their endpoints, $\sum_r |S_r| = |S_1 \cup \dots \cup S_j| \leq c_j(\mathcal{P} \cap \Lambda)$, whence $M_j(x, t) \leq c_j(\mathcal{P} \cap \Lambda)$. For the reverse inequality, let S attain $c_j(\mathcal{P} \cap \Lambda)$ and partition it into chains $S_1, \dots, S_j$. Since time increases strictly along an upgoing path, such a path is the graph of a 1-Lipschitz function on $[0, t]$ with the prescribed endpoints; thread S_r by one, say ϕ_r . Let $\psi_1(s) \leq \dots \leq \psi_j(s)$ be the increasing rearrangement of $\phi_1(s), \dots, \phi_j(s)$ at each time s . Rearrangement is non-expansive in the supremum norm, so each ψ_r is again 1-Lipschitz and has the same endpoints; the ψ_r do not cross, and at every time they occupy the same positions as the ϕ_r , so together they still meet every point of S . A perturbation supported off the finitely many points of S separates them away from $(0, 0)$ and (x, t) , exhibiting a j -bouquet of energy at least $|S|$. Hence $M_j(x, t) \geq c_j(\mathcal{P} \cap \Lambda)$, and equality holds.

Step 3: the lines are iterated last-passage values. Put $\mathcal{P}^{(1)} = \mathcal{P} \cap \Lambda$, and let $\mathcal{P}^{(i+1)}$ be the collision set of the single-line PNG evolution driven by $\mathcal{P}^{(i)}$. By the multiline rule the nucleations of L_{i+1} are the collisions of L_i , so by induction L_{i+1} is the single-line interface driven by $\mathcal{P}^{(i+1)}$, shifted down to its initial height $-i$. The identity $L_1 = h$ of Proposition 4.2 holds for the single-line interface of an arbitrary finite cloud; applied to $\mathcal{P}^{(i+1)}$ it gives $L_{i+1}(x, t) + i = h^{(i+1)}(x, t)$, where $h^{(i)}(x, t)$ denotes the length of a longest chain in $\mathcal{P}^{(i)} \cap \Lambda$. Thus

$$L_i(x, t) + i - 1 = h^{(i)}(x, t) \quad (i \geq 1), \quad h^{(1)} = h.$$

Step 4: the combinatorial kernel. There remains the identity

$$\sum_{i=1}^j h^{(i)}(x, t) = c_j(\mathcal{P} \cap \Lambda), \quad (*)$$

and here the argument meets classical combinatorics. After the rotation by 45° carrying the cone to the first quadrant, $\preceq$ becomes the coordinatewise order, $\mathcal{P} \cap \Lambda$ a finite planar set in general position, $\preceq$ -chains increasing subsequences, and c_j the greatest number of points met by j increasing paths. The collision step $\mathcal{P}^{(i)} \mapsto \mathcal{P}^{(i+1)}$ is the polynuclear form of the geometric Robinson–Schensted–Knuth correspondence [PS02]; under it the iterates $h^{(i)}(x, t)$ are the successive parts $\lambda_1 \geq \lambda_2 \geq \dots$ of a single partition, and $(*)$ is then Greene’s theorem [Gre74], that the largest union of j chains

has cardinality $\lambda_1 + \dots + \lambda_j$. We do not reproduce its proof; the inequality $\geq$ reads j disjoint chains off the first j rows of the shape, and $\leq$ is the matching dual bound.

Assembling the four steps,

$$\sum_{i=1}^j (L_i(x, t) + i - 1) = \sum_{i=1}^j h^{(i)}(x, t) = c_j(\mathcal{P} \cap \Lambda) = M_j(x, t). \quad \square$$

It is worth restating plainly what the identity says. The lower lines carry quantitative geometric meaning. The quantity $L_i(x, t) + i - 1$ measures how much the i^{th} line has risen above its initial level at (x, t) , and the sum of the first j such rises is exactly the maximum energy of a disjoint j -bouquet from $(0, 0)$ to (x, t) . Thus the multiline line ensemble packages the hierarchy of last passage data into a single ordered family of curves.

At this point the two principal messages of the chapter may be stated together. Geometrically, Poissonian last passage percolation gives rise to a line ensemble whose lower lines encode bouquet energies. Probabilistically, the same ensemble admits a continuous-time simple random walk Gibbs description: if one conditions on the neighbouring curves and on the induced entrance and exit data, then a finite block of curves is resampled as mutually avoiding continuous-time simple random walk bridges. These are two descriptions of one and the same object.

This is exactly the role of Poissonian last passage percolation in the broader course. It provides a pre-limiting model in which the key ideas are already visible in a clean and rigorous form. The line ensemble is both geometrically meaningful and probabilistically Gibbsian. In later lectures, one rescales this picture and is led to the Airy line ensemble, whose Brownian Gibbs property is the continuum counterpart of the discrete random-walk resampling rule proved here.

One should therefore think of the present chapter not as an isolated study of a particular growth model, but as a prototype. The Poisson cloud, the PNG dynamics, the multiline line ensemble, the bouquet interpretation, and the Gibbs property all survive, in transformed but recognisable form, in the scaling limit. That is the larger reason for dwelling on this model so carefully.

The Airy line ensemble and its Brownian Gibbs property

5.1. Brownian last passage percolation and the chapter's destination

We now turn from the Poissonian model of the previous chapter to the question of its scaling limit. Before introducing a new model, it is useful to pause and ask what sort of continuum object one should expect to arise.

The line ensemble under $\mathbb{P}_{\infty, T}$ is indexed by the spatial interval $[-T, T]$, and its uppermost curve records the PNG height profile at time T . If one wishes to zoom in on this random interface and see a non-trivial limit, two questions immediately arise. What deterministic shape governs the profile on the scale of order T ? And on what smaller spatial and vertical scales should one observe the random fluctuation around that shape?

It is best to answer these questions in the standard first-quadrant coordinates for planar last passage percolation. The connection with the PNG picture is as follows. The fixed-time slice $t = T$ in PNG corresponds, after the usual rotation by 45° , to the anti-diagonal line

$$u + v = \sqrt{2}T$$

in $\mathbb{R}_+^2$. A spatial point $x \in [-T, T]$ on the PNG profile corresponds to the point

$$\left(\frac{T-x}{\sqrt{2}}, \frac{T+x}{\sqrt{2}} \right)$$

on this anti-diagonal. Thus the PNG height at (T, x) is encoded by the Poissonian last passage value from the origin to this point.

The basic shape theorem for Poissonian last passage percolation [Joh00] says that the passage value from $(0, 0)$ to (u, v) has first-order size

$$2\sqrt{uv}.$$

It is useful to rephrase this in a way that already points toward the full shape function. In the present coordinates, the quantity $\sqrt{2A}$ gives the same first-order value whenever A denotes the area of the south-west to north-east oriented rectangle whose south-west corner is the origin and whose north-east corner is the given cone point. This is because an area-preserving linear dilation sends such a rectangle to a square, preserves the unit-intensity Poisson process law, and carries increasing paths to increasing paths. Thus the shape depends only on the rectangle area.

At the centre point $x = 0$, one has $u = v = T/\sqrt{2}$, so the first-order value is

$$2\sqrt{\frac{T}{\sqrt{2}} \cdot \frac{T}{\sqrt{2}}} = \sqrt{2}T.$$

More generally, at spatial location $x \in [-T, T]$, the corresponding endpoint is

$$\left(\frac{T-x}{\sqrt{2}}, \frac{T+x}{\sqrt{2}} \right),$$

and so the shape theorem gives the first-order profile

$$2\sqrt{\frac{T-x}{\sqrt{2}} \cdot \frac{T+x}{\sqrt{2}}} = \sqrt{2} \sqrt{T^2 - x^2}.$$

Thus the deterministic limit shape for the top curve is a semicircle.

Near its maximum at $x = 0$, this profile has the expansion

$$\sqrt{2} \sqrt{T^2 - x^2} = \sqrt{2} T \sqrt{1 - \frac{x^2}{T^2}} = \sqrt{2} T - \frac{x^2}{\sqrt{2} T} + o(T^{-1} x^2) \quad \text{when } |x| \ll T.$$

So a horizontal displacement of size x causes a deterministic drop of order x^2/T . If one chooses x on the scale $T^{2/3}$, then this drop is of order $T^{1/3}$. This is the KPZ 1:2:3 scaling: horizontal distances of order $T^{2/3}$ and vertical fluctuations of order $T^{1/3}$ are the scales on which the curvature of the limit shape and the randomness of the model should both remain visible.

To see a non-trivial local picture near the top of the profile, one must first shift vertically downward by the height of the maximum, namely by $\sqrt{2}T$, so that the part of the top curve near the centre is brought close to height zero. One must then zoom in anisotropically. The horizontal scale should be $T^{2/3}$, as the preceding discussion suggests, while the vertical scale should be $T^{1/3}$. This latter choice is also the one compatible with Brownian scaling: if a random curve is to have Brownian local behaviour after rescaling, then vertical fluctuations should be of the square root of horizontal size, and indeed

$$(T^{2/3})^{1/2} = T^{1/3}.$$

Accordingly, one is led to the centred scaling map

$$(x, y) \mapsto \left(T^{-2/3} x, T^{-1/3} (y - \sqrt{2} T) \right).$$

Under this magnification, the local random-walk type fluctuation of the prelimiting line ensemble should become Brownian, while the centred semicircle profile should become quadratic. In other words, one expects to see in the scaled picture a collection of mutually avoiding Brownian curves with a parabolic global shape. This is the same basic visual picture that already appeared in the toy model of the earlier lectures: Brownian fluctuation against a downward parabolic background.

This heuristic points toward the Airy line ensemble. Roughly speaking, one expects an N -indexed ordered ensemble of continuous curves whose top curve describes the local scaled fluctuation of the PNG interface, and whose parabolically shifted form should enjoy a Brownian resampling rule. This object is the *parabolic Airy line ensemble*; we shall often drop the qualifier and speak simply of the Airy line ensemble where no confusion can arise.

There is, however, an important practical difficulty in trying to reach this limit directly from the Poissonian line ensemble of the previous chapter. That ensemble satisfies the continuous-time simple random walk Gibbs property. This is already a very useful resampling rule, but it still reflects the microscopic jump structure of the discrete model. If one tries to pass to the continuum limit through this rule, one must carry those jumps all the way through the scaling argument. The limiting resampling mechanism one wants is Brownian from the outset.

This is why Brownian last passage percolation is so valuable. It is a prelimit integrable model in the same universality class, but its associated line ensembles already satisfy the exact Brownian Gibbs property. Thus Brownian last passage percolation realises, at finite scale, the very local resampling structure that one expects in the continuum limit. It is therefore not merely [irritating rhetorical

contrast a convenient alternative model. It is the natural setting in which the passage to the Airy line ensemble may be made while keeping the desired Brownian Gibbs rule exact at every finite stage.

The continuum object at stake is the Airy line ensemble: an ordered $\mathbb{N}$ -indexed ensemble of continuous curves whose uppermost member is the Airy_2 process and whose parabolically shifted form satisfies the Brownian Gibbs property. The aim of this chapter is to explain how Brownian last passage percolation leads to this ensemble. We begin by introducing the Brownian model and the line ensembles that it naturally generates.

5.2. Brownian last passage percolation and bouquet line ensembles

5.2.1. Brownian last passage values. In the preceding chapter, the object of direct geometric interest was the PNG height profile, but the real probabilistic leverage came only after that curve had been embedded into a larger line ensemble. There, the multiline PNG dynamics produced an ordered system of mutually avoiding curves with a useful Gibbs property, and the lower curves were then said, without proof, to encode bouquet energies in the Poissonian cloud. Thus the chapter already introduced an important theme: a curve first encountered through a growth model may be better understood after it is placed inside a richer random ensemble.

We now pass to Brownian last passage percolation. Here it is natural to begin from the path-energy picture itself. Let

$$B : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$$

be a family of independent two-sided standard Brownian motions. For integers $i \leq j$ and real numbers $x \leq y$, consider the collection of non-decreasing lists

$$\{z_k : k \in [i+1, j]\},$$

where each $z_k \in [x, y]$, with the convention that

$$z_i = x, \quad z_{j+1} = y.$$

Such a list encodes a staircase from (x, i) to (y, j) : on level k , one runs horizontally from z_k to z_{k+1} , and successive horizontal pieces are joined by vertical unit segments. The energy of the staircase is

$$\sum_{k=i}^j (B(k, z_{k+1}) - B(k, z_k)).$$

The Brownian last passage value from (x, i) to (y, j) is the maximum of this quantity over all such staircases. We denote it by

$$M_{(x,i) \rightarrow (y,j)}^1.$$

In the narrow-wedge setting, we write

$$M_n^1(t) = M_{(0,1) \rightarrow (t,n)}^1,$$

so that $M_n^1(t)$ is the maximum energy of a staircase from $(0, 1)$ to (t, n) . This is the Brownian counterpart of the ordinary point-to-point last passage value in the Poissonian model.

5.2.2. Staircase bouquets and the Brownian last passage line ensemble. As in the Poissonian chapter, the top curve alone is not the right object to isolate. The useful enlargement comes from allowing several disjoint optimising paths at once. In the present semi-discrete geometry, however, one must say a little carefully what this means.

In the Poissonian cloud, it is natural to speak of several up-right paths that start at the same point, end at the same point, and are otherwise disjoint. For staircases, this picture has to be implemented indirectly. If several staircases literally began from the same point $(0, 1)$ and ended at the same point (t, n) , then they could not be pairwise disjoint as subsets of the semi-discrete lattice. The right way to encode tied endpoints is to place the starting points at consecutive levels above one another at horizontal coordinate 0, and similarly the ending points at consecutive levels at horizontal coordinate t .

This does indeed represent common endpoints in the geometric sense relevant for last passage. A staircase is permitted to spend zero horizontal time on a level. Thus, at the left boundary, a path may jump upward immediately at time 0, before making any horizontal progress; and at the right boundary, it may defer its final vertical jumps until time t . In this way, a family of disjoint staircases with starting points

$$(0, 1), (0, 2), \dots, (0, j)$$

and ending points

$$(t, n - j + 1), (t, n - j + 2), \dots, (t, n)$$

should be understood as a semi-discrete surrogate for j paths that are tied together at both ends and otherwise disjoint.

Fix $n \in \mathbb{N}$, $j \in \llbracket 1, n \rrbracket$ and $t \geq 0$. A j -staircase bouquet from $(0, 1)$ to (t, n) is a j -tuple

$$\bar{\phi} = (\phi_1, \dots, \phi_j)$$

such that

$$\phi_r \in SC_{(0,r) \rightarrow (t, n-j+r)} \quad \text{for } r \in \llbracket 1, j \rrbracket,$$

and such that the horizontal segments belonging to distinct staircases are pairwise disjoint. Its energy is the sum of the energies of its constituent staircases. We write

$$M_n^j(t) = \sup \{ E(\bar{\phi}) : \bar{\phi} \text{ is a } j\text{-staircase bouquet from } (0, 1) \text{ to } (t, n) \}.$$

The passage from one staircase to many gives rise to a line ensemble. We define L_n by requiring that, for each $j \in \llbracket 1, n \rrbracket$,

$$\sum_{i=1}^j L_n(i, t) = M_n^j(t).$$

Thus the top curve is

$$L_n(1, t) = M_n^1(t),$$

while the lower curves measure the successive gains that become possible when one is allowed to use larger disjoint bouquets. In this way one obtains an ordered ensemble

$$L_n : \llbracket 1, n \rrbracket \times [0, \infty) \rightarrow \mathbb{R}.$$

This construction runs parallel to the bouquet picture from the Poissonian chapter, but here nothing more needs to be established: in the Brownian setting, the bouquet interpretation is built into the definition of the line ensemble.

5.2.3. Brownian watermelon. We now introduce the probabilistic object that will later be identified with the Brownian last passage ensemble. Informally, one would like to speak of n Brownian motions that start together and then evolve on $[0, \infty)$ without ever intersecting. This description is suggestive, but it is not itself a definition. There are two singular features. First, if one starts with independent Brownian motions, the event that they never intersect has probability zero, so one cannot simply condition on it in the naive sense. Second, if several Brownian motions are required to start from the same point, then the initial condition already lies on the boundary of the region in which strict non-intersection is supposed to hold.

The right state space is the open Weyl chamber

$$\mathcal{W}_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > \dots > x_n\}.$$

A point of $\mathcal{W}_n$ records a strictly ordered n -tuple of locations. If we run n independent Brownian motions from such a point, then they evolve until the first time two coordinates meet. At that moment the process exits $\mathcal{W}_n$, and one may think of it as being killed.

Conditioning a diffusion on a future event has a general shape, and the watermelon is an instance of it; it is worth setting that shape down first. Let X be a Brownian motion in $\mathbb{R}^d$, fix a later time S , and fix a Borel set A . Write

$$h(t, x) = \mathbb{P}_x(X_S \in A \mid X_t = x)$$

for the chance, seen from x at time t , that the path lands in A at time S ; this is the heat flow applied to $\mathbf{1}_A$. The conditioned dynamics are fixed by two observations. First, $h(t, X_t)$ is a martingale, being the conditional probability of a fixed event along the filtration; equivalently h is harmonic in space-time, $\partial_t h + \frac{1}{2} \Delta h = 0$. Second, conditioning is an application of Bayes' rule. Over an increment $[t, t + dt]$, the law of the next position y , given the present position x and the event $\{X_S \in A\}$, is the unconditioned transition reweighted by the chance of the event read from y and divided by the chance read from x ,

$$\mathbb{P}^h(X_{t+dt} \in dy \mid X_t = x) = \mathbb{P}(X_{t+dt} \in dy \mid X_t = x) \frac{h(t + dt, y)}{h(t, x)}.$$

The denominator $h(t, x)$ is the normalisation Bayes' rule demands; the numerator is the likelihood of the event seen from the candidate position.

The reweighting acts on the increment as a drift. Displacing from x by dx multiplies the chance of the event by $h(t + dt, x + dx)/h(t, x)$, whose first-order sensitivity to the displacement is carried by $\nabla \log h$; averaging this tilt against the Gaussian increment converts it into an added drift, and the conditioned motion solves

$$dX_t = dB_t + \nabla \log h(t, X_t) dt.$$

The conditioning appears as the gradient of the log-likelihood, a pull along the direction in which the target event grows more probable. This is the Doob h -transform, and its drift is $\nabla \log h$, the quantity h'/h in one dimension.

The same template governs conditioning on any event fixed by the future, with the chance of $\{X_S \in A\}$ replaced by the chance of the event at hand. For the watermelon that event is survival, and the discussion from here stands on its own; the arrival-in- A picture was explicit only to fix the shape of the construction, and need not be matched to it point for point.

Return to the n Brownian motions in $\mathcal{W}_n$, and impose survival: that no two coordinates have met by a horizon S . Writing $\tau_{\partial \mathcal{W}_n} = \inf\{t \geq 0 : X(t) \notin \mathcal{W}_n\}$ for the first exit time and $Q_u(x) := \mathbb{P}_x(\tau_{\partial \mathcal{W}_n} > u)$ for the survival probability over a duration u , the conditional probability of survival seen from

(t, x) is the remaining-survival probability $h(t, x) = Q_{S-t}(x)$, harmonic for the killed motion and vanishing on $\partial\mathcal{W}_n$. For finite S there is nothing singular: $Q_u(x) \in (0, 1)$ for $x \in \mathcal{W}_n$ and $u > 0$, so the conditioning is legitimate and the survived motion is the h -transform with drift $\nabla \log Q_{S-t}$.

The horizon cannot be sent to infinity inside the probability, since $Q_u(x) \downarrow 0$ as $u \rightarrow \infty$: two of the coordinates meet eventually, with certainty. What survives the limit is not the probability but its spatial shape. Brownian scaling gives $Q_u(x) = Q_1(x/\sqrt{u})$, so as $u \rightarrow \infty$ the effective starting point $x/\sqrt{u}$ retreats into the apex of the chamber. There the survival probability is governed by its leading harmonic mode, the unique polynomial that is harmonic on $\mathcal{W}_n$ and vanishes on the walls,

$$h_n(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j),$$

the Vandermonde determinant: $Q_1(y) \sim \kappa_n h_n(y)$ as $y \rightarrow 0$ in $\mathcal{W}_n$, for some $\kappa_n > 0$. Since h_n is homogeneous of degree $n(n-1)/2$ and diffusive scaling turns each degree into a half-power of time,

$$Q_u(x) \sim \kappa_n h_n(x) u^{-n(n-1)/4} \quad (u \rightarrow \infty).$$

The Vandermonde is the spatial profile of survival; the decay $u^{-n(n-1)/4}$ is its cost, the same for every starting configuration.

This factorisation is what carries the conditioning to the infinite horizon. The survival-to- S drift is $\nabla_x \log Q_{S-t}(x)$, and with $u = S - t$,

$$\log Q_u(x) = \log h_n(x) - \frac{n(n-1)}{4} \log u + \log \kappa_n + o(1).$$

The temporal term depends on u alone and is annihilated by ∇_x . Hence at each fixed (t, x) the drift converges, as $S \rightarrow \infty$, to the time-homogeneous field

$$\nabla \log h_n(x), \quad \partial_{x_i} \log h_n(x) = \sum_{j \neq i} \frac{1}{x_i - x_j}.$$

The finite-horizon conditionings therefore converge, and their limit is the Doob transform of killed Brownian motion by the Vandermonde. One sees here exactly how the Vandermonde enters: it is the part of the survival probability that the log-derivative keeps, while the vanishing total mass, a function of the elapsed time alone, cancels.

We record the two properties of h_n on which the transform rests. Every factor $x_i - x_j$ with $i < j$ is positive on $\mathcal{W}_n$, so $h_n > 0$ there; and h_n is harmonic, $\frac{1}{2}\Delta h_n = 0$ on $\mathcal{W}_n$, so that $h_n(X(t \wedge \tau_{\partial\mathcal{W}_n}))$ is a martingale and the value of h_n at the start equals its expectation at any later time before the motion leaves the chamber.

This harmonic function allows one to tilt the law of killed Brownian motion in a way that favours trajectories which stay well inside the chamber. Let $(P_t)_{t \geq 0}$ denote the semigroup of n independent Brownian motions killed upon exiting $\mathcal{W}_n$:

$$P_t f(x) = \mathbb{E}_x[f(X(t)) \mathbf{1}_{\{t < \tau_{\partial\mathcal{W}_n}\}}].$$

One now defines a new family of operators by

$$P_t^{h_n} f(x) = \frac{1}{h_n(x)} P_t(h_n f)(x).$$

This is the Doob h -transform of the killed process with respect to h_n . Two properties of h_n make it work. Positivity of h_n on $\mathcal{W}_n$ ensures that the prefactor $1/h_n(x)$ is well defined and that the reweighting keeps the operator positive. Harmonicity, together with the fact that h_n vanishes on

$\partial\mathcal{W}_n$, ensures that the killed semigroup fixes h_n , that is $P_t h_n = h_n$; hence $P_t^{h_n} \mathbf{1} = h_n^{-1} P_t h_n = \mathbf{1}$, so that $(P_t^{h_n})_{t \geq 0}$ is a conservative Markov semigroup rather than a merely sub-Markovian one.

To read this formula: under the killed law, paths are weighted only until they first collide. Under the transformed law, a trajectory that arrives at time t in a configuration where h_n is large receives extra weight proportional to that value. Since h_n is the product of all coordinate gaps, large values of h_n correspond to configurations in which the particles are well separated. In this way, the transformed law biases the process away from collision. One may regard it as the rigorous version of conditioning independent Brownian motions never to intersect.

The Markov process with semigroup $(P_t^{h_n})_{t \geq 0}$ is Dyson Brownian motion [Dys62], listed in decreasing order. Thus Dyson Brownian motion is the canonical ordered diffusion obtained from independent Brownian motions by conditioning them not to collide.

This construction starts the process from an interior point $x \in \mathcal{W}_n$, and so disposes of the first of the two singular features named at the outset. The second remains. The ensemble to which we shall compare the watermelon has a narrow-wedge flavour, all curves emerging from a common location at the left endpoint; but the fully coalesced configuration $(0, \dots, 0)$ lies on $\partial\mathcal{W}_n$, where h_n vanishes and the drift $\sum_{j \neq i} (x_i - x_j)^{-1}$ is singular, so it cannot serve as an ordinary initial condition for Dyson Brownian motion. The correct object is instead an *entrance law* from the boundary. Roughly speaking, this means a family of time- ε distributions in $\mathcal{W}_n$, for $\varepsilon > 0$, that is consistent with the Dyson semigroup and that should be interpreted as the law of the process immediately after time 0 when all particles have just emerged from a completely coalesced configuration. One may think of it as the limiting way to start the ordered diffusion from a point that lies on the boundary rather than in the chamber itself.

DEFINITION 5.1. For $n \in \mathbb{N}$, the n -curve Brownian watermelon is the ordered line ensemble whose law is that of Dyson Brownian motion with n curves, listed in decreasing order, started from the narrow-wedge entrance law at time 0. Equivalently, for every positive time, it is the ordered non-intersecting Brownian system obtained from killed Brownian motion in $\mathcal{W}_n$ by the Doob h -transform with $h = h_n$, interpreted with the entrance law from the boundary point $(0, \dots, 0)$.

Thus the Brownian watermelon is a genuine Brownian line ensemble, but one with a singular initial condition. The informal phrase “Brownian motions conditioned never to intersect” is a useful guide to intuition, but the mathematically correct construction runs through killed Brownian motion in the Weyl chamber, the harmonic function h_n , and the associated Doob transform. This distinction matters here because the object will shortly be compared with a line ensemble built from last passage energies. The point of the O’Connell–Yor theorem is that these two very different constructions lead to the same law.

5.2.4. The O’Connell–Yor identification. The point of introducing Brownian watermelon separately is that the following statement may now be read precisely.

THEOREM 5.2 (O’Connell–Yor [OY02]). For each $n \in \mathbb{N}$, the line ensemble

$$L_n : \llbracket 1, n \rrbracket \times [0, \infty) \rightarrow \mathbb{R}$$

defined through Brownian last passage bouquet energies has the same law as the n -curve Brownian watermelon.

This theorem is the key structural input from integrable probability. It says that the same random line ensemble admits two quite different descriptions. On the one hand, L_n is defined through optimal energies of staircase bouquets. On the other hand, it is a genuinely Brownian line ensemble, namely Dyson Brownian motion in line-ensemble form with narrow-wedge entrance law.

We will not prove Theorem 5.2. At a broad level, its proof belongs to the interaction between Brownian last passage percolation, variants of the Robinson–Schensted–Knuth correspondence—in the spirit of the bouquet identity, Proposition 4.1, proved late in Chapter 4—and the algebraic structure behind Dyson Brownian motion. For the present chapter, what matters is the conclusion: Brownian last passage percolation furnishes a prelimit line ensemble that is already Brownian in its local structure.

5.2.5. Geometric bouquets and Brownian watermelon. The bouquet language and the watermelon language refer to two different aspects of the same ensemble, and it is worth keeping their distinction in view.

The bouquet picture belongs to the last passage side of the story. It concerns disjoint energy-maximising staircases with tied endpoints, implemented semi-discretely by the shifted boundary levels described above. Its basic data are energetic and geometric: one studies maximisers, their disjointness, and the cumulative energy gains obtained by allowing more of them.

The watermelon picture belongs to the Brownian-motion side. It concerns an ordered family of mutually avoiding Brownian paths evolving from a singular initial configuration. Its basic data are probabilistic and dynamic: one studies a diffusion in the Weyl chamber, or equivalently a Doob-transformed Brownian system.

These two pictures should not be run together. They describe the same line ensemble in law, and at first sight they even look alike: in each case one sees a family of curves tied together at the two endpoints and forbidden to cross. Yet the two families are forged in completely different ways, and the difference surfaces in their local regularity. The watermelon curves are Brownian, and so are Hölder- $1/2^-$ continuous. The bouquet curves are assembled from energy-maximising staircases that collect Poisson points as they run from one endpoint to the other; they inherit the geometry of last passage geodesics, whose transversal fluctuations live in the KPZ $2/3$ world, and they are correspondingly Hölder- $2/3^-$ rather than $1/2^-$. The image of several strands separating and rejoining is thus a faithful guide to the global shape, but it must not be read as a statement about the fine structure of the paths.

In what follows, we shall use the O’Connell–Yor theorem to pass from the energetic definition of L_n to its Brownian description. That is the bridge that leads to the Brownian Gibbs property.

5.3. The Brownian Gibbs property

The O’Connell–Yor theorem identifies the Brownian last passage ensemble with a Brownian watermelon. We should now explain the local resampling rule enjoyed by this object.

The point is easiest to understand first for a finite system of mutually avoiding Brownian paths on a compact interval. One fixes the endpoint values of the paths, as well as any neighbouring curves that are to act as upper or lower boundaries, and then asks for the law of the curves in the interior of the interval. The answer is the Brownian Gibbs law: the curves are independent Brownian bridges before conditioning, and the only further ingredient is the requirement that they avoid one another and remain between the prescribed boundaries.

5.3.1. Mutually avoiding Brownian bridges on a compact interval. Fix a compact interval $I = [u, v] \subset \mathbb{R}$, a positive integer k , and vectors

$$\bar{x} = (x_1, \dots, x_k), \quad \bar{y} = (y_1, \dots, y_k)$$

with

$$x_1 > \dots > x_k, \quad y_1 > \dots > y_k.$$

Let

$$f, g : I \rightarrow \mathbb{R} \cup \{\pm\infty\}$$

be continuous functions satisfying

$$f(r) > g(r) \quad \text{for all } r \in I.$$

We think of f as an upper boundary and g as a lower boundary. Either boundary may be absent: this is represented by taking $f \equiv +\infty$ or $g \equiv -\infty$.

Let $\mathcal{B}_{k;\bar{x},\bar{y}}^I$ denote the law of k independent Brownian bridges

$$B_1, \dots, B_k : I \rightarrow \mathbb{R}$$

such that $B_i(u) = x_i$ and $B_i(v) = y_i$ for each $i \in \llbracket 1, k \rrbracket$.

The relevant geometric event is that these bridges avoid one another and also stay between the two boundaries. Write

$$\text{NoTouch}_{k,I}^{f,g} = \left\{ f(r) > B_1(r) > \dots > B_k(r) > g(r) \text{ for all } r \in I \right\}.$$

The conditional law

$$\mathcal{B}_{k;\bar{x},\bar{y}}^I(\cdot \mid \text{NoTouch}_{k,I}^{f,g})$$

is the law of a k -curve mutually avoiding Brownian bridge ensemble on I , with entrance data $\bar{x}$, exit data $\bar{y}$, upper boundary f , and lower boundary g .

It is convenient to name the associated acceptance probability:

$$A_{k,I}(\bar{x}, \bar{y}; f, g) = \mathcal{B}_{k;\bar{x},\bar{y}}^I(\text{NoTouch}_{k,I}^{f,g}).$$

Thus the mutually avoiding bridge law is simply the raw bridge law conditioned on the event $\text{NoTouch}_{k,I}^{f,g}$. In the special case $k = 1$, this is the probability that a single Brownian bridge stays between the two boundaries on the whole interval.

5.3.2. Why this is the right local law. The Brownian Gibbs property is a statement about conditional laws on compact intervals, and it is worth understanding why Brownian bridge measure appears so naturally.

Consider first a single Brownian motion on a time interval containing $I = [u, v]$. If one conditions on the path outside (u, v) , then the only new randomness left inside I is the fluctuation that joins the already revealed endpoint values at times u and v . That fluctuation has the law of a Brownian bridge. This is the finite-interval counterpart of the Lévy construction of Brownian motion: after the outside data and endpoint values are fixed, the missing middle piece is generated by fresh Gaussian input localised to the interval I .

It is useful to keep a concrete picture in mind. One may regard the bridge on I as built from an affine interpolation between its endpoint values together with a centred standard bridge. The latter can in turn be expanded using a countable family of independent Gaussian coefficients, for instance in a dyadic Lévy construction. If one were to redraw these Gaussian coefficients while holding fixed

the outside path and the endpoint values, one would regenerate the middle segment from its raw Brownian bridge law.

For a system of ordered curves, the same principle remains valid. Once the outside data and the endpoint values on I are fixed, the missing interior segments are, before any further conditioning, independent Brownian bridges. What distinguishes the ordered system is that not every redraw is admissible. In the top-curve case, the resampled bridge must stay above the frozen second curve. For a block of several curves, the resampled bridges must avoid one another and also stay between the neighbouring frozen curves above and below. Thus the Brownian Gibbs law is obtained by starting from raw Brownian bridge input and then conditioning on an acceptance event.

This is closely analogous to the earlier discussion of local Gaussian resampling. There too one freezes most of the object, redraws the Gaussian data responsible for one local piece, and then studies the geometric event that the redraw be compatible with the retained data. Here the compatibility condition is non-intersection.

5.3.3. Brownian Gibbs for finite ordered line ensembles. Let

$$\mathcal{L} = (\mathcal{L}(1, \cdot), \dots, \mathcal{L}(n, \cdot))$$

be an ordered n -curve line ensemble on a compact interval $J \subset \mathbb{R}$, so that

$$\mathcal{L}(1, r) > \mathcal{L}(2, r) > \dots > \mathcal{L}(n, r) \quad \text{for all } r \in \text{int}(J).$$

Fix integers $1 \leq k_1 \leq k_2 \leq n$ and a compact subinterval $I = [u, v] \subseteq J$.

We now describe the conditional law of the block of curves

$$\mathcal{L}(k_1, \cdot), \dots, \mathcal{L}(k_2, \cdot)$$

on the interval I , after everything else has been retained. The retained data consist simply of the ensemble outside the open strip $\llbracket k_1, k_2 \rrbracket \times (u, v)$. This already includes the endpoint values of the resampled curves at u and v , so no separate list of data is needed.

Let

$$\bar{x} = (\mathcal{L}(k_1, u), \dots, \mathcal{L}(k_2, u)), \quad \bar{y} = (\mathcal{L}(k_1, v), \dots, \mathcal{L}(k_2, v)).$$

Let the upper neighbouring curve be

$$f = \mathcal{L}(k_1 - 1, \cdot)|_I$$

when $k_1 > 1$, and $f \equiv +\infty$ when $k_1 = 1$. Let the lower neighbouring curve be

$$g = \mathcal{L}(k_2 + 1, \cdot)|_I$$

when $k_2 < n$, and $g \equiv -\infty$ when $k_2 = n$.

DEFINITION 5.3. The ensemble $\mathcal{L}$ is said to satisfy the *Brownian Gibbs property* if, for every choice of k_1, k_2 and $I = [u, v] \subseteq J$, the conditional law of

$$\mathcal{L}(k_1, \cdot), \dots, \mathcal{L}(k_2, \cdot)|_I$$

given the retained data outside $\llbracket k_1, k_2 \rrbracket \times (u, v)$ is

$$\mathcal{B}_{k; \bar{x}, \bar{y}}^I(\cdot | \text{NoTouch}_{k, I}^{f, g}), \quad k = k_2 - k_1 + 1.$$

In words: once the exterior data are held fixed, the missing block is redrawn from independent Brownian bridges with the induced endpoint values, conditioned to remain strictly ordered and to fit between the neighbouring frozen curves. This is the Brownian Gibbs property.

5.3.4. Brownian Gibbs for Brownian watermelon. We now return to the Brownian last passage line ensemble L_n . By the O’Connell–Yor theorem, it has the law of an n -curve Brownian watermelon. The pinching at time 0 is a global feature of that ensemble, but it does not create any difficulty for local resampling on intervals that stay away from the left endpoint. If

$$I = [u, v] \subset (0, \infty),$$

then the curves at times u and v are already strictly ordered, and the local problem is exactly of the finite bridge type just described.

Indeed, after conditioning on the ensemble outside a strip $\llbracket k_1, k_2 \rrbracket \times (u, v)$, the missing block on I is determined by two ingredients. First, its raw randomness is Brownian bridge randomness, in the sense explained above through the Lévy bridge picture. Second, the redraw must satisfy the geometric acceptance test that the curves avoid one another and remain between the neighbouring frozen curves. This is precisely the Brownian Gibbs mechanism.

PROPOSITION 5.4. For each $n \in \mathbb{N}$, the Brownian last passage line ensemble L_n satisfies the Brownian Gibbs property on compact subintervals of $(0, \infty)$.

We will use this proposition as an input. A careful proof for finite mutually avoiding Brownian bridge systems may be found, for example Lemma 2.13 of Dimitrov–Matetski [DM21]. For present purposes, the important point is the structure: local resampling is Brownian bridge resampling together with the acceptance event enforcing non-intersection.

5.3.5. Acceptance probability. Since acceptance probability will play an important role later, it is useful to record it in the same generality as the Brownian Gibbs law itself.

For a k -curve Brownian Gibbs resampling problem on a compact interval $I = [u, v]$, with entrance data $\bar{x}$, exit data $\bar{y}$, upper boundary f , and lower boundary g , the acceptance probability is

$$A_{k,I}(\bar{x}, \bar{y}; f, g) = \mathcal{B}_{k;\bar{x},\bar{y}}^I(\text{NoTouch}_{k,I}^{f,g}).$$

Thus $A_{k,I}$ is the conditional probability, under the raw independent Brownian bridge law, that the proposed redraw is admissible.

The top-curve case is the simplest and the one that will arise most often. Then $k = 1$, there is no internal avoidance to enforce, and if the top curve is being resampled inside an n -curve ensemble, the upper boundary is absent while the lower boundary is the frozen second curve. In that case,

$$A_{1,I}(x, y; +\infty, g)$$

is simply the probability that a Brownian bridge from x to y on I stays above g throughout the interval.

This quantity measures the strength of the rejection step in Brownian Gibbs resampling. The resampled segment is a Brownian bridge conditioned on passing the geometric non-intersection test, and $A_{k,I}$ is exactly the probability that it passes. That conditioning will later need quantitative control, and it is for that reason that the notation $A_{k,I}$ is worth introducing now.

5.4. KPZ scaling, regular ensembles, and the Airy line ensemble

The Brownian Gibbs property gives a local resampling rule for the finite- n Brownian last passage ensembles. The next task is to place those ensembles in their characteristic KPZ scaling window and to identify a class of possible subsequential limits. The opening section already indicated the underlying picture. The deterministic first-order profile is curved; after centring and rescaling, that

curvature becomes parabolic; and the local randomness remains Brownian, because the prelimiting ensembles satisfy the Brownian Gibbs property exactly. This section records the scaled ensembles, the notion of regularity that captures their key features, and the limiting object toward which they converge.

5.4.1. The 1/3 and 2/3 exponents. In the Poissonian model, the uppermost curve has semi-circular first-order shape, and near the origin the drop over horizontal displacement x is of order x^2/t . Choosing $x = t^{2/3}$ makes this deterministic drop of order $t^{1/3}$. This is the familiar KPZ balance: the natural horizontal scale is $t^{2/3}$, while the natural vertical scale is $t^{1/3}$.

The same exponents govern Brownian last passage percolation. One therefore seeks a scaling window in which horizontal displacements of order $n^{2/3}$ and vertical fluctuations of order $n^{1/3}$ are both visible at order one. In that window, the macroscopic curvature survives and becomes parabolic, while the local resampling structure remains Brownian because the prelimiting line ensembles already satisfy the Brownian Gibbs property.

5.4.2. The scaled ensemble. We now define the scaled Brownian last passage ensemble. Let

$$Q(x, y) = 2^{-1/2}(y - x)^2,$$

and recall the Brownian last passage line ensemble

$$L_n : \llbracket 1, n \rrbracket \times [0, \infty) \rightarrow \mathbb{R}.$$

The scaled ensemble

$$\mathcal{L}_n^{\text{sc}} : \llbracket 1, n \rrbracket \times \left[-\frac{1}{2}n^{1/3}, \infty\right) \rightarrow \mathbb{R}$$

is defined by

$$\mathcal{L}_n^{\text{sc}}(i, x) = 2^{1/2}n^{-1/3} \left(L_n(i, n + 2n^{2/3}x) - 2n - 2n^{2/3}x \right),$$

for $x \in \left[-\frac{1}{2}n^{1/3}, \infty\right)$ and $i \in \llbracket 1, n \rrbracket$.

Thus one centres the time variable at the characteristic location $t = n$, zooms horizontally by the $n^{2/3}$ -scale, and rescales vertically by the $n^{1/3}$ -scale. The subtraction of $2n + 2n^{2/3}x$ removes the leading deterministic growth of the top curve near that characteristic location. After this centring, the residual global profile in the scaled coordinates is parabolic.

A scaling figure may be placed here. Its purpose is simply to show that the unscaled last passage ensemble is being viewed through a narrow window around $t = n$, and that in the scaled coordinates the remaining first-order curvature is quadratic.

5.4.3. Regular ensembles. The scaled ensembles $\mathcal{L}_n^{\text{sc}}$ have three features that we shall want to retain in subsequential limits. They satisfy a Brownian Gibbs resampling rule. Their large-scale profile is parabolic. And the top curve enjoys one-point bounds strong enough to yield compactness on fixed windows. Following [Ham22], these features are packaged in the notion of a regular ensemble.

This notion was developed in [Ham22], where it is shown [a bit repetitive with end of preceding paragraph. that the edge-scaled Brownian last passage ensembles form a regular family. Conceptually, this is exactly the sort of behaviour that the opening section led us to expect. The earlier Poissonian discussion explained why KPZ scaling should reveal a parabolic global profile together with order-one fluctuations on compact windows. The regularity statement shows that this heuristic is indeed manifested quantitatively in the Brownian setting. Its proof ultimately harnesses the one-point estimates available for Brownian watermelon and transfers them to the scaled last passage ensembles.

The formal definition is more elaborate than what is strictly needed for the later construction of the Airy line ensemble, but it is useful to record it in full. The specific scales $n^{1/9}$ and $n^{1/3}$ arise naturally in the prelimiting Brownian last passage ensembles after KPZ scaling. For present purposes, the important point is that both scales diverge with n . Thus, on any fixed compact spatial interval, one is well inside the region on which the definition gives useful information.

DEFINITION 5.5. Consider a Brownian Gibbs ensemble of the form

$$\mathcal{L} : \llbracket 1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$$

defined on a probability space under the law $\mathbb{P}$. The number n of ensemble curves and the quantity z_n may take any values in $\mathbb{N}$ and $[0, \infty)$.

For two constants $C \geq c > 0$, the ensemble $\mathcal{L}$ is called (c, C) -regular if the following conditions are satisfied.

(1) **Endpoint escape.** One has

$$z_n \geq cn^{1/3}.$$

(2) **One-point lower tail.** If $z \geq -z_n$ satisfies $|z| \leq cn^{1/9}$, then

$$\mathbb{P}\left(\mathcal{L}(1, z) + 2^{-1/2}z^2 \leq -s\right) \leq C \exp\{-cs^{3/2}\}$$

for all $s \in [1, n^{1/3}]$.

(3) **One-point upper tail.** If $z \geq -z_n$ satisfies $|z| \leq cn^{1/9}$, then

$$\mathbb{P}\left(\mathcal{L}(1, z) + 2^{-1/2}z^2 \geq s\right) \leq C \exp\{-cs^{3/2}\}$$

for all $s \in [1, \infty)$.

The content of the definition is straightforward. The Brownian Gibbs property encodes the local bridge-resampling rule. The term $2^{-1/2}z^2$ records the parabolic curvature. The two one-point bounds say that, after this parabolic correction, the top curve typically remains on order-one scale over compact windows.

PROPOSITION 5.6. There exist positive constants c and C such that each of the scaled Brownian last passage line ensembles

$$\mathcal{L}_n^{\text{sc}} : \llbracket 1, n \rrbracket \times \left[-\frac{1}{2}n^{1/3}, \infty\right) \rightarrow \mathbb{R}, \quad n \in \mathbb{N},$$

is (c, C) -regular.

Thus the scaled Brownian last passage ensembles form a family whose local structure is Brownian, whose large-scale shape is parabolic, and whose top curve has enough one-point control to support compactness arguments.

5.4.4. The Airy line ensemble theorem. We now state the limiting theorem toward which the chapter is directed. It is important here to distinguish between two closely related ensembles.

The stationary object is the Airy line ensemble

$$\mathcal{A} = (\mathcal{A}(1, \cdot), \mathcal{A}(2, \cdot), \dots),$$

whose top curve is the Airy_2 process and whose law is invariant under horizontal shifts. The object that naturally appears from KPZ centring and rescaling, however, is its parabolic shift

$$\mathcal{L}(i, x) = \mathcal{A}(i, x) - 2^{-1/2}x^2.$$

It is this parabolic Airy line ensemble $\mathcal{L}$ that has the Brownian Gibbs property, and it is this ensemble to which the scaled Brownian last passage ensembles converge.

To state the convergence properly, one must specify not merely the finite dimensional distributions but the weak topology on a suitable curve space. Mere finite-dimensional convergence would leave room for pathological behaviour at exceptional times: for example, one could imagine thin upward spikes in the top curve that disappear at each fixed deterministic time but remain visible in the full path.

For fixed $k \in \mathbb{N}$ and $T > 0$, consider the space of k -curve continuous line ensembles on $[-T, T]$, that is,

$$C(\llbracket 1, k \rrbracket \times [-T, T], \mathbb{R}),$$

equipped with the topology of uniform convergence. A law on this space is then a probability measure on a Polish function space, and convergence of line ensembles means weak convergence of these probability measures. In particular, when we say that a sequence of line ensembles converges on compact sets, we mean that for each k and T , the laws of the restrictions of the top k curves to $[-T, T]$ converge weakly in this compact-uniform topology.

Equivalently, after passing to a suitable coupling, one may realise the convergence almost surely uniformly on $\llbracket 1, k \rrbracket \times [-T, T]$ for each fixed k and T . This is the functional mode of convergence needed for the Brownian Gibbs argument, and it is the one used in Corwin–Hammond [CH14].

THEOREM 5.7. Let $k \in \mathbb{N}$ and $T > 0$. Then the laws of the restrictions of the scaled Brownian last passage ensembles $\mathcal{L}_n^{\text{sc}}$ to

$$\llbracket 1, k \rrbracket \times [-T, T]$$

converge weakly, as $n \rightarrow \infty$, in the topology of uniform convergence on

$$C(\llbracket 1, k \rrbracket \times [-T, T], \mathbb{R}).$$

The resulting consistent family of limit laws defines an $\mathbb{N}$ -indexed ordered continuous line ensemble $\mathcal{L}$. Moreover, the ensemble

$$\mathcal{A}(i, x) = \mathcal{L}(i, x) + 2^{-1/2}x^2$$

is the Airy line ensemble. In particular, $\mathcal{A}(1, \cdot)$ is the Airy_2 process, and the parabolic Airy line ensemble $\mathcal{L}$ satisfies the Brownian Gibbs property.

Thus the scaled Brownian last passage ensembles converge not directly to the stationary Airy line ensemble, but to its parabolic shift. This is exactly what one should expect from the preceding scaling discussion: the first-order curvature survives in the limit as a quadratic term, and it is after restoring that quadratic term that one recovers the stationary Airy ensemble.

The principal aim of the rest of the chapter is to explain why the Brownian Gibbs structure survives this passage to the limit.

5.4.5. Acceptance probabilities on compact intervals. The acceptance quantity relevant to this limiting argument is not tied to the model interval $[-1, 1]$, nor to the resampling of a single curve. In general, one considers the Brownian Gibbs reconstruction problem for the top k curves on a compact interval $I = [\ell, r]$. Given the retained data outside the open middle over I , there is a raw k -curve Brownian bridge candidate law and an associated acceptance event requiring the candidate curves to avoid one another and to remain above the frozen $(k + 1)^{\text{st}}$ curve. We write $A_{k,I}^{(n)}$ for the corresponding conditional acceptance probability under the law $\mathbb{P}_n$ of the scaled ensemble $\mathcal{L}_n^{\text{sc}}$.

This is the natural quantity in full generality. For the proof, however, we shall first isolate the simplest case, namely the top-curve resampling problem on a fixed compact interval. Since the mechanism is already visible there, it is best to treat that model case in detail before indicating the generalisation.

Fix the interval $[-1, 1]$. For $n \in \mathbb{N}$, let $\mathbb{P}_n$ denote the law of $\mathcal{L}_n^{\text{sc}}$. Condition on the retained data consisting of the endpoint values

$$\mathcal{L}_n^{\text{sc}}(1, -1), \quad \mathcal{L}_n^{\text{sc}}(1, 1),$$

together with the second curve $\mathcal{L}_n^{\text{sc}}(2, \cdot)|_{[-1, 1]}$.

Under this conditioning, the missing top-curve segment is obtained by starting from a Brownian bridge with the given endpoint values and then conditioning that bridge to remain above the frozen second curve on $[-1, 1]$.

Let $\mathcal{B}$ denote this raw Brownian bridge law. Define the acceptance event

$$\text{NoTouch} = \{\text{the candidate bridge stays above } \mathcal{L}_n^{\text{sc}}(2, \cdot) \text{ on } [-1, 1]\},$$

and set

$$A_n = A_{1, [-1, 1]}^{(n)} = \mathcal{B}(\text{NoTouch}).$$

Thus A_n is the conditional acceptance probability for the model top-curve resampling problem on $[-1, 1]$. Since the retained data are random under $\mathbb{P}_n$, the quantity A_n is itself a random variable.

The general compact-window acceptance principle takes the following form.

PROPOSITION 5.8. Let $k \in \mathbb{N}$ and let $I \subset \mathbb{R}$ be a compact interval. For each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all sufficiently large n ,

$$\mathbb{P}_n(A_{k,I}^{(n)} \geq \delta) \geq 1 - \varepsilon.$$

The detailed argument below will focus first on the special case $(k, I) = (1, [-1, 1])$.

PROPOSITION 5.9. For each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{P}_n(A_n \geq \delta) \geq 1 - \varepsilon.$$

After proving Proposition 5.9, we shall indicate how the same mechanism extends to the general compact-window statement in Proposition 5.8.

5.4.6. Why the uniform estimate matters in the scaling limit. The need for Proposition 5.9 comes from the fact that the Brownian Gibbs property of the finite- n ensembles is not by itself enough to guarantee the same property for a subsequential limit. Under KPZ scaling, neighbouring curves are brought into close proximity on order-one windows. Since the ensembles are ordered, crossings are impossible, but that alone does not rule out a weaker degeneration in which the corridor available to the local Brownian bridge candidate becomes thinner and thinner as $n \rightarrow \infty$.

If such a degeneration were to occur, then the local conditional law in the prelimit could become an increasingly severe filtering of the raw Brownian bridge input. In the limit, one might then lose the robust bridge-resampling structure required by the Brownian Gibbs property. What must be controlled is quantitative: whether the corridor available to the candidate bridge stays non-degenerate on fixed compact windows as $n \rightarrow \infty$. The severity of the constraint, not its mere presence, is what governs whether the resampling structure survives the limit.

This is exactly what the uniform acceptance estimate prevents. It says that, with high probability, the local Brownian candidate is not being forced through an asymptotically microscopic gate. The rejection step remains effective but not degenerate at prelimit level. Consequently, the Brownian Gibbs resampling rule remains stable enough to pass to subsequential limits.

This is the main technical point that separates the finite- n Brownian Gibbs property of the scaled ensembles from the Brownian Gibbs property of the parabolic Airy line ensemble. The next section develops the resampling tools needed for the proof.

5.5. Five resampling tools

The later argument brings together five related ideas. The first is a stopping-domain version of the Brownian Gibbs property. The second is a monotonicity principle for avoiding Brownian bridge laws. The third is an instructive example, No Big Max, showing how Brownian Gibbs resampling can turn an exceptional event into something typical enough to expose it to one-point bounds. The fourth is the propagation of one-point lower-tail control from the top curve to lower curves. The fifth is a size-bias principle: conditioning on eventual success weights retained-data configurations in proportion to their success probability. We state these tools here in a form suited to later citation.

5.5.1. The strong domain Markov property. The Brownian Gibbs property is usually stated for deterministic compact intervals. In some arguments, however, it is useful to let the resampling interval depend on the already revealed exterior data. Such a random interval will be called a stopping domain. This is part of the mechanism in the next subsection: once a first hitting point is located, one wants to resample on the interval beginning there. The appropriate extension is the strong domain Markov property.

PROPOSITION 5.10 (Strong domain Markov property). Let $\mathcal{L}$ be a Brownian Gibbs line ensemble. Let $\mathfrak{l} \leq \mathfrak{r}$ be random variables such that the event $\{\mathfrak{l} \leq \mathfrak{r}\}$ and the values of $\mathfrak{l}, \mathfrak{r}$ are measurable with respect to the sigma-field generated by the ensemble outside the random interval $(\mathfrak{l}, \mathfrak{r})$. Then, conditional on this exterior sigma-field, the missing curve segments on $[\mathfrak{l}, \mathfrak{r}]$ have the Brownian Gibbs law determined by their endpoint values and by the adjacent frozen curves.

This is the spatial analogue of the strong Markov property for Brownian motion. No new resampling rule is being introduced. One begins with the usual Brownian Gibbs property on deterministic intervals, approximates the stopping domain $[\mathfrak{l}, \mathfrak{r}]$ from outside by dyadic intervals whose endpoints are already determined by the exterior data, applies the deterministic resampling rule there, and

then passes to the limit using continuity of the curves and of the non-intersecting Brownian bridge law as the boundary data vary. We will use this extension without proof; see Lemma 2.5 of *AiryLE*.

5.5.2. Monotonicity of avoiding bridge laws. We will use repeatedly the fact that avoiding bridge ensembles are monotone in their endpoint data and in their neighbouring boundary curves. Since this is the first place where the point enters in a serious way, it is worth stating it cleanly and giving a proof sketch.

Fix a compact interval $[a, b]$, a positive integer k , and two collections of endpoint data

$$\bar{x} = (x_1, \dots, x_k), \quad \bar{y} = (y_1, \dots, y_k),$$

and

$$\bar{x}' = (x'_1, \dots, x'_k), \quad \bar{y}' = (y'_1, \dots, y'_k),$$

all lying in the Weyl chamber, so that

$$x_1 > \dots > x_k, \quad y_1 > \dots > y_k,$$

and likewise for $\bar{x}'$ and $\bar{y}'$. Let

$$f, g, f', g' : [a, b] \rightarrow \mathbb{R} \cup \{\pm\infty\}$$

be continuous functions such that $f \geq f'$ and $g \geq g'$ pointwise on $[a, b]$. Here f is the upper boundary and g the lower boundary; the same convention applies to f' and g' . We also assume that the endpoint data are ordered coordinatewise:

$$x_i \geq x'_i \quad \text{and} \quad y_i \geq y'_i \quad \text{for each } i \in \llbracket 1, k \rrbracket.$$

Let $\mathcal{B}_{k;\bar{x},\bar{y}}^{[a,b]}[f, g]$ denote the law of k independent Brownian bridges on $[a, b]$ from $\bar{x}$ to $\bar{y}$, conditioned to avoid one another and to remain between the boundaries f and g . Similarly define $\mathcal{B}_{k;\bar{x}',\bar{y}'}^{[a,b]}[f', g']$.

LEMMA 5.11 (Monotonicity of avoiding bridge ensembles). Under the preceding hypotheses, there exists a coupling of

$$\mathcal{Q} \sim \mathcal{B}_{k;\bar{x},\bar{y}}^{[a,b]}[f, g] \quad \text{and} \quad \mathcal{Q}' \sim \mathcal{B}_{k;\bar{x}',\bar{y}'}^{[a,b]}[f', g']$$

such that, almost surely,

$$\mathcal{Q}_i(t) \geq \mathcal{Q}'_i(t) \quad \text{for all } i \in \llbracket 1, k \rrbracket \text{ and all } t \in [a, b].$$

In particular, any event that is increasing in the curves has probability at least as large under $\mathcal{B}_{k;\bar{x},\bar{y}}^{[a,b]}[f, g]$ as under $\mathcal{B}_{k;\bar{x}',\bar{y}'}^{[a,b]}[f', g']$.

PROOF SKETCH. We first establish the corresponding statement for avoiding random-walk bridges, and then pass to Brownian bridges by diffusive scaling.

Consider a discrete-time or continuous-time nearest-neighbour random-walk bridge model on a fine spatial mesh inside $[a, b]$, with the same endpoint data and with the same conditioning of mutual avoidance and confinement between upper and lower boundaries. For each of the two data sets, the admissible path collections form a finite partially ordered state space under pointwise comparison of the paths.

On each state space one runs the same local Markov chain. One may take a single-site Glauber dynamics: attach independent exponential clocks to the space-time lattice sites where a local update is permitted, and when a clock rings, attempt to raise or lower the relevant path value by one mesh unit, provided that doing so preserves the bridge constraints, the ordering of the k paths, and the

boundary conditions. This dynamics is irreducible on the finite admissible state space, and its invariant law is exactly the conditioned random-walk bridge measure.

The key point is that the two dynamics may be coupled using the same underlying clock process and the same attempted local moves. Start the two systems from their lowest admissible configurations. Because the endpoint data and lower boundary in the primed system lie no higher than those in the unprimed system, and because likewise the upper boundary in the primed system lies no lower, the initial configurations may be chosen in ordered form: the unprimed one lies pointwise above the primed one.

We claim that this ordering is preserved by the coupled dynamics. Indeed, suppose that at some moment the two path ensembles are ordered. If a proposed local move is admissible in the lower system, then the same move is also admissible in the higher one unless it would violate one of the ordering or boundary constraints. But those constraints are weaker in the higher system: its endpoints are no lower, its lower boundary is no higher, and its upper boundary is no lower. A direct inspection of the local move therefore shows that the coupled update cannot destroy the pointwise ordering. Thus the order is preserved for all times.

Since the coupled chain is irreducible on a finite state space, each marginal converges to its unique invariant law, namely the corresponding conditioned random-walk bridge law. Passing to the limit along a sequence of times tending to infinity, one obtains a monotone coupling of the two conditioned random-walk bridge measures.

To pass from random walks to Brownian bridges, refine the mesh and apply the usual diffusive scaling. The unconditioned bridge laws converge to Brownian bridge laws, and the conditioning events converge as well. Because the monotone couplings are tight and preserve pointwise order at every mesh level, one may pass to a subsequential limit and obtain a coupling of the two avoiding Brownian bridge ensembles with the same order-preservation property. This yields the desired monotonicity.

For the applications in this chapter, one usually needs only a simple special case: lowering the endpoint data and removing or lowering the lower boundary can only decrease the conditioned bridge ensemble. That is exactly the form invoked in the proof of No Big Max. $\square$

5.5.3. No Big Max. The one-point bounds in the definition of a regular ensemble control the upper tail of the top curve at any fixed spatial location. The next proposition shows how Brownian Gibbs resampling converts this into a bound on the maximum over a compact interval. It is a simple but instructive use of the Brownian Gibbs property: a large value somewhere in an interval is an exceptional event under the original law, but after one resamples from the first place where that largeness occurs, it becomes typical enough to force a substantial value at a fixed point, where one-point information is available.

PROPOSITION 5.12 (No Big Max). Let $\mathcal{L} : \llbracket 1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$ be a (c, C) -regular ensemble. Assume that

$$n \geq (2c)^{-18} \quad \text{and} \quad t \in [2^{7/2}, 2n^{1/3}].$$

Then

$$\mathbb{P} \left(\sup_{x \in [-1, 1]} \left(\mathcal{L}(1, x) + 2^{-1/2} x^2 \right) \geq t \right) \leq 6C \exp \{ -2^{-6} c t^{3/2} \}.$$

PROOF. Set

$$\text{High}(s) = \left\{ \sup_{x \in [-1, 1]} \mathcal{L}(1, x) \geq s \right\}.$$

Since $2^{-1/2}x^2 \leq 2^{-1/2} < 1$ for $x \in [-1, 1]$, the event in the proposition is contained in $\text{High}(t-1)$. Because $t \geq 2^{7/2} > 2$, we have $t-1 \geq t/2$. It is therefore enough to prove that

$$\mathbb{P}(\text{High}(s)) \leq 6C \exp \left\{ -2^{-9/2} c s^{3/2} \right\} \quad (5)$$

for all $s \in [2^{5/2}, 2n^{1/3}]$; the proposition then follows by applying (5) with $s = t/2$.

Assume that $\text{High}(s)$ occurs. Let

$$X = \inf \{x \in [-1, 1] : \mathcal{L}(1, x) \geq s\}.$$

On $\text{High}(s)$, the quantity X lies in $[-1, 1]$. It is the first place in $[-1, 1]$ where the top curve reaches the level s . We now resample the top curve on the interval $[X, 2]$. By the strong domain Markov property, conditional on the exterior data this resampled segment has Brownian Gibbs law with left endpoint $\mathcal{L}(1, X)$, right endpoint $\mathcal{L}(1, 2)$, and lower boundary $\mathcal{L}(2, \cdot)$.

Now also impose the event

$$\mathcal{L}(1, 2) \geq -s/3.$$

At this point, monotonicity applies. If we make the task easier by lowering the left endpoint to exactly s , lowering the right endpoint to exactly $-s/3$, and removing the lower boundary altogether, then the resulting comparison law is stochastically dominated by the original one. Thus the resampled top curve on $[X, 2]$ stochastically dominates the ordinary Brownian bridge from s to $-s/3$.

In this way, under the conditional law on $[X, 2]$, the resampled top curve stochastically dominates the ordinary Brownian bridge on $[X, 2]$ from s to $-s/3$. Evaluate this comparison bridge at the point 1. Its mean there is the affine interpolation between (X, s) and $(2, -s/3)$, namely

$$\frac{2-1}{2-X} s + \frac{1-X}{2-X} \left(-\frac{s}{3} \right) = \frac{2+X}{3(2-X)} s.$$

Since $X \in [-1, 1]$, this quantity is at least $s/9$. A Brownian bridge has symmetric fluctuations about its affine interpolation, and so the probability that its value at 1 is at least its mean is $1/2$. Consequently, on the event $\text{High}(s) \cap \{\mathcal{L}(1, 2) \geq -s/3\}$, Brownian Gibbs resampling produces a top-curve value at 1 exceeding $s/9$ with conditional probability at least $1/2$.

Since Brownian Gibbs resampling preserves the law of the ensemble,

$$\mathbb{P}(\mathcal{L}(1, 1) \geq s/9) \geq \frac{1}{2} \mathbb{P}(\text{High}(s) \cap \{\mathcal{L}(1, 2) \geq -s/3\}). \quad (6)$$

Hence

$$\mathbb{P}(\text{High}(s)) \leq \mathbb{P}(\mathcal{L}(1, 2) < -s/3) + 2\mathbb{P}(\mathcal{L}(1, 1) \geq s/9). \quad (7)$$

We now invoke regularity at the fixed spatial points 1 and 2. Since $n \geq (2c)^{-18}$, both points lie in the allowed one-point region $|z| \leq cn^{1/9}$.

For the upper-tail term,

$$\mathbb{P}(\mathcal{L}(1, 1) \geq s/9) \leq \mathbb{P}(\mathcal{L}(1, 1) + 2^{-1/2} \geq s/9) \leq C \exp \left\{ -c(s/9 - 2^{-1/2})^{3/2} \right\}.$$

Because $s \geq 2^{5/2}$, one has $s/9 - 2^{-1/2} \geq s/18$, and therefore

$$\mathbb{P}(\mathcal{L}(1, 1) \geq s/9) \leq C \exp \left\{ -c 2^{-9/2} s^{3/2} \right\}. \quad (8)$$

For the lower-tail term, note that

$$\mathcal{L}(1, 2) < -s/3 \implies \mathcal{L}(1, 2) + 2^{-1/2} \cdot 2^2 < -s/3 + 2^{3/2}.$$

Since $s \geq 2^{5/2}$, the right-hand side is at most $-s/12$. Hence regularity at the spatial point 2 gives

$$\mathbb{P}(\mathcal{L}(1, 2) < -s/3) \leq C \exp \{ -c 12^{-3/2} s^{3/2} \} \leq C \exp \{ -c 2^{-9/2} s^{3/2} \}. \quad (9)$$

Substituting (8) and (9) into (7) yields

$$\mathbb{P}(\text{High}(s)) \leq 3C \exp \{ -2^{-9/2} c s^{3/2} \}.$$

This proves (5), and hence the proposition. $\square$

This proof is worth remembering because its mechanism recurs later. A large maximum on $[-1, 1]$ is exceptional under the original law. But once the first hitting point X is revealed and Brownian Gibbs resampling is applied on $[X, 2]$, that same largeness forces the resampled curve to have a substantial chance of being large at the fixed point 1, where one-point control is available. The exceptional event has been turned, by lawful resampling, into something typical enough to be measured at a deterministic location.

5.5.4. One-point control for lower curves. A regular ensemble comes with one-point control only for its uppermost curve: after parabolic correction, curve 1 has upper and lower tails of order $e^{-cs^{3/2}}$. For the acceptance argument later in the chapter, however, one also needs comparable one-point control for lower indexed curves. This is not an additional hypothesis. It is a consequence of Brownian Gibbs resampling together with the regularity input already available for curve 1.

The proof in *AiryPoly* is inductive in the curve index and proceeds through a stronger statement: one first proves that each fixed lower curve is unlikely to dip very far below its parabolic scale anywhere on a compact interval, and then deduces the one-point estimate as a corollary. We record only the one-point consequence, but it is useful to sketch the interval argument, since that is where the real mechanism lies.

PROPOSITION 5.13 (One-point lower-tail control for lower curves). Fix $k \in \mathbb{N}$. Let $\mathcal{L}$ be a (c, C) -regular ensemble. Then there exist positive constants c_k and C_k , depending only on k , c , and C , such that, throughout the regularity window,

$$\mathbb{P}(\mathcal{L}(k, x) + 2^{-1/2}x^2 \leq -s) \leq C_k e^{-c_k s^{3/2}} \quad \text{for all } s \geq 1.$$

A corresponding upper-tail estimate is immediate from ordering, since $\mathcal{L}(k, \cdot) \leq \mathcal{L}(1, \cdot)$. The lower tail is the substantive part.

The mechanism behind the proposition is that Brownian Gibbs resampling carries one-point control downward. One-point control along the top curve supplies a ceiling at the parabolic scale; the resampling manufactures from it a floor for the curves beneath, one index at a time, so that each lower curve is pinned to the same parabola as the top. Before carrying this out it is worth recalling why that parabola is present at all, since every estimate below is read against it. Earlier the parabola entered as the deterministic limit shape of the droplet; the resampling rule supplies a second, independent reason it cannot be absent.

Ask first what the Brownian Gibbs property requires of a single curve with nothing beneath it. The rule is this: take any compact interval, erase the curve there, and redraw it as a Brownian bridge between the two surviving endpoints; the law of the curve is to come back unchanged. No curve

obeys it. Let $\mathcal{L}$ be a candidate. Its height $\mathcal{L}(0)$ has some distribution, so a fixed unit interval I of heights carries $\mathbb{P}(\mathcal{L}(0) \in I) = c > 0$. Resample on $[-T, T]$. Conditional on the two endpoints, the redrawn $\mathcal{L}(0)$ is the midpoint of a Brownian bridge of length $2T$, a Gaussian of variance $T/2$, whose density is at most $(\pi T)^{-1/2}$ at every height, whatever the endpoints. The resampled height therefore lands in I with probability at most $(\pi T)^{-1/2}$, and for T large this falls below c . The law cannot return unchanged, so no single curve carries a Brownian Gibbs law.

Put a curve beneath it, and to see the effect plainly take that lower curve flat. The top curve is then a Brownian bridge on $[-T, T]$ conditioned to stay above a fixed level. Holding a long bridge to one side of a level does not press it against the level; it lifts it, the height at the centre standing of order $\sqrt{T}$ above the barrier by entropic repulsion. That height marches off with T , so once again no law is left fixed. A flat floor is too slack to hold the curve in place.

What is needed is a floor that, on every long interval, leaves the resampled bridge sitting an order-one height above it, neither crushed against it nor lifted away. The floor must fall away from the centre, fast enough to make room for the bridge's sag and no faster. A downward parabola does this: against $-2^{-1/2}x^2$ the conditioned bridge has a non-degenerate limit, and the curve, corrected by $+2^{-1/2}x^2$, is an order-one object—the scale on which the one-point bounds are framed.

That the floor is a parabola, and not some other falling curve, follows from one feature of the rule. A Brownian bridge is unchanged in law, up to its endpoints, by the addition of an affine function: adding $a + bx$ returns a bridge, the linear part absorbed into the boundary values. The resampling therefore feels the background only through what no affine function can remove, namely its curvature. For the rule to pose the same problem on every long window, as a spatially homogeneous ensemble demands, the curvature must be the same on every window, hence constant; and a profile of constant second derivative is a parabola. The value $2^{-1/2}$ of that curvature is set by the Brownian scale rather than by this argument: in Cameron–Martin terms the parabola is the deterministic shift whose tilt of the bridge is the same at every point, and the size of that uniform tilt is what the diffusion coefficient fixes.

One further fact is worth carrying into the argument, where intervals other than $[-T, T]$ appear. Adding an affine function to all curves at once leaves the Brownian Gibbs law invariant: it shifts a curve and the curve below it together, carries bridges to bridges, and preserves every ordering constraint. A parabola with a linear function added is again a parabola, of the same curvature and a displaced vertex. So a resampling set on $[-2T, 0]$ or $[0, 2T]$ is, after adding the right linear function and translating, the resampling set on $[-T, T]$, with the parabola symmetric about the midpoint and the resampled curve brought into alignment. The argument below uses this freedom to centre each step where the parabolic correction is balanced.

We sketch the argument in the form relevant here.

Step 1: the top curve on a compact interval. Fix $T \geq 1$. For the induction, one needs not only the regular one-point bound for curve 1, but also a compact-interval version:

$$\mathbb{P}\left(\inf_{x \in [-T, T]} (\mathcal{L}(1, x) + 2^{-1/2}x^2) \leq -s\right) \leq CTe^{-cTs^{3/2}}.$$

This is obtained from regularity and Brownian Gibbs as follows.

At each integer $m \in [-T, T]$, regularity gives

$$\mathbb{P}(\mathcal{L}(1, m) + 2^{-1/2}m^2 \leq -s) \leq Ce^{-cs^{3/2}}.$$

There are only $O(T)$ such integers, so by a union bound,

$$\mathbb{P}\left(\min_{m \in [-T, T] \cap \mathbb{Z}} (\mathcal{L}(1, m) + 2^{-1/2} m^2) \leq -s\right) \leq CT e^{-cs^{3/2}}. \quad (10)$$

Now fix one unit interval $[m, m+1] \subset [-T, T]$. Conditional on the exterior data, the top curve on $[m, m+1]$ has Brownian Gibbs law. By Lemma 5.11, its conditional law is bounded below by that of an ordinary Brownian bridge with the same endpoint values and no lower boundary. Hence, if the endpoint values both exceed $-s$ after parabolic correction, then the probability that the bridge drops below $-2s$ somewhere in the unit interval is at most $e^{-c's^2}$ for a universal $c' > 0$. Summing over the $O(T)$ many unit intervals yields

$$\mathbb{P}\left(\inf_{x \in [-T, T]} (\mathcal{L}(1, x) + 2^{-1/2} x^2) \leq -2s, \min_{m \in [-T, T] \cap \mathbb{Z}} (\mathcal{L}(1, m) + 2^{-1/2} m^2) > -s\right) \leq C'T e^{-c's^2}. \quad (11)$$

Combining (10) and (11) gives the desired compact-interval lower-tail control for curve 1.

Step 2: the second curve on side intervals. The next step, and the heart of the induction, is to explain how control of curve 1 forces curve 2 to be present on neighbouring side intervals.

Fix $T \geq 5$, and consider the left and right side intervals

$$[-2T, -T] \quad \text{and} \quad [T, 2T].$$

We discuss the right interval $[T, 2T]$; the left-hand argument is identical. Following *AiryPoly*, define the bad event

$$\text{Low}_2^{[T, 2T]} = \left\{ \sup_{x \in [T, 2T]} (\mathcal{L}(2, x) + Q(x)) \leq -\frac{1}{2}LT^2 - T^{1/2} \right\},$$

where $Q(x) = 2^{-1/2}x^2$, and where $L > 0$ is a fixed large constant. On this event, curve 2 lies uniformly far below its parabolic scale throughout the whole side interval.

To exploit this, one intersects with the event that the endpoint values of curve 1 at T and $2T$ are not unusually low. Write

$$G_1(s) = \left\{ \mathcal{L}(1, s) + Q(s) \geq -T^{1/2} \right\}, \quad s \in \{T, 2T\},$$

and set

$$A_2^{[T, 2T]} = G_1(T) \cap G_1(2T) \cap \text{Low}_2^{[T, 2T]}.$$

The complement of $G_1(T) \cap G_1(2T)$ already has very small probability by the one-point lower-tail control for curve 1, so the main task is to bound $\mathbb{P}(A_2^{[T, 2T]})$.

On the event $A_2^{[T, 2T]}$, resample curve 1 on $[T, 2T]$. Conditional on the exterior data, this resampled curve has the law of a Brownian bridge from some endpoint values u, v satisfying

$$u \geq -Q(T) - T^{1/2}, \quad v \geq -Q(2T) - T^{1/2},$$

conditioned not to touch the lower barrier

$$f(x) = -Q(x) - \frac{1}{2}LT^2 - T^{1/2}, \quad x \in [T, 2T].$$

The important point is that this barrier is low. Let B denote the free Brownian bridge on $[T, 2T]$ from u to v . After subtracting the affine interpolant between the endpoints and rescaling time linearly from $[T, 2T]$ to $[0, 1]$, one obtains a standard Brownian bridge B^{br} . The event that the

free bridge avoids the barrier f contains the event that the standard bridge stays below 1. For a standard Brownian bridge, one has the exact formula

$$\mathbb{P}\left(\sup_{0 \leq s \leq 1} B^{\text{br}}(s) > x\right) = e^{-2x^2}, \quad x > 0.$$

In particular,

$$\mathbb{P}\left(\sup_{0 \leq s \leq 1} B^{\text{br}}(s) \leq 1\right) = 1 - e^{-2}.$$

Hence

$$\mathcal{B}_{1;u,v}^{[T,2T]}(\text{NoTouch}_f) \geq 1 - e^{-2}.$$

So conditioning on NoTouch_f is not very costly. By Bayes' theorem, any event E measurable with respect to the bridge on $[T, 2T]$ satisfies

$$\mathcal{B}_{1;u,v}^{[T,2T]}(E \mid \text{NoTouch}_f) = \frac{\mathcal{B}_{1;u,v}^{[T,2T]}(E \cap \text{NoTouch}_f)}{\mathcal{B}_{1;u,v}^{[T,2T]}(\text{NoTouch}_f)} \leq (1 - e^{-2})^{-1} \mathcal{B}_{1;u,v}^{[T,2T]}(E),$$

and $(1 - e^{-2})^{-1} < 2$.

Now consider the midpoint $3T/2$. Let ℓ be the affine interpolation between the endpoint data (T, u) and $(2T, v)$. Because u and v are near their parabolic values, the quantity

$$\ell(3T/2) + \frac{1}{4}LT^2$$

still lies well below the parabolic scale at $3T/2$. More precisely, using the secant-versus-parabola gap,

$$Q(3T/2) - \frac{1}{2}(Q(T) + Q(2T)) = 2^{-7/2}T^2.$$

Thus a free bridge from u to v has exponentially small probability to be as high as

$$-Q(3T/2) - 2^{-9/2}T^2$$

at the midpoint:

$$\mathcal{B}_{1;u,v}^{[T,2T]}(B(3T/2) \geq -Q(3T/2) - 2^{-9/2}T^2) \leq \exp\{-2^{-8}T^3\}.$$

Applying the Bayes bound above therefore yields

$$\begin{aligned} \mathbb{P}\left(\mathcal{L}(1, 3T/2) \geq -Q(3T/2) - 2^{-9/2}T^2 \mid \mathbf{A}_2^{[T,2T]}\right) \\ \leq (1 - e^{-2})^{-1} \exp\{-2^{-8}T^3\}. \end{aligned}$$

For $T \geq 5$, the right-hand side is at most $1/2$. Hence

$$\mathbb{P}(\mathbf{A}_2^{[T,2T]}) \leq 2\mathbb{P}\left(\mathcal{L}(1, 3T/2) \geq -Q(3T/2) - 2^{-9/2}T^2\right).$$

The one-point lower-tail estimate for curve 1 now gives

$$\mathbb{P}(\mathbf{A}_2^{[T,2T]}) \leq 2C \exp\{-c 2^{-27/4}T^3\}.$$

Since

$$\mathbb{P}(-\mathbf{G}_1(T)) + \mathbb{P}(-\mathbf{G}_1(2T)) \leq 2Ce^{-cT^{3/4}},$$

one concludes that

$$\mathbb{P}(\mathbf{Low}_2^{[T,2T]}) \leq C_1 e^{-c_1 T^3}.$$

The same argument applies on $[-2T, -T]$.

Thus, with probability at least $1 - C_2 e^{-c_2 T^3}$, curve 2 is not very low on either side interval.

Step 3: from the sides back to the centre. Define the complementary side events

$$\text{NoLow}_2^{[-2T, -T]} \quad \text{and} \quad \text{NoLow}_2^{[T, 2T]},$$

and let $\mathbf{E}$ be their intersection together with the event that curve 1 is not too low on $[-2T, 2T]$. On $\mathbf{E}$, define random times $\sigma_1 \in [-2T, -T]$ and $\sigma_2 \in [T, 2T]$ by

$$\sigma_1 = \inf \left\{ x \in [-2T, -T] : \mathcal{L}(2, x) + Q(x) \geq -2^{-7/2}T^2 - T^{1/2} \right\},$$

and

$$\sigma_2 = \sup \left\{ x \in [T, 2T] : \mathcal{L}(2, x) + Q(x) \geq -2^{-7/2}T^2 - T^{1/2} \right\}.$$

These are well-defined on $\mathbf{E}$, and the strip $\llbracket 2 \rrbracket \times (\sigma_1, \sigma_2)$ is a stopping domain.

Resample curve 2 on $[\sigma_1, \sigma_2]$. Its endpoint heights are now moderate by construction, and curve 1 stays safely above on the whole larger window by the good event $\mathbf{E}$. By monotonicity, the conditional law of the resampled second curve is bounded below by that of an ordinary Brownian bridge with the same endpoints and a constant lower boundary. Such a bridge has a uniformly positive probability to stay above

$$-Q(0) - CT^2$$

throughout $[-T, T]$. Therefore, if the actual second curve made a deeper dip on $[-T, T]$, then by Brownian Gibbs resampling one would create an event of uniformly positive conditional probability forcing the same dip for the comparison bridge. Since this is impossible, one obtains

$$\mathbb{P} \left(\inf_{x \in [-T, T]} (\mathcal{L}(2, x) + Q(x)) \leq -CT^2 \right) \leq C_3 e^{-c_3 T^3}.$$

Evaluating at a single point gives the one-point lower-tail estimate for curve 2.

The same two-stage argument then iterates from curve $k-1$ to curve k .

Step 4: the continuing induction. The same two-stage argument iterates. To control curve k , one first uses the already-established compact-interval control for curve $k-1$ to show that curve k must rise to a moderate level on each of the two side intervals; otherwise curve $k-1$, when resampled there, would have a positive chance to track a secant line that sits order T^2 below the parabola at the midpoint, contradicting its inductive lower-tail bound. One then scans in from left and right to the first and last such points, resamples curve k across the resulting random interval, and compares it from below with an ordinary Brownian bridge with moderate endpoints and a constant lower boundary. This yields the same type of compact-interval lower-tail estimate for curve k , and hence the one-point bound in Proposition 5.13.

For the present chapter, the important conclusion is simply the proposition: once k is fixed, every one of the top k curves enjoys one-point lower-tail control of the same stretched-exponential type as the top curve.

5.5.5. Conditioning on success and size bias. The final tool is a simple Radon–Nikodym identity with a useful geometric interpretation. It is not specific to Brownian Gibbs ensembles, but it is the mechanism by which the later argument promotes information under an auxiliary law into information under the true law obtained by conditioning on eventual success.

LEMMA 5.14 (Size-bias identity). Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, let $Y : \Omega \rightarrow S$ be a random element, and let $Z : S \rightarrow [0, 1]$ be measurable. Suppose that, given $Y = y$, an auxiliary random

experiment succeeds with probability $Z(y)$. Let ν denote the law of Y after conditioning on eventual success. Then ν is absolutely continuous with respect to the law μ_Y of Y , and

$$\frac{d\nu}{d\mu_Y}(y) = \frac{Z(y)}{\int Z d\mu_Y}. \quad (12)$$

Equivalently, for every measurable $E \subseteq S$,

$$\nu(E) = \frac{\int_E Z d\mu_Y}{\int Z d\mu_Y}. \quad (13)$$

PROOF. For measurable $E \subseteq S$,

$$\nu(E) = \mu(Y \in E \mid \text{success}) = \frac{\mu(Y \in E, \text{success})}{\mu(\text{success})}.$$

Conditioning on Y gives

$$\mu(Y \in E, \text{success}) = \int_E Z d\mu_Y \quad \text{and} \quad \mu(\text{success}) = \int Z d\mu_Y.$$

This yields (13), and hence (12). $\square$

One may think of this identity geometrically. Regard S as the base space and Z as a non-negative height function on it. Form the region

$$\mathcal{R}_Z = \{(y, u) \in S \times [0, 1] : 0 \leq u \leq Z(y)\}.$$

Sampling from ν is equivalent to throwing a dart uniformly into the region under the graph of Z and then forgetting the vertical coordinate. Base points y with larger height $Z(y)$ support taller columns, occupy more volume under the graph, and are therefore more likely to be seen under the true law. This is the size-bias picture.

The next corollary is the abstract estimate that will later be applied in a planar setting.

COROLLARY 5.15 (Dart-fall estimate). In the setting of Lemma 5.14, let $a > 0$ and $\eta \in (0, 1]$. If $G \subseteq S$ is measurable and satisfies

$$\mu_Y(G) \geq \eta \quad \text{and} \quad Z(y) \geq a \text{ for every } y \in G, \quad (14)$$

then

$$\int Z d\mu_Y \geq a\eta \quad (15)$$

and, for every $\rho \in (0, 1]$,

$$\nu(Z \leq \rho a\eta) \leq \rho. \quad (16)$$

PROOF. The lower bound (15) is immediate:

$$\int Z d\mu_Y \geq \int_G Z d\mu_Y \geq a \mu_Y(G) \geq a\eta.$$

Now let $E_\rho = \{Z \leq \rho a\eta\}$. By the size-bias identity,

$$\nu(E_\rho) = \frac{\int_{E_\rho} Z d\mu_Y}{\int Z d\mu_Y} \leq \frac{\rho a\eta \mu_Y(E_\rho)}{a\eta} \leq \rho.$$

This proves (16). $\square$

The point is simple but useful. If the graph of Z contains a block of height at least a over a base set of μ_Y -mass at least η , then the total volume under the graph is at least $a\eta$. A dart thrown uniformly under that graph is therefore unlikely to land in the thin floor layer $\{Z \leq \rho a\eta\}$: the chance is at most ρ . Later, the base space S will be two-dimensional, the retained data will determine a height function Z given by an acceptance probability, and this corollary will formalise the statement that the true law cannot place much mass where that acceptance probability is much smaller than the natural scale of the volume under the graph.

5.6. Missing closed middle reconstruction

Recall that

$$\mathcal{L}_n : \llbracket 1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$$

denotes the scaled n -curve ensemble. In this section we set up the reconstruction that underlies the later acceptance argument. We do this first in the model case $k = 1$.

Fix numbers

$$\ell < r$$

in the interior of the spatial domain of $\mathcal{L}_n$. Later we will take

$$(\ell, r) = (-1, 1),$$

but it is useful to write the discussion first in slightly greater generality.

The interval $[\ell, r]$ will be called the *closed middle*. The adjacent intervals

$$[\ell - 1, \ell] \quad \text{and} \quad [r, r + 1]$$

will be called the side intervals. The basic situation is this: the second curve is frozen on the whole stretch $[\ell - 1, r + 1]$, while for the top curve the data on the side intervals are retained in standard-bridge form, but the two inner endpoint heights

$$\mathcal{L}_n(1, \ell), \quad \mathcal{L}_n(1, r),$$

together with the bridge across $[\ell, r]$, are treated as missing.

If $a < b$ lie in the domain of a continuous function $f : [a, b] \rightarrow \mathbb{R}$, write $\ell_f^{[a, b]}$ for the affine function on $[a, b]$ agreeing with f at a and b . The standard bridge induced by f on $[a, b]$ is

$$f^{[a, b]}(x) = f(x) - \ell_f^{[a, b]}(x), \quad x \in [a, b].$$

Thus $f^{[a, b]}$ vanishes at the endpoints. In particular, the standard bridge ensemble induced by $\mathcal{L}_n$ on $[a, b]$ is

$$\mathcal{L}_n^{[a, b]}(i, x) = \mathcal{L}_n(i, x) - \ell_n^{[a, b]}(i, x), \quad (i, x) \in \llbracket 1, n \rrbracket \times [a, b],$$

where $\ell_n^{[a, b]}(i, \cdot)$ is the affine interpolant of the i th curve between a and b .

5.6.1. Retained data and missing data. We now specialise to $k = 1$. Let $\mathcal{F}$ be the sigma-field generated by the following retained data:

- the second curve $\mathcal{L}_n(2, \cdot)$ on the full spatial domain $[-z_n, \infty)$;
- the top curve $\mathcal{L}_n(1, \cdot)$ outside the open interval (ℓ, r) ;
- the left side standard bridge

$$\mathcal{L}_n^{[\ell-1, \ell]}(1, \cdot);$$

- the right side standard bridge

$$\mathcal{L}_n^{[r, r+1]}(1, \cdot).$$

The conditional law given this sigma-field will be denoted by $\mathbb{P}_{\mathcal{F}}$.

Thus the missing information consists of exactly three pieces:

- the top-curve height at ℓ ,
- the top-curve height at r ,
- the standard bridge of the top curve across $[\ell, r]$.

We call this the *missing closed middle* because the missing data include the endpoint heights at ℓ and r , not just the open interval (ℓ, r) .

Equivalently, the missing data may be viewed either as the whole top-curve segment on $[\ell, r]$, or as the pair of endpoint heights together with the middle standard bridge. The latter presentation is the useful one for what follows, because the side constraints will depend only on the endpoint heights, whereas the middle constraint will depend on the middle bridge itself.

5.6.2. Reconstruction from endpoint data and a middle bridge. Suppose that one is given proposed endpoint heights $x, y \in \mathbb{R}$ and a continuous function $b : [\ell, r] \rightarrow \mathbb{R}$ vanishing at ℓ and r . From the triple (x, y, b) , one reconstructs a candidate top curve on the whole domain. We write this reconstructed curve as

$$\mathcal{L}_n^{\text{re}}(x, y, b) : [-z_n, \infty) \rightarrow \mathbb{R}.$$

Outside the side-and-middle window $[\ell - 1, r + 1]$, the top curve is left unchanged. On the side intervals, one restores the retained side standard bridges using the proposed endpoint heights x and y . On the middle interval, one uses the affine interpolation between x and y together with the proposed bridge b . Concretely,

$$\mathcal{L}_n^{\text{re}}(x, y, b)(s) = \begin{cases} \mathcal{L}_n(1, s), & s \in [-z_n, \ell - 1] \cup (r + 1, \infty), \\ \mathcal{L}_n^{[\ell-1, \ell]}(1, s) + (\ell - s)\mathcal{L}_n(1, \ell - 1) + (s - \ell + 1)x, & s \in [\ell - 1, \ell], \\ \frac{r - s}{r - \ell}x + \frac{s - \ell}{r - \ell}y + b(s), & s \in [\ell, r], \\ \mathcal{L}_n^{[r, r+1]}(1, s) + (r + 1 - s)y + (s - r)\mathcal{L}_n(1, r + 1), & s \in [r, r + 1]. \end{cases} \quad (17)$$

The formula on $[\ell - 1, \ell]$ restores the affine part joining the known outer endpoint $\mathcal{L}_n(1, \ell - 1)$ to the proposed inner value x ; the formula on $[r, r + 1]$ does the analogous thing on the right. Thus the missing closed middle is reconstructed from two endpoint heights together with a bridge across $[\ell, r]$.

5.6.3. The Wiener candidate. To describe the actual conditional law under $\mathbb{P}_{\mathcal{F}}$, it is convenient to begin with the candidate law obtained by ignoring the ordering constraint.

Let

$$u = \mathcal{L}_n(1, \ell - 1), \quad v = \mathcal{L}_n(1, r + 1).$$

These quantities are $\mathcal{F}$ -measurable. Consider Brownian bridge law on the full interval $[\ell - 1, r + 1]$ from u to v ; denote it by

$$\mathcal{B}_{1;u,v}^{[\ell-1,r+1]}.$$

Let

$$W : [\ell - 1, r + 1] \rightarrow \mathbb{R}$$

be a sample from this law. We call W the *Wiener candidate*.

Its endpoint heights at the inner points are $W(\ell)$ and $W(r)$, and its middle standard bridge is $W^{[\ell,r]}$. Thus W determines a candidate triple

$$(W(\ell), W(r), W^{[\ell,r]}),$$

and hence a reconstructed curve

$$\mathcal{L}_n^{\text{re}}(W(\ell), W(r), W^{[\ell,r]}).$$

It is useful that the Wiener candidate lives on the full interval $[\ell - 1, r + 1]$, not only on $[\ell, r]$. The side pieces and the middle piece are therefore parts of one Brownian bridge rather than independent objects assembled afterwards. This is exactly the right raw candidate for the missing top-curve segment before the requirement of staying above the frozen second curve is imposed.

5.6.4. Side tests and middle test. Given a candidate triple (x, y, b) , or equivalently a reconstructed curve $\mathcal{L}_n^{\text{re}}(x, y, b)$, there are three conditions to check.

The *left side test* is that

$$\mathcal{L}_n^{\text{re}}(x, y, b)(s) > \mathcal{L}_n(2, s) \quad \text{for all } s \in [\ell - 1, \ell].$$

The *right side test* is that

$$\mathcal{L}_n^{\text{re}}(x, y, b)(s) > \mathcal{L}_n(2, s) \quad \text{for all } s \in [r, r + 1].$$

The *middle test* is that

$$\mathcal{L}_n^{\text{re}}(x, y, b)(s) > \mathcal{L}_n(2, s) \quad \text{for all } s \in [\ell, r].$$

The important point is that the two side tests depend only on the retained side bridge data together with the endpoint heights x and y . The middle test depends on the actual bridge across $[\ell, r]$. In terms of the Wiener candidate, the middle test is simply

$$W(s) > \mathcal{L}_n(2, s) \quad \text{for all } s \in [\ell, r]. \tag{18}$$

5.6.5. Moving the side bridges. The side tests admit a simple geometric description that prepares the corners analysis.

Consider the left side interval $[\ell - 1, \ell]$. The retained data determine the standard-bridge shape there and the outer endpoint value $\mathcal{L}_n(1, \ell - 1)$. If one varies the inner endpoint value x , the entire left side curve moves affinely: the bridge shape is unchanged, but its tilt is altered so that it now ends at height x at the point ℓ . The same remark applies on the right when one varies y .

For each side there is therefore a threshold phenomenon. On the left, if x is taken sufficiently large, the reconstructed side curve stays above the frozen second curve on the whole interval $[\ell - 1, \ell]$; if x is lowered far enough, that property fails. Because increasing x raises the reconstructed curve pointwise, the set of admissible x -values is an upper half-line. The corresponding threshold is the

left corner. Similarly, on the right, the admissible y -values form an upper half-line, with threshold given by the right corner.

Thus the side constraints may be compressed into inequalities on the two endpoint heights. That is the reason for separating the missing closed middle into endpoint data and a middle bridge.

5.6.6. The conditional law in usable form. We may now summarise the model $k = 1$ reconstruction in the form that will be used below.

Under $\mathbb{P}_{\mathcal{F}}$, the law of the missing closed middle is obtained from the Wiener candidate by conditioning on three tests: the left side test, the right side test, and the middle test. Equivalently, one first samples a Brownian bridge across $[\ell - 1, r + 1]$, then asks whether the resulting reconstructed top curve stays above the frozen second curve on the two side intervals and on the middle interval. The accepted candidate has exactly the conditional law of the missing top-curve data.

The important point is that these three constraints are not on the same footing. Once the corners are introduced, the two side tests depend only on the endpoint pair

$$(\mathcal{L}_n(1, \ell), \mathcal{L}_n(1, r)),$$

while the middle test depends on the bridge across $[\ell, r]$. Thus the model conditional law separates naturally into two stages: first a condition on the bead pair, and then, given that pair, a bridge-avoidance condition across the middle interval.

This is exactly the decomposition used later in the proof of the model acceptance estimate. The side-tested bead law supplies the base measure on bead space; the middle test contributes the acceptance probability; and the true conditional law is obtained from the former by weighting with the latter.

5.7. Corners and the geometry of side constraints

The previous section separated the missing closed middle into three pieces: the left endpoint height, the right endpoint height, and the bridge across the middle interval. It also showed that the side tests depend only on the retained side geometry together with the two endpoint heights. We now make this dependence explicit in the model case $k = 1$.

We continue to work with

$$\ell = -1, \quad r = 1.$$

Thus the left side interval is $[-2, -1]$, the right side interval is $[1, 2]$, and the retained-data sigma-field $\mathcal{F}$ contains the frozen second curve, the top curve outside $(-1, 1)$, and the two side standard bridges.

5.7.1. The corner quantities. Consider first the left side interval. For a proposed endpoint height $x \in \mathbb{R}$ at -1 , the retained left-side data determine the candidate path

$$s \mapsto \mathcal{L}_n^{[-2, -1]}(1, s) + (-1 - s)\mathcal{L}_n(1, -2) + (s + 2)x, \quad s \in [-2, -1].$$

This is exactly the left-hand piece of the reconstruction from the previous section. Some values of x produce a path that stays strictly above the frozen second curve on the whole interval $[-2, -1]$, and some do not. Since increasing x raises the candidate path pointwise, the admissible set is an upper half-line.

We define the left corner by

$$\text{Corner}^\ell = \inf \left\{ x \in \mathbb{R} : \mathcal{L}_n^{[-2, -1]}(1, s) + (-1 - s)\mathcal{L}_n(1, -2) + (s + 2)x > \mathcal{L}_n(2, s) \text{ for all } s \in [-2, -1] \right\}.$$

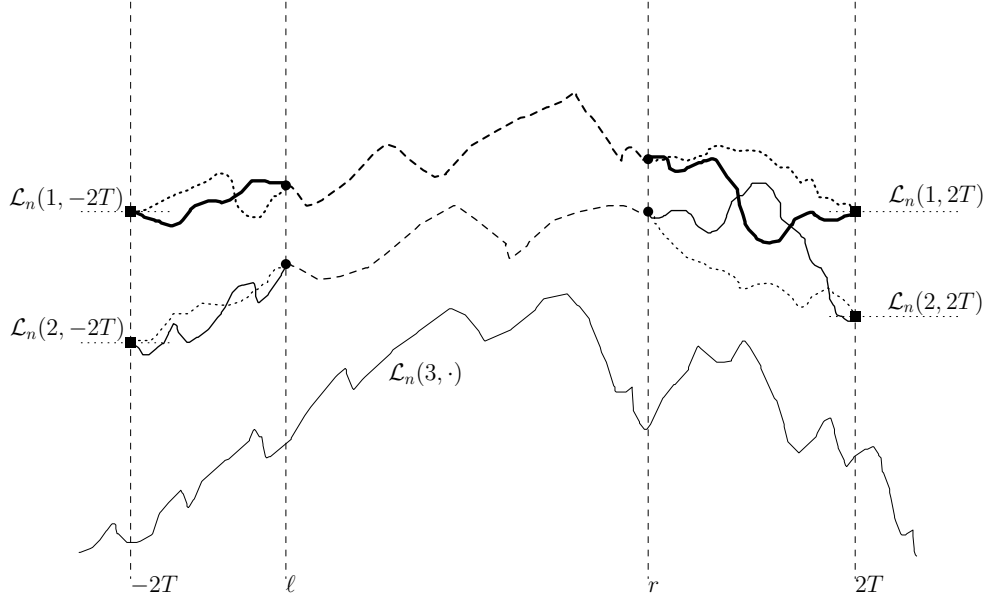


FIGURE 5.1. The retained data consist of the frozen second curve, the top curve outside the closed middle, and the two side standard bridges. The missing data are the endpoint heights at ℓ and r together with the bridge across $[\ell, r]$. A Wiener candidate proposes a Brownian bridge across $[\ell - 1, r + 1]$; the side and middle tests then ask whether the reconstructed curve stays above the frozen second curve.

Thus Corner^ℓ is the threshold height at -1 above which the left side test is passed.

Similarly, for a proposed endpoint height $y \in \mathbb{R}$ at 1 , the retained right-side data determine the candidate path

$$s \mapsto \mathcal{L}_n^{[1,2]}(1, s) + (2 - s)y + (s - 1)\mathcal{L}_n(1, 2), \quad s \in [1, 2].$$

The admissible values of y again form an upper half-line, and we define the right corner by

$$\text{Corner}^r = \inf \left\{ y \in \mathbb{R} : \mathcal{L}_n^{[1,2]}(1, s) + (2 - s)y + (s - 1)\mathcal{L}_n(1, 2) > \mathcal{L}_n(2, s) \text{ for all } s \in [1, 2] \right\}.$$

By construction, both Corner^ℓ and Corner^r are $\mathcal{F}$ -measurable.

5.7.2. The corner lemma. The usefulness of the corner quantities is that they compress the side tests to endpoint inequalities.

LEMMA 5.16 (Corner lemma). In the model case $k = 1$, let x_1 and y_1 denote the endpoint heights at -1 and 1 . Then, conditional on $\mathcal{F}$,

- (1) the left side test on $[-2, -1]$ is passed whenever

$$x_1 > \text{Corner}^\ell;$$

- (2) the right side test on $[1, 2]$ is passed whenever

$$y_1 > \text{Corner}^r.$$

Conversely, if the left side test is passed, then $x_1 \geq \text{Corner}^\ell$, and if the right side test is passed, then $y_1 \geq \text{Corner}^r$.

PROOF. We prove the left-hand statement; the right-hand one is identical.

Once the retained data $\mathcal{F}$ are fixed, the left-side candidate path depends on the unknown endpoint height x_1 only through the affine term joining the known value $\mathcal{L}_n(1, -2)$ to x_1 at -1 . If $x_1 \leq x'_1$, then the candidate path corresponding to x'_1 lies pointwise above the one corresponding to x_1 . Hence admissibility is monotone in x_1 : once the strict inequality with the frozen second curve holds at some height, it holds at every larger height.

It follows that the set of admissible x_1 -values is an upper half-line, whose infimum is Corner^ℓ . Therefore every $x_1 > \text{Corner}^\ell$ is admissible, while admissibility at x_1 implies $x_1 \geq \text{Corner}^\ell$. $\square$

This is the only point at which the distinction between strict and weak inequalities matters. Later the endpoint data will be distributed according to absolutely continuous laws, so the difference between $x_1 > \text{Corner}^\ell$ and $x_1 \geq \text{Corner}^\ell$, and likewise on the right, will not affect any probability statements.

LEMMA 5.17 (Corners are dominated by the realised endpoint heights). Almost surely,

$$\text{Corner}^\ell \leq \mathcal{L}_n(1, -1) \quad \text{and} \quad \text{Corner}^r \leq \mathcal{L}_n(1, 1).$$

Consequently, there exist positive constants c and C , independent of n , such that

$$\mathbb{P}_n(\text{Corner}^\ell > t) + \mathbb{P}_n(\text{Corner}^r > t) \leq Ce^{-ct^{3/2}} \quad \text{for all } t \geq 1.$$

PROOF. By definition, Corner^ℓ is the infimum of the values x for which the left-side candidate path stays strictly above the frozen second curve on $[-2, -1]$. The realised top-curve segment $\mathcal{L}_n(1, \cdot)|_{[-2, -1]}$ itself is admissible, because the ensemble is strictly ordered. Its endpoint height at -1 is $\mathcal{L}_n(1, -1)$. Therefore

$$\text{Corner}^\ell \leq \mathcal{L}_n(1, -1).$$

The same argument gives

$$\text{Corner}^r \leq \mathcal{L}_n(1, 1).$$

The tail bound then follows from the one-point upper-tail estimates for the top curve at the spatial points -1 and 1 . $\square$

5.7.3. What the corners achieve. Before corners are introduced, the side constraints are pathwise conditions on the two side intervals: the reconstructed top curve must stay above the frozen second curve on $[-2, -1]$ and on $[1, 2]$. After corners are introduced, these same conditions are encoded by threshold inequalities on the two endpoint heights:

$$x_1 > \text{Corner}^\ell \quad \text{and} \quad y_1 > \text{Corner}^r,$$

up to the harmless issue of equality just noted.

This is the form needed later. The missing closed middle consists of two endpoint heights together with the bridge across $[-1, 1]$. The side geometry has now been packaged into the random thresholds Corner^ℓ and Corner^r , while the remaining pathwise condition is the middle test on $[-1, 1]$.

So the reconstruction problem has been reorganised into two parts:

- a condition on the endpoint pair (x_1, y_1) , namely that it lie above the two corner thresholds;
- a bridge-avoidance condition across the middle interval.

That separation is what later makes the acceptance analysis tractable.

5.8. A uniform lower bound on acceptance probability

We now prove Proposition 5.9, the model case $(k, I) = (1, [-1, 1])$ of the compact-window acceptance principle.

The argument has a simple structure. First, on a favourable retained-data event, the side-tested bead law assigns definite mass to a region of bead space lying well above the corners. For such bead pairs, the middle test is easy: the candidate bridge starts and ends high above the frozen second curve. Second, the true conditional law is obtained from the side-tested bead law by weighting with the middle acceptance probability. This is exactly the size-bias mechanism recorded earlier, and the dart-fall estimate converts the positive mass of the favourable bead region into a high-probability lower bound on the actual acceptance probability.

PROOF OF PROPOSITION 5.9. Fix $\epsilon \in (0, 1)$. We begin by defining a favourable event on the retained data. Let

$$F_t = \left\{ \mathcal{L}_n(1, -2) \geq -t, \mathcal{L}_n(1, 2) \geq -t, \text{Corner}^\ell \leq t, \text{Corner}^r \leq t, \sup_{x \in [-1, 1]} \mathcal{L}_n(2, x) \leq t \right\}.$$

By the one-point lower-tail bounds at -2 and 2 , by Lemma 5.17, and by Proposition 5.12, there exist positive constants c_1 and C_1 , independent of n , such that

$$\mathbb{P}_n(F_t^c) \leq C_1 e^{-c_1 t^{3/2}}.$$

Fix a realisation Y of the retained data in the event F_t . All probability measures below are conditional on this Y .

Let ν_Y be the law of the bead pair (x_1, y_1) under the Wiener candidate. Let

$$S_Y = \{x_1 \geq \text{Corner}^\ell, y_1 \geq \text{Corner}^r\}$$

be the side-test event. By the corner lemma, this is exactly the event that the two side constraints are satisfied. Let

$$\nu_Y^* = \nu_Y(\cdot | S_Y)$$

be the side-tested law of the bead pair.

For a side-admissible pair (x, y) , let $A_Y(x, y)$ denote the conditional probability that the middle Brownian bridge on $[-1, 1]$ from x to y stays above the frozen second curve. Thus A_n is the random variable $A_Y(x_1, y_1)$ when the bead pair is sampled from the true conditional law of the ensemble.

We now identify a region of bead space on which A_Y is uniformly positive. Define

$$G_t = [2t, 3t] \times [2t, 3t].$$

Since $Y \in F_t$, we have $\text{Corner}^\ell \leq t$ and $\text{Corner}^r \leq t$, so $G_t \subseteq S_Y$. Hence

$$\nu_Y^*(G_t) = \frac{\nu_Y(G_t)}{\nu_Y(S_Y)} \geq \nu_Y(G_t).$$

Under ν_Y , the bead pair is the two-point marginal of a Brownian bridge on $[-2, 2]$ whose endpoint values at -2 and 2 are at least $-t$. Its covariance matrix is fixed, and its mean vector is bounded below coordinatewise by $(-t, -t)$ on F_t . Standard Gaussian estimates therefore give a constant $C_2 > 0$, independent of n , Y , and t , such that

$$\nu_Y^*(G_t) \geq \nu_Y(G_t) \geq e^{-C_2 t^2}. \quad (19)$$

Now consider $(x, y) \in G_t$. Then both endpoints of the middle bridge lie in $[2t, 3t]$, while on F_t the frozen second curve on $[-1, 1]$ stays below t . Thus the middle test is certainly passed whenever the standard bridge part stays above $-t/2$. For $t \geq 2$, the latter event has probability bounded below by a universal positive constant. Hence there exists $a_0 > 0$, independent of n , Y , and t , such that

$$A_Y(x, y) \geq a_0 \quad \text{for all } (x, y) \in G_t. \quad (20)$$

Let ν'_Y denote the conditional law of the bead pair under the actual ensemble, given the retained data Y . This is obtained from the side-tested law ν_Y^* by conditioning on the middle test. Equivalently, by the size-bias identity from the tools section,

$$\frac{d\nu'_Y}{d\nu_Y^*}(x, y) = \frac{A_Y(x, y)}{Z_Y}, \quad Z_Y = \int A_Y d\nu_Y^*. \quad (21)$$

The normalising constant Z_Y is the average middle acceptance probability under the side-tested law. By (19) and (20),

$$Z_Y \geq \int_{G_t} A_Y d\nu_Y^* \geq a_0 \nu_Y^*(G_t) \geq a_0 e^{-C_2 t^2}. \quad (22)$$

We now invoke the abstract dart-fall estimate from the tools section. Apply Corollary 5.15 with base space the bead-pair space, base measure ν_Y^* , height function $(x, y) \mapsto A_Y(x, y)$, and favourable set G_t . The hypotheses hold with

$$a = a_0, \quad \eta = e^{-C_2 t^2},$$

in view of (19) and (20). Therefore, for any $\rho \in (0, 1]$,

$$\nu'_Y(A_Y \leq \rho a_0 e^{-C_2 t^2}) \leq \rho. \quad (23)$$

Now choose $\rho = \epsilon/2$, and define

$$\delta_t = \frac{\epsilon a_0}{2} e^{-C_2 t^2}.$$

Then, for every retained-data realisation $Y \in F_t$,

$$\nu'_Y(A_Y \leq \delta_t) \leq \epsilon/2. \quad (24)$$

Since ν'_Y is the conditional law of the bead pair under the true ensemble given Y , we may now average (24) over the retained data:

$$\mathbb{P}_n(A_n \leq \delta_t) \leq \mathbb{P}_n(F_t^c) + \mathbb{P}_n(A_n \leq \delta_t; F_t) \leq \mathbb{P}_n(F_t^c) + \epsilon/2.$$

Finally choose $t = t(\epsilon)$ large enough that $C_1 e^{-c_1 t^{3/2}} \leq \epsilon/2$. Then $\mathbb{P}_n(A_n \leq \delta_t) \leq \epsilon$, and therefore

$$\mathbb{P}_n(A_n \geq \delta_t) \geq 1 - \epsilon.$$

This proves Proposition 5.9. □

The model mechanism is now complete. The missing closed middle has been decomposed into the bead pair and the middle bridge, the side constraints have been compressed into the corner inequalities, and the actual conditional law has been expressed as a size-biased version of the side-tested bead law. The rest of the argument is devoted to extending this mechanism: first to the top k curves in the same geometry, and then to general compact intervals.

5.8.1. Bounds on the extreme corners. Before turning to the k -curve version of the model acceptance argument, we record the only quantitative corner information that will later be needed in the definition of the favourable event F_t . The point is that the top corner is controlled from above by the realised top curve, while the bottom corner is controlled from below by the realised $(k+1)^{\text{st}}$ curve. Since these neighbouring curves already satisfy one-point bounds, the same is true for the extreme corners.

We continue in the model geometry $[-2, -1] \cup [-1, 1] \cup [1, 2]$. For fixed $k \in \mathbb{N}$, let

$$\text{Corner}_1^\ell > \cdots > \text{Corner}_k^\ell \quad \text{and} \quad \text{Corner}_1^r > \cdots > \text{Corner}_k^r$$

denote the left and right corner vectors.

LEMMA 5.18 (Extreme corners are controlled by neighbouring curves). Almost surely,

$$\text{Corner}_1^\ell \leq \mathcal{L}_n(1, -1), \quad \text{Corner}_1^r \leq \mathcal{L}_n(1, 1), \quad (25)$$

and

$$\text{Corner}_k^\ell \geq \mathcal{L}_n(k+1, -1), \quad \text{Corner}_k^r \geq \mathcal{L}_n(k+1, 1). \quad (26)$$

PROOF. We begin with the top corners. The realised side configuration of the top k curves is admissible. In particular, the realised endpoint value $\mathcal{L}_n(1, -1)$ is an admissible choice for the top left bead, and $\mathcal{L}_n(1, 1)$ is an admissible choice for the top right bead. Since Corner_1^ℓ and Corner_1^r are defined as the infima of the corresponding admissible sets, this gives (25).

For the bottom corners, consider the left side. Any admissible choice of the k^{th} left bead must yield a side bridge that stays strictly above the frozen $(k+1)^{\text{st}}$ curve on $[-2, -1]$. Evaluating at the right endpoint of that interval, one sees that the bead height itself must exceed $\mathcal{L}_n(k+1, -1)$. Since Corner_k^ℓ is the infimum of the admissible k^{th} left bead heights, it follows that

$$\text{Corner}_k^\ell \geq \mathcal{L}_n(k+1, -1).$$

The same argument on the right gives

$$\text{Corner}_k^r \geq \mathcal{L}_n(k+1, 1).$$

This proves (26). □

The preceding lemma turns the needed corner bounds into one-point bounds for curves 1 and $k+1$. Since the regularity hypothesis gives one-point control for the top curve, and Proposition 5.13 gives the same for each fixed lower curve, we obtain the following.

PROPOSITION 5.19 (Tail bounds for the extreme corners). Fix $k \in \mathbb{N}$. Then there exist positive constants c_k and C_k , depending only on k and on the regularity parameters, such that for all $t \geq 1$,

$$\mathbb{P}(\text{Corner}_1^\ell > t) + \mathbb{P}(\text{Corner}_1^r > t) \leq C_k e^{-c_k t^{3/2}}, \quad (27)$$

$$\mathbb{P}(\text{Corner}_k^\ell < -t) + \mathbb{P}(\text{Corner}_k^r < -t) \leq C_k e^{-c_k t^{3/2}}. \quad (28)$$

PROOF. By Lemma 5.18,

$$\mathbb{P}(\text{Corner}_1^\ell > t) \leq \mathbb{P}(\mathcal{L}_n(1, -1) > t), \quad \mathbb{P}(\text{Corner}_1^r > t) \leq \mathbb{P}(\mathcal{L}_n(1, 1) > t).$$

Since the spatial points ± 1 lie in the regularity window for all large n , the one-point upper-tail estimate for curve 1 yields (27), after absorbing the fixed parabolic correction at ± 1 into the constants.

Likewise,

$$\mathbb{P}(\text{Corner}_k^\ell < -t) \leq \mathbb{P}(\mathcal{L}_n(k+1, -1) < -t), \quad \mathbb{P}(\text{Corner}_k^r < -t) \leq \mathbb{P}(\mathcal{L}_n(k+1, 1) < -t),$$

and Proposition 5.13, applied to curve $k+1$, gives (28), again after absorbing the fixed parabolic correction at ± 1 into the constants. $\square$

In particular, with probability at least $1 - C_k e^{-c_k t^{3/2}}$, the top left and right corners lie below t , while the bottom left and right corners lie above $-t$. This is the form of control that will enter the favourable event F_t in the next subsection.

5.8.2. How the model proof extends to the general k -curve setting. We continue to work in the model geometry

$$[-2, -1] \cup [-1, 1] \cup [1, 2],$$

but now with the top k curves rather than only the top curve. The purpose of this subsection is to explain how the proof of the model acceptance estimate extends to this fixed- k setting. The underlying mechanism is the same as in the case $k = 1$: one conditions on retained data, studies the Wiener candidate under the conditional law $\mathbb{P}_{\mathcal{F}}$, identifies a favourable event F_t on the retained data, and then, on F_t , finds a region of bead space on which the middle acceptance probability is uniformly positive.

What changes is the geometry. There are now k side bridges on the left and k on the right. The side constraints are encoded not by two corners but by two corner vectors. The middle bridge is now a k -curve Brownian bridge ensemble, and the favourable bead region must respect the Weyl-chamber ordering.

Retained data and missing data. The retained data are the natural k -curve analogue of the $k = 1$ case: the frozen $(k+1)^{\text{st}}$ curve, the top k curves outside $(-1, 1)$, and the side standard bridges of those k curves on the side intervals $[-2, -1]$ and $[1, 2]$. The missing data are: the left bead vector

$$\bar{x} = (x_1, \dots, x_k),$$

the right bead vector

$$\bar{y} = (y_1, \dots, y_k),$$

and the k -curve standard bridge ensemble on $[-1, 1]$. The bead vectors must satisfy the Weyl-chamber inequalities

$$x_1 > \dots > x_k, \quad y_1 > \dots > y_k.$$

Given the retained data Y , let ν_Y denote the law of the bead pair $(\bar{x}, \bar{y})$ under the raw Wiener candidate, and let ν_Y^* denote the law after the side tests are imposed. The actual conditional law of the bead pair is then obtained from ν_Y^* by weighting with the middle acceptance probability, just as before.

Corners and the side-admissible region. On the left, one may lower the k th side bridge until it just ceases to stay above the frozen $(k+1)$ st curve. This determines the lowest admissible bead height for the k th curve. Having fixed that, one lowers the $(k-1)$ st side bridge until it just ceases to stay above the k th, and so on. In this way one obtains an ordered left corner vector

$$\text{Corner}_1^\ell > \dots > \text{Corner}_k^\ell.$$

The same construction on the right yields

$$\text{Corner}_1^r > \dots > \text{Corner}_k^r.$$

These corner vectors describe the lower edge of the side-admissible region. In particular, the set of admissible left bead vectors is an upper set in the Weyl chamber: if a vector is admissible, then any coordinatewise larger vector that still lies in the Weyl chamber is also admissible. Likewise on the right. Thus the role of the corners is the same as before, but now in k dimensions.

The favourable event F_t . As in the $k = 1$ argument, the proof begins by introducing a favourable event F_t , depending on a large positive parameter t . This event is $\mathcal{F}$ -measurable: it is visible to the witness, since it is determined by the retained data alone. Its purpose is to guarantee that the Wiener candidate has room to behave well.

For fixed k , let F_t be the event that the following three conditions hold.

First, the outer endpoint data of the top k curves at -2 and 2 are controlled:

$$\mathcal{L}_n(1, -2) \leq t, \quad \mathcal{L}_n(1, 2) \leq t,$$

and

$$\mathcal{L}_n(k, -2) \geq -t, \quad \mathcal{L}_n(k, 2) \geq -t.$$

Since the curves are ordered, this places all intermediate outer endpoint values in the strip $[-t, t]$.

Second, the extreme corners are controlled:

$$\text{Corner}_1^\ell \leq t, \quad \text{Corner}_1^r \leq t,$$

and

$$\text{Corner}_k^\ell \geq -t, \quad \text{Corner}_k^r \geq -t.$$

Third, the frozen lower boundary on the middle interval is not too high:

$$\sup_{u \in [-1, 1]} \mathcal{L}_n(k+1, u) \leq t.$$

On F_t , the side-admissible region begins at controlled height, the top and bottom of the bead stack are controlled at the outer points, and the frozen $(k+1)$ st curve on $[-1, 1]$ lies low enough that the top k curves may be sent through a stack of disjoint corridors.

The probability of F_t^c is small uniformly in n . Indeed, the outer-point bounds follow from the one-point estimates for curves 1 and k , the corner bounds follow from Proposition 5.19, and the bound on the $(k+1)$ st curve on $[-1, 1]$ follows from the corresponding No Big Max estimate. Thus, for fixed k , there exist positive constants c_k and C_k such that

$$\mathbb{P}_n(F_t^c) \leq C_k e^{-c_k t^{3/2}}.$$

The favourable region G_t in bead space. Assume now that the retained data Y lie in F_t . We define a region $G_t \subset \mathbb{R}^k \times \mathbb{R}^k$ of bead pairs for which the side tests are certainly passed and the middle test has a definite chance to be passed.

Choose k pairwise disjoint corridors, stacked one above another, each of vertical width t . For definiteness, let

$$J_i = [(3i-1)t, 3it], \quad i \in \llbracket 1, k \rrbracket,$$

and let J_i^{mid} denote the middle third of J_i . The full stack therefore reaches height of order kt , not kt^2 , in the model interval $[-1, 1]$. The kt^2 -scale issue belongs to the later extension to a longer interval, where one must also clear the parabolic drop.

We then let G_t be the set of bead pairs $(\bar{x}, \bar{y})$ such that, for each i ,

$$x_i \in J_i^{\text{mid}} \quad \text{and} \quad y_i \in J_i^{\text{mid}}.$$

Thus the i th curve is assigned the i th corridor, both at the left bead and at the right bead.

Because $Y \in F_t$, the top left and right corners lie below t , while the bottom left and right corners lie above $-t$. In particular, the whole stacked region G_t lies strictly above the side-admissible threshold. Since the side-admissible sets are upper sets in the Weyl chamber, it follows that every bead pair in G_t passes the side tests.

There is an important geometric point here. The curves are not being moved independently. Raising the bead of the k th curve on the left may force the beads above it to be raised as well, in order to preserve strict ordering and side admissibility. The stacked-corridor choice respects this. It places the k th curve in the lowest lane, the $(k-1)$ st in the next lane, and so on, so that the Weyl-chamber constraints and the side constraints are simultaneously satisfied.

Side-tested mass of G_t . Under the raw law ν_Y , the $2k$ beads are jointly Gaussian. Their covariance matrix depends only on the Brownian bridge geometry. On the event F_t , the mean vector lies in a region of size $O(t)$, because the outer endpoint data are all in $[-t, t]$. Since k is fixed and G_t has positive Lebesgue volume at distance only $O(kt)$ from that mean vector, standard Gaussian estimates give

$$\nu_Y^*(G_t) \geq \nu_Y(G_t) \geq e^{-C_k t^2}$$

for some constant C_k .

Uniform positivity of middle acceptance on G_t . Now take $(\bar{x}, \bar{y}) \in G_t$. For each i , let L_i be the affine interpolant on $[-1, 1]$ joining x_i to y_i . Since both endpoints lie in the same corridor J_i^{mid} , the graph of L_i stays within the full corridor J_i for the whole interval. Because the corridors are stacked and disjoint, the affine interpolants

$$L_1, \dots, L_k$$

remain strictly ordered, with gaps of order t between neighbouring curves. Moreover, the lowest interpolant L_k lies at height of order t , while on F_t the frozen $(k+1)$ st curve stays below t . Thus there is a definite margin between the bottom corridor and the lower boundary.

If each standard Brownian bridge fluctuation stays within a sup-norm tube of radius $t/10$ about zero, then the k reconstructed curves remain in their respective corridors, stay strictly ordered, and remain above the frozen lower boundary. Since k is fixed, the probability of this event for k independent standard Brownian bridges is bounded below by a positive constant a_k . Hence

$$A_Y(\bar{x}, \bar{y}) \geq a_k \quad \text{for all } (\bar{x}, \bar{y}) \in G_t.$$

Conclusion. At this point the rest of the proof is unchanged. The normalising constant

$$Z_Y = \int A_Y d\nu_Y^*$$

is bounded below on F_t , because A_Y is bounded below on G_t and $\nu_Y^*(G_t)$ is at least $e^{-C_k t^2}$. The true conditional bead law is obtained from ν_Y^* by size-biasing with A_Y . The dart-fall estimate then implies that very small middle acceptance probabilities are rare under the true conditional law. Averaging over Y proves the compact-window acceptance estimate for the top k curves in the model geometry

$$[-2, -1] \cup [-1, 1] \cup [1, 2].$$

5.8.3. From $[-1, 1]$ to a general compact interval. We now explain what changes when the middle interval is a general compact interval $I = [\ell, r]$. This is the second stage in the extension from the model case to Proposition 5.8. The extra structure is simple and concrete. Over a longer interval, the parabolic curvature creates a deterministic drop of order T^2 , and the corridor stack must be placed high enough to clear that drop.

By translating and rescaling space, it is enough to treat the symmetric configuration

$$[-T-1, -T] \cup [-T, T] \cup [T, T+1]$$

for some fixed $T > 0$. The side intervals still have length one; only the middle interval has changed. The retained-data and missing-data decomposition is exactly as before. Conditioned on the retained data Y , one first imposes the side tests and obtains a side-tested law ν_Y^* on the entrance and exit vectors for the top k curves. One then imposes the middle test across $[-T, T]$, whose conditional probability is the acceptance probability. Thus the true conditional law is again obtained from ν_Y^* by weighting with the middle acceptance probability.

The new point is the following elementary geometric fact. Let

$$\mathbf{p}(x) = 2^{-1/2}x^2.$$

Fix $a, b \in \mathbb{R}$, and let L be the affine function on $[-T, T]$ joining $(-T, a)$ to (T, b) . Then, for every $x \in [-T, T]$,

$$L(x) - \mathbf{p}(x) \geq \min\{a - \mathbf{p}(-T), b - \mathbf{p}(T)\} - 2^{-1/2}T^2.$$

Indeed, the secant line to the parabola between $-T$ and T is constant, equal to $2^{-1/2}T^2$, while the parabola itself falls to 0 at the origin. Thus the largest possible drop of the parabola below its endpoint level across $[-T, T]$ is $2^{-1/2}T^2$. In particular, if the endpoint heights are placed $2^{-1/2}T^2 + C$ above the local parabolic baseline, then the whole affine interpolant remains at least C above that baseline throughout the interval.

This is the only genuinely new feature compared with the model interval $[-1, 1]$. In the model case the parabolic drop is an $O(1)$ quantity and is absorbed into the constants. Over $[-T, T]$, it must be tracked explicitly.

With this in mind, the favourable retained-data event is defined exactly as before, but relative to the local parabolic level at $\pm T$. Fix $t > 0$. Let $F_{t,T}$ be the event that the following hold.

First, the outer endpoint data of the top k curves at $-T-1$ and $T+1$ are controlled relative to the local parabolic scale:

$$\mathcal{L}_n(1, -T-1) + \mathbf{p}(T+1) \leq t, \quad \mathcal{L}_n(1, T+1) + \mathbf{p}(T+1) \leq t,$$

and

$$\mathcal{L}_n(k, -T-1) + \mathbf{p}(T+1) \geq -t, \quad \mathcal{L}_n(k, T+1) + \mathbf{p}(T+1) \geq -t.$$

Since the curves are ordered, all intermediate outer endpoint values then also lie in the strip $[-t, t]$ after parabolic correction.

Second, the extreme corners at $-T$ and T are controlled:

$$\text{Corner}_1^\ell \leq \mathbf{p}(T) + t, \quad \text{Corner}_1^r \leq \mathbf{p}(T) + t,$$

and

$$\text{Corner}_k^\ell \geq \mathbf{p}(T) - t, \quad \text{Corner}_k^r \geq \mathbf{p}(T) - t.$$

Third, the frozen $(k+1)^{\text{st}}$ curve on $[-T, T]$ lies below the level

$$\mathfrak{p}(T) + t.$$

On $F_{t,T}$, the side-admissible region begins at controlled height, the top and bottom of the bead stack at the outer points are controlled, and the frozen $(k+1)^{\text{st}}$ curve on $[-T, T]$ lies low enough that the top k curves may be sent through a stack of disjoint corridors.

The probability of $F_{t,T}^c$ is small uniformly in n . Indeed, the outer-point bounds follow from the one-point estimates for curves 1 and k , the corner bounds follow from Proposition 5.19, and the bound on the $(k+1)^{\text{st}}$ curve on $[-T, T]$ follows from the compact No Big Max estimate applied to curve $k+1$. Thus, for fixed k and fixed T , there exist positive constants $c_{k,T}$ and $C_{k,T}$ such that

$$\mathbb{P}_n(F_{t,T}^c) \leq C_{k,T} e^{-c_{k,T} t^{3/2}}.$$

Assume now that the retained data Y lie in $F_{t,T}$. We define the favourable region $G_{t,T} \subset \mathbb{R}^k \times \mathbb{R}^k$ by placing the bead coordinates in a stack of k disjoint corridors, exactly as in the model case, but shifted upward by the amount needed to clear the parabolic drop. More precisely, let

$$H_T = 2^{-1/2} T^2,$$

and for $i \in \llbracket 1, k \rrbracket$ set

$$J_{i,T} = [H_T + (3i-1)t, H_T + 3it].$$

Let $J_{i,T}^{\text{mid}}$ denote the middle third of $J_{i,T}$. Then $G_{t,T}$ is the set of bead pairs $(\bar{x}, \bar{y})$ such that, for every i ,

$$x_i \in J_{i,T}^{\text{mid}} \quad \text{and} \quad y_i \in J_{i,T}^{\text{mid}}.$$

Thus the i th curve is assigned the i th corridor, and the whole corridor stack has been lifted by the additive amount H_T . Since $Y \in F_{t,T}$, the top corners lie at most t above the local parabolic level $\mathfrak{p}(T)$, while the bottom corners lie at most t below it. Thus the stacked region $G_{t,T}$, obtained by lifting the model corridors by the amount $H_T = \mathfrak{p}(T)$, lies strictly above the side-admissible threshold. Therefore every bead pair in $G_{t,T}$ passes the side tests.

Now fix $(\bar{x}, \bar{y}) \in G_{t,T}$, and let L_i denote the affine interpolant on $[-T, T]$ joining x_i to y_i . Because both endpoints lie in the same corridor $J_{i,T}^{\text{mid}}$, the preceding parabolic estimate shows that L_i remains inside the full corridor $J_{i,T}$ throughout $[-T, T]$. Since the corridors are stacked and disjoint, the interpolants

$$L_1, \dots, L_k$$

remain strictly ordered, with gaps of order t between neighbouring curves. Moreover, the lowest interpolant L_k remains a distance of order t above the frozen $(k+1)^{\text{st}}$ curve, again by the definition of $F_{t,T}$.

Consequently, if each of the k standard Brownian bridge fluctuations stays within a fixed sup-norm tube of radius $t/10$ about zero, then the reconstructed middle curves stay in their respective corridors, remain strictly ordered, and stay above the frozen $(k+1)^{\text{st}}$ curve. This event has probability bounded below by a positive constant $a_{k,T}$ depending only on k and T . Hence, for all $(\bar{x}, \bar{y}) \in G_{t,T}$,

$$A_Y(\bar{x}, \bar{y}) \geq a_{k,T}.$$

It remains to control the side-tested mass of $G_{t,T}$. Under the raw law ν_Y , the $2k$ beads are jointly Gaussian. On the event $F_{t,T}$, the mean vector lies in a region whose coordinates are each within

$O_T(t)$ of the local parabolic baseline. Since $G_{t,T}$ is centred at height $H_T + O_k(t)$, the Gaussian cost of placing all $2k$ beads in these prescribed corridors is

$$\nu_Y^*(G_{t,T}) \geq \nu_Y(G_{t,T}) \geq \exp\{-C_{k,T}(t + T^2)^2\}.$$

Since T is fixed once the interval I is fixed, this is of the same form as before, namely $\exp\{-O_{k,T}(t^2)\}$.

At this point the rest of the argument is unchanged. The normalising constant

$$Z_Y = \int A_Y d\nu_Y^*$$

is bounded below on $F_{t,T}$, because A_Y is bounded below on $G_{t,T}$ and $\nu_Y^*(G_{t,T})$ has positive lower bound there. The true conditional bead law is obtained from ν_Y^* by size-biasing with A_Y . The same dart-fall estimate then shows that very small middle acceptance probabilities are rare under the true conditional law. Averaging over the retained data proves Proposition 5.8.

The model argument already contains the main mechanism. The general compact interval differs only in that the corridor stack must be lifted by the parabolic-drop scale $2^{-1/2}T^2$, and that one needs one-point control for the top $k + 1$ curves rather than only for the top two.

5.9. The identifying finite-dimensional convergence input

Before turning to the construction of the Airy line ensemble, it is useful to isolate the extra input on which that construction rests. The Brownian Gibbs machinery developed so far gives tightness and later allows one to pass the Brownian Gibbs property to subsequential limits. It does not by itself identify those limits. For that, one needs an independent convergence theorem from integrable probability: the edge-scaled non-intersecting Brownian bridge ensemble has finite-dimensional distributions converging to those of the multi-line Airy process.

The aim of the present section is not to prove that theorem in full detail. Rather, it is to give a detailed proof guide to why it is true, and to explain the chain of ideas on which it rests:

$$\begin{aligned} \text{Karlin–McGregor} &\longrightarrow \text{space-time determinantal structure} \longrightarrow \text{extended kernel asymptotics} \\ &\longrightarrow \text{finite-dimensional convergence.} \end{aligned}$$

The one-time Karlin–McGregor computation is an instructive warm-up, but it is not by itself enough for the needed multi-time statement. The real input is the extended determinantal kernel. Before turning to the Brownian bridge ensemble itself, we therefore begin by reviewing the general concepts that make the later argument intelligible.

5.9.1. General principles: Karlin–McGregor, determinantal point processes, and orthogonal polynomials. Several ideas organize the argument to come. The first is the Karlin–McGregor formula for non-intersecting Markov paths. The second is the notion of a determinantal point process. The third is the role of orthogonal polynomials, and in the present Brownian setting, of the Hermite functions. These three ideas are closely related, and it is worth setting them beside one another before applying them to the Brownian bridge ensemble.

A determinantal point process on a space S with reference measure μ is a random locally finite subset $\Xi \subseteq S$ whose correlation functions are given by determinants of a kernel. Concretely, this means that there exists a function

$$K : S \times S \rightarrow \mathbb{R}$$

such that, for each $m \geq 1$, the m -point correlation density of Ξ takes the form

$$\rho_m(x_1, \dots, x_m) = \det[K(x_i, x_j)]_{i,j=1}^m.$$

This definition becomes useful once one thinks of the kernel as the kernel of an integral operator. Many probabilistic questions about the point process can then be reformulated as analytic questions about that operator. In particular, convergence of determinantal point processes is often proved by showing that the associated kernels converge in a strong operator sense.

A basic source of such kernels comes from orthogonal polynomials. Suppose that $\mu(dx) = w(x) dx$ is a measure on $\mathbb{R}$, and that

$$p_0, p_1, p_2, \dots$$

are orthonormal polynomials in $L^2(\mu)$. This means that

$$\int_{\mathbb{R}} p_r(x) p_s(x) w(x) dx = \delta_{rs}.$$

From these functions one forms the Christoffel–Darboux kernel

$$K_N(x, y) = \sum_{r=0}^{N-1} p_r(x) p_r(y).$$

Why is this kernel natural? If $f \in L^2(\mu)$, then

$$\int_{\mathbb{R}} K_N(x, y) f(y) \mu(dy) = \sum_{r=0}^{N-1} p_r(x) \int_{\mathbb{R}} p_r(y) f(y) \mu(dy),$$

which is simply the orthogonal expansion of the projection of f onto the span of the first N polynomials. Thus K_N is the integral kernel of the orthogonal projection onto this N -dimensional subspace.

This kernel governs a natural N -particle ensemble. Consider the density on ordered N -tuples

$$x_1 < \dots < x_N$$

given by

$$\frac{1}{Z_N} \Delta_N(x)^2 \prod_{i=1}^N w(x_i),$$

where

$$\Delta_N(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$$

is the Vandermonde determinant. The standard determinant identity

$$\Delta_N(x) = \det(x_i^{j-1})_{1 \leq i, j \leq N}$$

shows that, after orthonormalising the monomials against the weight w , this density may be rewritten as

$$\frac{1}{N!} \det[K_N(x_i, x_j)]_{i, j=1}^N.$$

So the corresponding point process is determinantal with kernel K_N . This is the basic orthogonal-polynomial mechanism behind many exactly solvable particle systems.

In the present setting, the relevant orthogonal polynomials are the Hermite polynomials. Their associated orthonormal functions

$$\phi_r(x) = c_r H_r(x) e^{-x^2/2}, \quad r \geq 0,$$

form an orthonormal basis of $L^2(\mathbb{R})$. These functions are important for two related reasons. First, they are what one obtains by orthonormalising polynomials against the Gaussian weight, so they

describe the one-time Hermite ensemble. Second, they diagonalise the harmonic oscillator. This matters for Brownian motion because, after the natural Gaussian conjugation of Brownian bridge transition kernels, the relevant semigroup is governed by that operator. Thus the Hermite functions connect the one-time orthogonal-polynomial picture to the many-time Brownian bridge picture.

The relevance to non-intersecting Brownian paths begins with the Karlin–McGregor formula. Let

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

denote the one-dimensional Brownian transition density. If

$$u_1 < \cdots < u_n, \quad v_1 < \cdots < v_n,$$

then the transition density from $u = (u_1, \dots, u_n)$ to $v = (v_1, \dots, v_n)$ for n independent Brownian motions killed on collision is

$$\det[p_t(u_i, v_j)]_{i,j=1}^n.$$

Equivalently, this determinant gives the density for n Brownian paths to go from u to v in time t without intersecting.

For $n = 2$, the formula has a very concrete interpretation. Consider two independent Brownian motions started from $u_1 < u_2$. The density that they end at $v_1 < v_2$ after time t is

$$p_t(u_1, v_1)p_t(u_2, v_2).$$

Among these pairs of paths, some never intersect and some do. If two paths intersect, then after the first meeting time one may swap their remaining portions. This gives a measure-preserving bijection between the set of path pairs from (u_1, u_2) to (v_1, v_2) that intersect and the set of unrestricted path pairs from (u_1, u_2) to (v_2, v_1) . Thus the density of non-intersecting pairs is

$$p_t(u_1, v_1)p_t(u_2, v_2) - p_t(u_1, v_2)p_t(u_2, v_1),$$

which is precisely the 2×2 determinant

$$\det \begin{pmatrix} p_t(u_1, v_1) & p_t(u_1, v_2) \\ p_t(u_2, v_1) & p_t(u_2, v_2) \end{pmatrix}.$$

For general n , one can view the determinant as an inclusion–exclusion version of the same reflection idea. If one expands the determinant as

$$\det[p_t(u_i, v_j)]_{i,j=1}^n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n p_t(u_i, v_{\sigma(i)}),$$

then each term corresponds to n independent Brownian paths from the ordered starting points u_i to the endpoint assignment $v_{\sigma(i)}$. The contributions of path families with intersections cancel in pairs under a sign-reversing involution based on the first collision. The only surviving families are the non-intersecting ones, and these must end with the same ordering with which they began. This is the content of Karlin–McGregor.

There is already a lesson here. Karlin–McGregor expresses the law of the non-intersecting system by determinants of one-particle kernels. Orthogonal polynomials, in turn, often convert such determinant formulas into projection kernels. In one-time questions this leads directly to determinantal point processes of orthogonal-polynomial type. In the Brownian bridge setting, this is exactly what happens: the one-time law becomes a Hermite ensemble.

This formula is already enough to analyse one-time marginals of non-intersecting Brownian bridge ensembles. What it does not immediately provide is a clean description of the joint law at several times. One can certainly write such a law by iterating the formula and using the Markov property, but the result is a product of determinants whose asymptotic analysis is cumbersome. Moreover, our eventual interest is not in all N particles at finitely many times, but rather in the edge and the top few curves. In principle one could integrate out the lower $N - k$ particles at each time, but such a repeated marginalisation obscures the structure and is poorly suited to asymptotics. The right next step is therefore not brute-force integration, but a structural reorganization: the many-time law may be rewritten as a determinantal point process on space-time. That is the bridge from Karlin–McGregor to the extended kernel.

5.9.2. Brownian bridge ensembles, the ϵ -parameter, and the one-time asymptotics.

We now turn to the finite- N ensemble whose edge-scaled finite-dimensional limits will later identify the Airy line ensemble.

Fix $N \in \mathbb{N}$. For $\epsilon > 0$, let

$$a^{(N,\epsilon)} = \epsilon(N-1, N-2, \dots, 0) \in \mathbb{R}^N.$$

For each $\epsilon > 0$, consider the law of N Brownian bridges on the time interval $[0, 2]$ from $a^{(N,\epsilon)}$ to $a^{(N,\epsilon)}$, conditioned on mutual non-intersection. This is a perfectly ordinary non-intersecting Brownian bridge ensemble: both endpoint vectors are strictly ordered, so the law is absolutely continuous with respect to a product Brownian bridge measure conditioned on a non-collision event.

The parameter ϵ plays an important regularising role. As long as $\epsilon > 0$, the starting and ending points are distinct, and the ensemble is defined in the usual way. The object of real interest, however, is the singular limit in which the endpoint vectors collapse. It is in this sense that the term *Brownian watermelon* is most naturally used: one lets

$$\epsilon \downarrow 0$$

and obtains a limiting law in which all curves are tied together at the endpoints. The finite- ϵ bridge ensembles are therefore a convenient access route to a singular limiting measure.

The simplest useful quantity to compute is the joint density of the N particles at a single intermediate time, say at time 1. Let

$$Y = (Y_1, \dots, Y_N), \quad Y_1 < \dots < Y_N,$$

denote the midpoint vector. By the Markov property and the Karlin–McGregor formula, its density is proportional to

$$\det[p_1(a_i, y_j)]_{i,j=1}^N \det[p_1(y_i, a_j)]_{i,j=1}^N.$$

Since the two determinants are transposes of one another, the numerator is the square of one determinant.

This determinant may be analysed directly. Writing

$$a_i = (N - i)\epsilon,$$

one has

$$p_1(a_i, y_j) = (2\pi)^{-1/2} \exp\left\{-\frac{(a_i - y_j)^2}{2}\right\}.$$

From row i one factors out $e^{-a_i^2/2}$, and from column j one factors out $e^{-y_j^2/2}$. This gives

$$\det[p_1(a_i, y_j)]_{i,j=1}^N = (2\pi)^{-N/2} \exp\left\{-\frac{1}{2} \sum_{i=1}^N a_i^2\right\} \exp\left\{-\frac{1}{2} \sum_{j=1}^N y_j^2\right\} \det[e^{a_i y_j}]_{i,j=1}^N.$$

Now the last determinant is a Vandermonde determinant in the variables

$$e^{\epsilon y_1}, \dots, e^{\epsilon y_N}.$$

Indeed,

$$\det[e^{a_i y_j}]_{i,j=1}^N = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e^{\epsilon y_1} & e^{\epsilon y_2} & \cdots & e^{\epsilon y_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(N-1)\epsilon y_1} & e^{(N-1)\epsilon y_2} & \cdots & e^{(N-1)\epsilon y_N} \end{pmatrix} = \prod_{1 \leq i < j \leq N} (e^{\epsilon y_j} - e^{\epsilon y_i}).$$

For fixed y ,

$$e^{\epsilon y_j} - e^{\epsilon y_i} = \epsilon(y_j - y_i) + O(\epsilon^2),$$

uniformly on compact sets. Hence

$$\det[e^{a_i y_j}]_{i,j=1}^N = \epsilon^{\binom{N}{2}} \Delta_N(y) (1 + o(1)),$$

where

$$\Delta_N(y) = \prod_{1 \leq i < j \leq N} (y_j - y_i) = \det(y_i^{j-1})_{1 \leq i, j \leq N}.$$

It follows that

$$\det[p_1(a_i, y_j)]_{i,j=1}^N = (2\pi)^{-N/2} \epsilon^{\binom{N}{2}} \exp\left\{-\frac{1}{2} \sum_{j=1}^N y_j^2\right\} \Delta_N(y) (1 + o(1)),$$

again locally uniformly in y .

Squaring this determinant and normalising, one obtains in the limit

$$\frac{1}{C_N} \Delta_N(y)^2 \exp\left\{-\sum_{i=1}^N y_i^2\right\}$$

on the Weyl chamber $y_1 < \cdots < y_N$. This is precisely the Hermite orthogonal-polynomial ensemble. In particular, the one-time law of the non-intersecting bridge ensemble has already entered the general orthogonal-polynomial framework discussed above: the Vandermonde square records the repulsion between particles, while the Gaussian weight identifies the Hermite family as the relevant basis.

This computation is worth carrying out in detail for two reasons. First, it is a direct and intelligible application of Karlin–McGregor and the small- ϵ expansion, so the reader can see concretely where the Vandermonde structure comes from. Second, it explains why determinantal structure should be expected: one is led naturally to a Hermite orthogonal-polynomial ensemble, and hence to a projection kernel built from Hermite functions.

At the same time, it is important to see what this one-time computation does *not* give. It identifies the law of the full particle configuration at one time, but it does not yet provide the many-time law in a form suitable for asymptotics, and it certainly does not by itself give the joint law of only the top few curves at several times in any manageable way. For that, one must pass from the one-time orthogonal-polynomial picture to a genuine space-time determinantal description. It is to that reorganization that we now turn.

5.9.3. From Karlin–McGregor to the extended kernel. The one-time Karlin–McGregor computation already shows the essential algebraic feature of the non-intersecting bridge ensemble: determinants appear because one starts with independent Brownian propagation and then subtracts away the crossing configurations. At several times, however, the same computation begins to produce a quantity that is both richer and less convenient. It is richer because it now describes the full ordered particle system at finitely many times. It is less convenient because our ultimate interest is not in all N particles, but in the edge and in the top few curves. In principle, one could iterate Karlin–McGregor and then integrate out the lower $N - k$ particles at each chosen time. Formally this would recover the law of the top k particles at those times. As a route to asymptotics, however, it is poor: one would be marginalising a large determinant at each level, and it is hard to see from such a formula why a stable edge limit should emerge. The point of the extended determinantal description is precisely to reorganise the same information into a form that is robust under asymptotic analysis.

Fix times $0 < t_1 < \dots < t_m < 2$. For each $r \in \{1, \dots, m\}$, let

$$x^{(r)} = (x_1^{(r)} < \dots < x_N^{(r)})$$

denote the ordered particle positions at time t_r . Repeated use of the Markov property and the Karlin–McGregor formula gives a joint density proportional to

$$\det[p_{t_1}(a_i, x_j^{(1)})] \det[p_{t_2-t_1}(x_i^{(1)}, x_j^{(2)})] \cdots \det[p_{2-t_m}(x_i^{(m)}, a_j)]. \quad (29)$$

This is already a very concrete formula. The first determinant propagates the initial data to time t_1 ; each intermediate determinant propagates from one time slice to the next; and the final determinant propagates from time t_m to the terminal data. Thus the many-time law is described entirely in terms of single-particle transition kernels, but in a form that is not yet well adapted to asymptotics or to retaining only the edge data.

A more useful perspective is to regard (29) as the starting point of a general determinantal mechanism. It is helpful to formulate this mechanism in a slightly more flexible setting.

Suppose one has functions

$$\phi_i : \mathbb{R} \rightarrow \mathbb{R}, \quad \psi_j : \mathbb{R} \rightarrow \mathbb{R}, \quad W_r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

for $1 \leq i, j \leq N$ and $1 \leq r \leq m-1$, and consider on $(x^{(1)}, \dots, x^{(m)}) \in (\mathbb{R}^N)^m$ a density proportional to

$$\det[\phi_i(x_j^{(1)})]_{i,j=1}^N \left(\prod_{r=1}^{m-1} \det[W_r(x_i^{(r)}, x_j^{(r+1)})]_{i,j=1}^N \right) \det[\psi_i(x_j^{(m)})]_{i,j=1}^N. \quad (30)$$

The Eynard–Mehta theorem says that the corresponding point configuration on the discrete–continuous space $\{1, \dots, m\} \times \mathbb{R}$ is determinantal.

The basic reason is multilinear algebra. When one computes a correlation function, one integrates out the variables that are not being observed. Each such integration meets a product of two determinants, and the key identity is Andréief’s formula

$$\int_{\mathbb{R}^N} \det[f_i(x_j)]_{i,j=1}^N \det[g_i(x_j)]_{i,j=1}^N dx_1 \cdots dx_N = N! \det \left[\int_{\mathbb{R}} f_i(x) g_j(x) dx \right]_{i,j=1}^N.$$

This identity is itself very natural. Expanding both determinants as sums over permutations produces a double sum. After integration, the variables separate, and because the integrand is symmetric in the x_j , all terms with the same relative permutation contribute equally. The sum then

recombines into the determinant on the right-hand side. Thus the effect of integrating out an entire time slice is to replace two adjacent determinants by a single matrix product. Iterating this procedure through the successive time slices leaves behind one determinantal kernel governing all correlation functions.

So the determinantal structure is already latent in (30). The Eynard–Mehta theorem packages the repeated application of Andréief’s identity into a general statement, but the underlying mechanism is exactly this repeated collapsing of determinants under integration.

In the Brownian bridge setting, one simply takes

$$\phi_i(x) = p_{t_1}(a_i, x), \quad W_r(x, y) = p_{t_{r+1}-t_r}(x, y), \quad \psi_j(y) = p_{2-t_m}(y, a_j).$$

Then (30) is exactly (29). Thus the many-time non-intersecting bridge ensemble is a determinantal point process on $\{t_1, \dots, t_m\} \times \mathbb{R}$. The practical value of this reformulation is that the marginalisation over the lower particles is now built into the formalism. One studies a single space-time kernel rather than repeatedly integrating large determinant expressions by hand.

Let us indicate how that kernel arises. Define the $N \times N$ matrix

$$A_{ij} = p_2(a_i, a_j).$$

This is the matrix of total Brownian bridge weights from the initial positions to the terminal positions. The Eynard–Mehta formula then gives the space-time kernel

$$K_N^{\text{ext}}(s, x; t, y) = -\mathbf{1}_{s < t} p_{t-s}(x, y) + \sum_{i,j=1}^N p_s(a_i, x) (A^{-1})_{ij} p_{2-t}(y, a_j). \quad (31)$$

The two terms in this formula have a clear interpretation.

The term

$$-\mathbf{1}_{s < t} p_{t-s}(x, y)$$

is the free Brownian propagation from time s to time t . It records the time-ordering of the process and is present in every extended kernel of this type.

The second term incorporates the bridge boundary conditions at times 0 and 2 together with the non-intersection constraint. The matrix inverse A^{-1} appears because, after the repeated use of Andréief’s identity, one must invert the matrix of pairings between the left boundary functions and the right boundary functions. In this way, (31) is the expression that arises by the algebra of the many-time Karlin–McGregor formula.

Formula (31) already identifies the many-time Brownian bridge ensemble as determinantal. For asymptotic purposes, however, it is useful to rewrite the kernel in a basis adapted to the Gaussian structure. This is where the Hermite functions enter.

The one-time density computed in the previous subsection is the Hermite orthogonal-polynomial ensemble. Correspondingly, the many-time kernel may be expressed in terms of Hermite functions. Let

$$\phi_r(x) = c_r H_r(x) e^{-x^2/2}, \quad r \geq 0, \quad (32)$$

denote the normalised Hermite functions, so that $\{\phi_r\}_{r \geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$ diagonalising the harmonic oscillator. These are the functions already implicit in the one-time orthogonal-polynomial picture; in the many-time setting they become the natural spectral basis. After the standard Gaussian conjugation, the Brownian bridge semigroup acts diagonally in the basis (32). This is the structural reason that the kernel admits a spectral expansion.

More concretely, one may rewrite (31) in the form

$$K_N^{\text{ext}}(s, x; t, y) = -\mathbf{1}_{s < t} p_{t-s}(x, y) + \sum_{r=0}^{N-1} e^{\lambda_r(s-t)} \phi_r(x) \phi_r(y), \quad (33)$$

where the λ_r are the corresponding harmonic-oscillator eigenvalues, with the precise normalisation depending on the chosen Brownian scaling and conjugation. The point to retain is that the finite-rank correction in (31) has become the spectral projection onto the first N Hermite modes.

This Hermite representation is the natural starting point for the edge asymptotics. The soft edge is the point where the oscillatory region of the Hermite family gives way to its rapidly decaying region, and it is this transition that the Airy analysis resolves. We now discuss that transition more carefully.

5.9.4. Edge scaling and the extended Airy kernel. Once the extended Hermite kernel has been identified, the remaining task is asymptotic. The relevant scaling is the soft-edge scaling for the top of the Hermite spectrum. For the one-time Hermite ensemble, the particle cloud fills an interval whose right edge is asymptotically located at $\sqrt{2N}$. To examine fluctuations there, one writes

$$x = \sqrt{2N} + \frac{\xi}{\sqrt{2} N^{1/6}}, \quad y = \sqrt{2N} + \frac{\eta}{\sqrt{2} N^{1/6}}, \quad (34)$$

and, in the space-time setting, one simultaneously rescales time differences on the $N^{-1/3}$ scale, equivalently on the $N^{2/3}$ scale in the original unscaled temporal variable. Vertical fluctuations are then of order $N^{-1/6}$ in the one-time Hermite normalisation, which is the same edge scale that becomes order one in the Brownian last passage line-ensemble scaling used later in the notes.

The meaning of the soft edge is visible already at the level of the Hermite functions ϕ_r . They satisfy the harmonic-oscillator eigenvalue equation, and their behaviour changes qualitatively with the spatial location. In the bulk they oscillate, with zeros spaced on the natural microscopic scale. Beyond the edge they decay rapidly, because the Gaussian factor overwhelms the polynomial growth. Near the edge one enters a turning-point regime for this second-order differential equation. After the rescaling (34), the Hermite equation degenerates to the Airy equation

$$f''(\xi) = \xi f(\xi).$$

Thus the Airy function appears as the model profile describing the transition between oscillation and decay.

This is the classical Plancherel–Rotach picture for Hermite asymptotics. One may think of it this way: the zeros of the Hermite polynomials accumulate up to the spectral edge, but beyond that edge the oscillations cease and are replaced by decay. The Airy function interpolates between these two regimes. On the negative side, $\text{Ai}(\xi)$ oscillates with slowly varying amplitude; on the positive side, it decays super-exponentially. Its zeros lie on the negative half-line and mark the oscillatory side of the transition. This turning-point description is not peculiar to Hermite polynomials. In many soft-edge limits for orthogonal-polynomial ensembles, the same Airy profile governs the transition, which is one reason the Airy kernel is so widespread in edge asymptotics.

At one time, the Plancherel–Rotach asymptotics lead to the convergence of the Hermite kernel to the Airy kernel. In a form consistent with (34), this says that

$$\frac{1}{\sqrt{2} N^{1/6}} K_N \left(\sqrt{2N} + \frac{\xi}{\sqrt{2} N^{1/6}}, \sqrt{2N} + \frac{\eta}{\sqrt{2} N^{1/6}} \right) \longrightarrow K^{\text{Ai}}(\xi, \eta),$$

where

$$K^{\text{Ai}}(\xi, \eta) = \int_0^\infty \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda.$$

At several times, the same turning-point analysis applied to the extended Hermite kernel yields convergence to the extended Airy kernel

$$K^{\text{Ai,ext}}(s, \xi; t, \eta).$$

In standard form, this kernel is

$$K^{\text{Ai,ext}}(s, \xi; t, \eta) = \begin{cases} \int_0^\infty e^{-\lambda(s-t)} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda, & s \geq t, \\ -\int_{-\infty}^0 e^{-\lambda(s-t)} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda, & s < t. \end{cases}$$

Setting $s = t$ recovers the one-time Airy kernel K^{Ai} .

This formula is structurally parallel to the Hermite one. The time-ordering term persists, while the finite Hermite spectral projection is replaced by the corresponding Airy projection. Thus the edge limit preserves the determinantal structure in a refined form: the finite- N space-time process is encoded by a projection onto the first N Hermite modes, and after soft-edge magnification that projection converges to the Airy projection. The limiting space-time object is therefore again determinantal, now with the extended Airy kernel.

5.9.5. From kernel convergence to finite-dimensional convergence. At this point the identifying input needed later in the notes is close at hand. The general theory of determinantal point processes says that, when the kernels converge in a sufficiently strong sense—here trace-class convergence on bounded windows is the natural framework—the associated point processes converge weakly. This is the place where the abstract determinantal-process theory enters, and where the asymptotic analysis of the kernel is converted into a probabilistic convergence statement.

Applied here, the conclusion is the following. Fix finitely many times

$$t_1, \dots, t_m$$

and fix $k \in \mathbb{N}$. Consider the top k particles of the edge-scaled Brownian bridge ensemble at those times. The convergence of the extended Hermite kernel to the extended Airy kernel implies that their joint law converges, as $N \rightarrow \infty$, to a limiting law. These limiting joint laws are exactly the finite-dimensional distributions of the multi-line Airy process.

This is the external identifying input used in the next section. It does not come from the Brownian Gibbs and acceptance analysis. Rather, it comes from the determinantal formulas and their asymptotics. The next section will take this finite-dimensional convergence as given and combine it with tightness and with the compact-window acceptance estimate in order to upgrade the multi-line Airy process from a compatible family of finite-dimensional distributions to a continuous non-intersecting line ensemble with the Brownian Gibbs property.

5.9.6. What this section has and has not done. Let us end by being explicit about the division of labour.

This section has explained why the relevant finite-dimensional convergence should be true, and what analytic route proves it:

$$\begin{aligned} \text{Karlin-McGregor} &\longrightarrow \text{Eynard-Mehta} \longrightarrow \text{extended Hermite kernel} \\ &\longrightarrow \text{extended Airy kernel} \longrightarrow \text{finite-dimensional convergence.} \end{aligned}$$

It has also explained why the one-time Vandermonde density, while illuminating, is not itself enough for the multi-time problem, and why the Airy function appears naturally as the turning-point profile of the Hermite family at the soft edge.

What it has not done is prove the kernel asymptotics in full detail. Those belong to the integrable-probability analysis of the determinantal kernel. For the purposes of these notes, the important point is to know exactly what the external input is, how it arises from the determinantal formalism, and how it fits into the later construction of the Airy line ensemble.

5.10. Construction of the Airy line ensemble

We now explain how Proposition 5.8 enters the passage from the scaled Brownian last passage ensembles to the Airy line ensemble.

The preceding section supplied the identifying finite-dimensional convergence input. What remains is to combine that input with the Brownian Gibbs technology developed earlier in the chapter. More precisely, there are three tasks. First, one must prove tightness of the top finitely many curves on each fixed compact interval. Second, one must use the already known finite-dimensional convergence to identify the law of every subsequential finite-window limit. Third, one must show that the Brownian Gibbs property passes to the limit. The compact-window acceptance estimate is the key new tool in the first and third steps: it gives quantitative control of local Brownian bridge reconstruction, and in this way prevents the limiting line ensemble from being obtained through a degenerate filtering of the prelimit Brownian input.

5.10.1. From the multi-line Airy process to the Airy line ensemble. It is helpful to separate two closely related objects.

Historically, the first limiting object is the *multi-line Airy process*: a compatible family of finite-dimensional distributions arising from edge scaling [PS02]. At that stage one does not yet have a continuous $\mathbb{N}$ -indexed line ensemble. Finite-dimensional distributions alone do not provide continuity, nor do they by themselves yield a line ensemble with Brownian Gibbs structure.

The Airy line ensemble is stronger [CH14]. It is a continuous ordered line ensemble whose finite-dimensional distributions agree with those of the multi-line Airy process. This section explains how to pass from the former type of object to the latter.

We must also distinguish between the Airy line ensemble

$$\mathcal{A} = (\mathcal{A}(1, \cdot), \mathcal{A}(2, \cdot), \dots),$$

which is invariant under spatial translation, and its parabolic shift

$$\mathcal{L}(i, x) = \mathcal{A}(i, x) - 2^{-1/2}x^2.$$

The scaled Brownian last passage ensembles converge to the parabolic ensemble $\mathcal{L}$. It is $\mathcal{L}$, not $\mathcal{A}$, that satisfies the Brownian Gibbs property. The ensemble $\mathcal{A}$ is recovered by adding back the parabola.

Thus the task of the section is this: construct a continuous ordered limiting line ensemble $\mathcal{L}$, identify its finite-dimensional distributions with those of the multi-line Airy process, and show that $\mathcal{L}$ satisfies the Brownian Gibbs property. The translation-invariant ensemble $\mathcal{A}$ is then defined by

$$\mathcal{A}(i, x) = \mathcal{L}(i, x) + 2^{-1/2}x^2.$$

5.10.2. Compactness on a fixed finite window. Fix $k \in \mathbb{N}$ and a compact interval $I = [-T, T] \subset \mathbb{R}$. We show that the laws of the restrictions of the top k curves of $\mathcal{L}_n^{\text{sc}}$ to I form a tight family in

$$C([1, k] \times I, \mathbb{R}),$$

equipped with the topology of uniform convergence.

The issue is compactness in path space. For this, one needs two ingredients: control of the curve values at finitely many deterministic points, and control of oscillation between those points.

Choose a mesh

$$-T = x_0 < x_1 < \cdots < x_m = T.$$

For each fixed $i \leq k$ and each mesh point x_r , the one-point bounds for curve i imply that, after parabolic correction, $\mathcal{L}_n^{\text{sc}}(i, x_r)$ has uniformly small tails. Hence, for every $\varepsilon > 0$, there exists $M < \infty$ such that

$$\sup_n \mathbb{P}_n \left(\max_{1 \leq i \leq k} \max_{0 \leq r \leq m} |\mathcal{L}_n^{\text{sc}}(i, x_r)| > M \right) \leq \varepsilon.$$

Thus the top k curves are unlikely to take very large values at the mesh points.

We next control oscillation on each interval $[x_r, x_{r+1}]$. Fix one such interval. Proposition 5.8 implies that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all sufficiently large n , the Brownian Gibbs reconstruction problem for the top k curves on $[x_r, x_{r+1}]$ has acceptance probability at least δ with probability at least $1 - \varepsilon$. On that event, and conditional on the retained data, the missing segment is obtained from k independent Brownian bridges with the corresponding random endpoint data by conditioning on an event of probability at least δ . Therefore, for any oscillation event E_r depending only on the reconstructed segment,

$$\mathbb{P}_n(E_r \mid \text{retained data}) \leq \delta^{-1} \mathbb{P}_n^{\text{bridge}}(E_r \mid \text{same endpoints}),$$

where the right-hand side refers to the raw bridge law.

Now restrict to the high-probability event that all mesh-point values lie in $[-M, M]$. On this event, the endpoint data for every bridge segment lie in a fixed compact set. Standard Brownian bridge estimates therefore imply that, for every $\eta > 0$,

$$\sup_{\substack{\text{endpoint data} \\ |\text{all endpoints}| \leq M}} \mathbb{P}_n^{\text{bridge}} \left(\sup_{1 \leq i \leq k} \text{osc}_{[x_r, x_{r+1}]} \mathcal{B}_i > \eta \right) \longrightarrow 0$$

as the mesh size $\max_r |x_{r+1} - x_r|$ tends to zero. Since there are only finitely many mesh intervals, a union bound shows that, for fine enough mesh, the top k curves have uniformly small oscillation on every mesh interval with high probability.

Thus, with high probability, the top k curves are uniformly bounded on the mesh and have uniformly small oscillation between mesh points. By the Arzelà–Ascoli criterion, this yields tightness of the top k curves on I in the uniform topology.

5.10.3. Identification of finite-window subsequential limits. It follows that every sequence $n_j \rightarrow \infty$ has a further subsequence, which we do not relabel, such that the restriction of the top k curves of $\mathcal{L}_{n_j}^{\text{sc}}$ to I converges in law in

$$C([1, k] \times I, \mathbb{R})$$

to some continuous k -line ensemble.

At this stage, continuity of the limit is automatic, because the ambient space consists of continuous functions. Likewise, the limit is ordered in the weak sense that its curves do not cross:

$$Q_1(x) \geq Q_2(x) \geq \cdots \geq Q_k(x) \quad \text{for each } x \in I$$

almost surely. This follows because the prelimit ensembles are strictly ordered for every n , and the order relation is closed under uniform convergence.

What makes the limit *the* Airy object rather than merely some limiting line ensemble is the separate identifying convergence input. The finite-dimensional distributions of the restrictions of $\mathcal{L}_n^{\text{sc}}$ are already known to converge to those of the multi-line Airy process by the determinantal analysis just described. Hence every subsequential finite-window limit has the same finite-dimensional distributions. Since the ambient space is a space of continuous functions, this determines the law uniquely. Thus, for each pair (k, I) , there is a unique finite-window law arising as the limit of the restrictions of $\mathcal{L}_n^{\text{sc}}$.

5.10.4. Assembly of the global parabolic Airy line ensemble. We now pass from these finite-window laws to a single $\mathbb{N}$ -indexed line ensemble. Choose the increasing sequence of windows

$$I_m = [-m, m], \quad m \in \mathbb{N}.$$

For each m , the preceding discussion provides a unique law on

$$C([1, m] \times I_m, \mathbb{R})$$

for the top m curves on I_m . These laws are compatible under the obvious restriction maps, because the prelimit laws are compatible and the finite-window limits are unique. Standard projective-consistency arguments therefore yield a probability law on the space of $\mathbb{N}$ -indexed continuous line ensembles on $\mathbb{R}$ whose restriction to each

$$[1, m] \times I_m$$

has the prescribed law.

Equivalently, one may phrase the same construction by a diagonal subsequence argument: successively pass to subsequences so that convergence holds on $[1, m] \times I_m$ for each m , and then take the diagonal subsequence. Uniqueness of the finite-window limit laws shows that the result does not depend on the choice of diagonalisation.

We denote the resulting continuous ordered $\mathbb{N}$ -indexed line ensemble by $\mathcal{L}$. Its finite-dimensional distributions are precisely those of the multi-line Airy process after the parabolic centring already built into the scaling of $\mathcal{L}_n^{\text{sc}}$. Thus $\mathcal{L}$ is the *parabolic Airy line ensemble*.

The stationary Airy line ensemble is then defined by

$$\mathcal{A}(i, x) = \mathcal{L}(i, x) + 2^{-1/2}x^2.$$

In particular, $\mathcal{A}(1, \cdot)$ is the Airy₂ process.

It remains to show that $\mathcal{L}$ satisfies the Brownian Gibbs property.

5.10.5. Passing the Brownian Gibbs property to the limit. Fix a compact interval $J \subset \mathbb{R}$, a compact subinterval $[a, b] \subseteq J$, and integers

$$1 \leq i \leq j.$$

We consider Brownian Gibbs resampling of the block of curves $[i, j]$ on $[a, b]$. To retain the genuine lower boundary for this block, one works with the top $j + 1$ curves. Thus we consider the restriction

of $\mathcal{L}_n^{\text{sc}}$ to the top $j + 1$ curves on J , and likewise the restriction of the limiting ensemble $\mathcal{L}$ to the top $j + 1$ curves on J .

By the preceding discussion, after passing to a subsequence if necessary, these top $j + 1$ curves converge in law on J to the corresponding restriction of $\mathcal{L}$. Since

$$C(\llbracket 1, j + 1 \rrbracket \times J, \mathbb{R})$$

with the uniform topology is Polish, the Skorokhod representation theorem allows us to place the prelimit sequence and its limit on a common probability space so that this convergence holds almost surely and uniformly on J .

We now realise the auxiliary bridge candidates on this same probability space in a common-noise way. A Brownian bridge on $[a, b]$ with endpoints (u, v) may be written uniquely as

$$\beta_{u,v}(t) = \frac{b-t}{b-a} u + \frac{t-a}{b-a} v + \beta^\circ(t), \quad t \in [a, b],$$

where β° is a standard centred Brownian bridge on $[a, b]$. For a block of $j - i + 1$ curves, we therefore generate an infinite sequence of candidate bridge ensembles by taking an infinite sequence of independent centred bridge ensembles and then, for each n and for the limit, adding to them the appropriate random affine interpolants determined by the corresponding endpoint data. Thus the prelimit and limiting candidates are built from the same underlying centred bridge noise.

For each n , run the usual rejection procedure for the block $\llbracket i, j \rrbracket$ on $[a, b]$: accept the first candidate that stays below the $(i - 1)^{\text{st}}$ curve when $i > 1$, above the $(j + 1)^{\text{st}}$ curve, and internally non-intersecting, and then insert that accepted candidate in place of the original block. Define the analogous rejection procedure for the limiting ensemble $\mathcal{L}$ using the same common sequence of centred bridge noises.

The first issue is to ensure that both rejection procedures terminate quickly with high probability. Proposition 5.8 implies that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all large n , the prelimit acceptance probability is at least δ with probability at least $1 - \varepsilon$. For the limit, the relevant acceptance probability is a continuous function of the endpoint data and neighbouring boundary curves at every strictly ordered configuration. Since the prelimit boundary data converge uniformly to the limiting data, the limiting acceptance probability is therefore also bounded below by, say, $\delta/2$ on the same high-probability event, for all sufficiently large n .

Choose $M \in \mathbb{N}$ so large that

$$(1 - \delta/2)^M \leq \varepsilon.$$

Then, on this high-probability event, both rejection procedures accept within the first M trials with probability at least $1 - \varepsilon$.

The second issue is to compare the accept–reject decisions for the first M candidates. Fix one of these candidates. Because the centred bridge noises are common and the affine endpoint parts converge uniformly, the candidate bridge ensembles themselves converge uniformly on $[a, b]$. The surrounding boundary curves also converge uniformly. Therefore, if this candidate is accepted in the limiting problem because all the relevant inequalities are strict, then it is accepted in the prelimit problem for all large n ; and similarly, if it is rejected in the limit because some inequality fails strictly, then it is rejected in the prelimit problem for all large n . The only way stabilisation could fail is if, in the limiting problem, the candidate were exactly tangent to one of the neighbouring boundary curves or if two candidate curves were exactly tangent to one another. Such knife-edge

events have probability zero under the auxiliary bridge law. Therefore, for each fixed candidate among the first M , its acceptance or rejection is eventually constant in n .

It follows that, on the event that both procedures accept within the first M trials, the index of the first accepted candidate is eventually the same for the prelimit and the limit. Hence the resampled prelimit ensembles converge almost surely, uniformly on J , to the corresponding resampled limiting ensemble.

For each finite n , however, the resampled and unresampled ensembles have the same law by the Brownian Gibbs property of $\mathcal{L}_n^{\text{sc}}$. Let f be a bounded continuous test function on the space of continuous $(j+1)$ -line ensembles on J . Then

$$\mathbb{E}[f(\mathcal{L}_n^{\text{sc}})] = \mathbb{E}[f(\tilde{\mathcal{L}}_n^{\text{sc}})]$$

for every n , where $\tilde{\mathcal{L}}_n^{\text{sc}}$ denotes the resampled ensemble. By the coupling and the convergence of the rejection procedures, both sides converge along the chosen subsequence to the corresponding expectations for the limiting original and limiting resampled ensembles. Hence those two limiting laws coincide. Since the block $\llbracket i, j \rrbracket$, the interval $[a, b]$, and the ambient compact set J were arbitrary, this is precisely the Brownian Gibbs property for the limiting line ensemble $\mathcal{L}$.

5.10.6. Conclusion. We have shown that for each $k \in \mathbb{N}$ and each compact interval I , the top k curves of the scaled Brownian last passage ensembles converge on I to a unique continuous ordered limit whose finite-dimensional distributions agree with those of the multi-line Airy process. These finite-window limits assemble into a unique $\mathbb{N}$ -indexed continuous line ensemble $\mathcal{L}$, the parabolic Airy line ensemble.

We have also shown that $\mathcal{L}$ satisfies the Brownian Gibbs property. The stationary Airy line ensemble is therefore obtained by restoring the parabola:

$$\mathcal{A}(i, x) = \mathcal{L}(i, x) + 2^{-1/2}x^2.$$

Its top curve is the Airy_2 process, and its finite-dimensional distributions are those of the multi-line Airy process. This proves Theorem 5.7.

Brownian Gibbs applied: from ensemble to landscape, and KPZ fractal geometry

6.1. From the Airy line ensemble to the directed landscape

We have now seen two closely related pictures of Brownian last passage percolation. On the one hand, there is the finite- n model itself, in which energy is gathered by following Brownian curves and jumping upward from one curve to the next. On the other hand, after soft-edge scaling, there is the Airy line ensemble, which records the unit-order fluctuations of the top curves of the system. In this section, we explain how to pass from this line-ensemble picture to the random metric object that governs the full KPZ scaling limit, namely the directed landscape.

It is useful to begin with the simplest geodesic problem. Consider Brownian last passage from $(0, 0)$ to (n, n) . The maximizing path moves forward in time and may jump upward from one Brownian curve to the next, collecting the increments it encounters. Its energy is of order $2n$, and after subtracting this deterministic growth and dividing by $n^{1/3}$, one obtains a random fluctuation of unit order. In the scaling limit, that fluctuation is described by the Tracy–Widom GUE distribution [TW94, Joh00]—or, more relevantly for us, by the value at zero of the uppermost curve in the parabolic Airy line ensemble.

This single endpoint problem already suggests a richer one. Rather than fixing the endpoint at $(n, 1)$, we may allow it to move horizontally on the natural $n^{2/3}$ scale. In scaled coordinates, one is then led to consider geodesics from $(0, 0)$ to $(x, 1)$ for varying $x \in \mathbb{R}$. This is the scale on which endpoint perturbations produce non-trivial correlations in energy and non-degenerate geodesic geometry. If one traces such geodesics backwards from different upper endpoints, then, for nearby endpoints, they typically merge at a macroscopic intermediate height before continuing together to the root. Thus one should picture not merely a collection of energy values, but a rooted family of coalescing scaled geodesics carrying those energies.

This geodesic picture is part of what the Airy line ensemble encodes. As one scans along the upper boundary at scaled time 1, the energy profile of paths from the root $(0, 0)$ to $(x, 1)$ is given by the parabolic Airy process, that is, by the top curve of the parabolic Airy line ensemble. Moreover, if one considers k disjoint paths with common starting point and common endpoint, then the maximal k -tuple energy is encoded by the sum of the top k curves. This is the scaling-limit counterpart of the bouquet picture already studied: O’Connell and Yor’s Theorem 5.2 identifies finite- n bouquet energies with the Brownian watermelon, and the soft-edge limit of the latter is the Airy line ensemble.

In this way, the Airy line ensemble may be understood as describing a rooted last passage picture. Fix a starting point, say the origin. Then the ensemble records the energies of reaching all upper endpoints. In the prelimit, these energies are realised by geodesics, and these geodesics coalesce as they are traced back towards the root. Thus the rooted Airy picture is not only an energy profile: it

is also the scaling limit of a geometric picture in which many endpoint geodesics share long common trunks before branching apart near their endpoints.

The next natural question is what happens if the starting point is varied as well as the endpoint. If we begin not at $(0, 0)$ but at $(x, 0)$, and ask for the energy of reaching $(x + y, 1)$, we should again expect a scaled energy profile of Airy type. In effect, for each choice of lower root x , one has a rooted Airy picture describing the energies of travel across the strip from that root to all upper endpoints. As x varies, these rooted pictures should be imagined as a coupled family of geodesic trees, or informally as a forest rooted along the lower boundary.

(In fact, this picture of trees rooted on the real line is attractive but not quite accurate. There are exceptional pairs of points for which there is more than one geodesic on the route $(x, 0) \rightarrow (y, 1)$. Notably, for almost every $x \in \mathbb{R}$, there is a countable set of locations y at which uniqueness is violated, with a pair of geodesics existing. The elements of the pair are ordered, with a shared course out of the origin until a certain moment at which the left geodesic departs from the route pursued by the right one, the pair coalescing once more only at the terminal location $(y, 1)$. Such phenomena encode interesting fractal structures such as devil's staircases or distribution functions of random Cantor sets. But, at least for now, this is an aside. The trees' picture is roughly accurate and may help to picture the constructions involved.)

These rooted pictures arise from the same Brownian environment and are coupled non-trivially. As the lower root varies from x to x' , it is natural to ask about the joint distribution of the energetic profiles, and also about how the corresponding geodesic trees fit together. With two lower points x, x' and two upper points y, y' fixed, the scaled geodesics that run from $(x, 0)$ to $(y, 1)$, and from $(x', 0)$ to $(y', 1)$, are typically unique; one may show uniqueness, for example, by resampling a suitable portion of the underlying Brownian environment. These geodesics may merge for part of their journey before separating again near the endpoints. Thus even across one strip, the geometry is already a coupled picture of many rooted trees, not a single one.

The object that records this coupled two-endpoint picture across one strip is the Airy sheet. It is the random field that gives the jointly defined scaled energies of travelling from x at time 0 to y at time 1, for all $x, y \in \mathbb{R}$. If the Airy line ensemble describes the rooted view from a fixed starting point, then the Airy sheet stitches together all these rooted views into a single two-endpoint object. It should be thought of as the KPZ scaling limit of Brownian last passage energies across one strip.

The directed landscape is the corresponding object in full space-time. It is the random field that assigns an energy to travel from (x, s) to (y, t) for arbitrary real x, y, s, t with $s < t$. Thus the Airy line ensemble, the Airy sheet, and the directed landscape stand in a natural progression. The Airy line ensemble describes rooted energies from one fixed lower point. The Airy sheet describes all energies across a single strip. The directed landscape describes all energies between arbitrary space-time points.

From this point of view, the central task is plain. One must build the Airy sheet. Once the strip problem has been solved, the full landscape may be assembled by composition. Indeed, if one knows the energy of travelling across a strip, then to travel from (x, s) to (y, t) one may divide the time interval $[s, t]$ into narrower strips and optimise over the intermediate spatial points at which the path crosses the interfaces between them. In this sense, the construction of the directed landscape reduces to the construction of the Airy sheet and to the rule by which strip energies compose.

In principle, constructing the Airy sheet appears to require a mysteriously coupled family of Airy line ensembles, one rooted at every point of the lower boundary. In fact, as we explain in this chapter,

it is possible to build the Airy sheet from just one such ensemble. The leading-order fluctuations of Brownian LPP can be captured within a single Airy ensemble window. We will develop some intuition for why this is possible later in the section.

6.2. The melon map

Let $f = (f_1, \dots, f_n)$ be a vector of continuous functions on $[0, \infty)$, with $f_i(0) = 0$ for each i . We regard f_i as the i^{th} line of a last passage environment, with paths moving to the right in the horizontal coordinate and upward through the line indices, from line n towards line 1.

For $t \geq 0$, define the one-path last passage value

$$f[(0, n) \rightarrow (t, 1)] = \sup_{0=t_0 \leq t_1 \leq \dots \leq t_n=t} \sum_{i=1}^n (f_i(t_i) - f_i(t_{i-1})).$$

A choice of jump times $0 = t_0 \leq t_1 \leq \dots \leq t_n = t$ determines a càdlàg path $\pi : [0, t] \rightarrow \{1, \dots, n\}$, which follows line i on the interval $[t_{i-1}, t_i]$. Its weight is

$$f(\pi) = \sum_{i=1}^n (f_i(t_i) - f_i(t_{i-1})).$$

Thus $f[(0, n) \rightarrow (t, 1)]$ is the maximum weight of such a path.

We will also need multi-path values. Two upright paths are called disjoint if they do not share a horizontal segment. We allow them to share starting or ending points, and more generally to meet at isolated points. For two ordered k -tuples of points

$$\mathbf{u} = (u_1, \dots, u_k), \quad \mathbf{v} = (v_1, \dots, v_k), \quad u_i, v_i \in \mathbb{R} \times \{1, \dots, n\},$$

let $Q[\mathbf{u} \rightarrow \mathbf{v}]$ be the set of disjoint k -tuples $\pi = (\pi_1, \dots, \pi_k)$ such that $\pi_i : u_i \rightarrow v_i$. The weight of π is the sum of the weights of its components,

$$f(\pi) = \sum_{i=1}^k f(\pi_i),$$

and the corresponding multi-path last passage value is

$$f[\mathbf{u} \rightarrow \mathbf{v}] = \max_{\pi \in Q[\mathbf{u} \rightarrow \mathbf{v}]} f(\pi).$$

If $Q[\mathbf{u} \rightarrow \mathbf{v}] = \emptyset$, we set $f[\mathbf{u} \rightarrow \mathbf{v}] = -\infty$. An optimiser, when it exists, is called a disjoint k -path optimiser.

For a point u , write

$$u^k = (u, \dots, u)$$

for the k -tuple consisting of k copies of u . The melon map W sends $f = (f_1, \dots, f_n)$ to a new n -tuple

$$Wf = (Wf_1, \dots, Wf_n)$$

defined by the successive differences

$$Wf_k(t) = f[(0, n)^k \rightarrow (t, 1)^k] - f[(0, n)^{k-1} \rightarrow (t, 1)^{k-1}], \quad (35)$$

for $k = 1, \dots, n$, with the convention

$$f[(0, n)^0 \rightarrow (t, 1)^0] = 0.$$

Thus the sum of the top k melon curves records the optimal k -path bouquet value:

$$\sum_{i=1}^k Wf_i(t) = f[(0, n)^k \rightarrow (t, 1)^k].$$

This is often the most useful way to remember the definition: the melon map repackages the whole collection of rooted bouquet energies into a line ensemble.

This is the same melon map used earlier to construct Dyson Brownian motion from independent Brownian motions. Namely, if $B = (B_1, \dots, B_n)$ is a vector of independent Brownian motions started at zero, then WB has the law of n -Dyson Brownian motion.

One immediate consequence of the definition is that $Wf(0) = 0$. A less obvious, but fundamental, property is that the curves of Wf are ordered.

PROPOSITION 6.1. For all $t \geq 0$ and $k = 1, \dots, n-1$,

$$Wf_k(t) \geq Wf_{k+1}(t).$$

PROOF. Write

$$L_r(t) = f[(0, n)^r \rightarrow (t, 1)^r], \quad r = 0, 1, \dots, n,$$

with the convention $L_0(t) = 0$. By the definition of the melon map,

$$Wf_k(t) = L_k(t) - L_{k-1}(t).$$

It is therefore enough to prove the concavity inequality

$$L_{k+1}(t) + L_{k-1}(t) \leq 2L_k(t), \quad k = 1, \dots, n-1.$$

Let

$$\Pi^+ = (\pi_1^+, \dots, \pi_{k+1}^+)$$

be an optimising $(k+1)$ -tuple in $Q[(0, n)^{k+1} \rightarrow (t, 1)^{k+1}]$, and let

$$\Pi^- = (\pi_1^-, \dots, \pi_{k-1}^-)$$

be an optimising $(k-1)$ -tuple in $Q[(0, n)^{k-1} \rightarrow (t, 1)^{k-1}]$. Together these two bouquets contain $2k$ paths, with common lower and upper endpoints. The idea is to combine them and redistribute their traces into two new bouquets of size k . At each time slice, order the $2k$ traces by their line index, assign the odd ordered traces to one new bouquet and the even ordered traces to the other. The only point requiring verification is that this time-slice sorting really produces upright, mutually disjoint paths rather than a collection of disconnected pieces.

For each $s \in [0, t]$, consider the multiset of line indices occupied at time s by the paths in Π^+ and Π^- :

$$\mathcal{M}(s) = \{\pi_1^+(s), \dots, \pi_{k+1}^+(s), \pi_1^-(s), \dots, \pi_{k-1}^-(s)\}.$$

List its elements in increasing order, counted with multiplicity,

$$a_1(s) \leq a_2(s) \leq \dots \leq a_{2k}(s).$$

Define two k -tuples of paths $T' = (\tau'_1, \dots, \tau'_k)$ and $T = (\tau_1, \dots, \tau_k)$ by taking respectively the odd and even ordered traces:

$$\tau'_j(s) = a_{2j-1}(s), \quad \tau_j(s) = a_{2j}(s), \quad j = 1, \dots, k.$$

We now check that these are legitimate disjoint k -tuples. The sorting operation does not create any new trace; it only relabels pieces of the original paths. As s varies, an ordered trace can switch from following a path of Π^+ to following a path of Π^- only at a time when the two traces meet. Such a switch is therefore just a concatenation at a common point, and the resulting τ_j 's and τ_j' 's are again upright paths from $(0, n)$ to $(t, 1)$.

It remains to check that the paths within each new bouquet are disjoint. For $0 < s < t$, the paths in each of the original bouquets Π^+ and Π^- are strictly ordered. Hence, for every j ,

$$a_j(s) < a_{j+2}(s).$$

Indeed, among the three entries $a_j(s), a_{j+1}(s), a_{j+2}(s)$, at least two come from the same original bouquet, and those two must be strictly ordered. It follows that

$$\tau_1'(s) < \tau_2'(s) < \cdots < \tau_k'(s) \quad \text{and} \quad \tau_1(s) < \tau_2(s) < \cdots < \tau_k(s)$$

for $0 < s < t$. Thus both T' and T belong to $Q[(0, n)^k \rightarrow (t, 1)^k]$.

Finally, the sorting has not changed the total collection of horizontal line-segments being traversed; it has only divided them differently between two bouquets. Therefore

$$f(\Pi^+) + f(\Pi^-) = f(T') + f(T).$$

Since T' and T are both admissible k -tuples, each has weight at most $L_k(t)$. Hence

$$L_{k+1}(t) + L_{k-1}(t) = f(\Pi^+) + f(\Pi^-) = f(T') + f(T) \leq 2L_k(t).$$

This proves the desired concavity inequality, and hence $Wf_k(t) \geq Wf_{k+1}(t)$. □

Due to the ordering of the lines, we immediately get the following rooted encoding.

COROLLARY 6.2. For every $n \geq 1$ and $t \geq 0$,

$$f[(0, n) \rightarrow (t, 1)] = Wf[(0, n) \rightarrow (t, 1)].$$

PROOF. Since $Wf(0) = 0$ and the curves of Wf are ordered, the optimal path in the environment Wf jumps immediately to the top curve and then follows that curve until time t . Hence

$$Wf[(0, n) \rightarrow (t, 1)] = Wf_1(t).$$

By the definition of the melon map,

$$Wf_1(t) = f[(0, n) \rightarrow (t, 1)].$$

This proves the corollary. □

The same idea applies when the common lower point is not 0. If $a \in \mathbb{R}$, write

$$f_i^{[a]}(u) = f_i(a + u) - f_i(a), \quad u \geq 0.$$

Then the rooted passage value from a to $b \geq a$ through the curves $f_1, \dots, f_n$ is

$$f^{[a]}[(0, n) \rightarrow (b - a, 1)],$$

and so is again encoded by the top curve of the corresponding transformed system. Thus, for each fixed lower root a , one obtains a rooted profile in the upper endpoint by applying the same transform to the shifted environment seen from a .

Applying the map W every time we shift our starting endpoint a seems to be a very tedious procedure. However, it turns out that we only need to apply the map W once by the following theorem.

THEOREM 6.3 (RSK Isometry). Let $f = (f_1, \dots, f_n)$ be any cadlag functions on $[0, \infty)$, and let $Wf = (Wf_1, \dots, Wf_n)$ be its image under the melon map. Then, for any admissible $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{<}^k$,

$$f[(\mathbf{x}, n) \rightarrow (\mathbf{y}, 1)] = Wf[(\mathbf{x}, n) \rightarrow (\mathbf{y}, 1)].$$

Here $(\mathbf{x}, n) = ((x_1, n), \dots, (x_k, n))$, and similarly for $(\mathbf{y}, 1)$.

PROOF SKETCH. The statement for cadlag functions is obtained from a discrete version by approximating the functions by step functions and then passing to the limit. We therefore first explain the discrete statement.

Let M be an $n \times m$ array of real weights. We write a lattice point as (r, ℓ) , where $r \in \{1, \dots, m\}$ is the horizontal coordinate and $\ell \in \{1, \dots, n\}$ is the line index. Thus $(1, n)$ is the lower-left corner and $(m, 1)$ is near the upper-right corner. Upright paths move to the right in the r -coordinate and from larger line index towards smaller line index.

For tuples $\mathbf{u} = (u_1, \dots, u_k)$ and $\mathbf{v} = (v_1, \dots, v_k)$ of lattice points, let $Q[\mathbf{u} \rightarrow \mathbf{v}]$ denote the set of disjoint k -tuples $\pi = (\pi_1, \dots, \pi_k)$, where π_i runs from u_i to v_i . The zero-temperature, or last-passage, value is

$$L_M(\mathbf{u}; \mathbf{v}) = \max_{\pi \in Q[\mathbf{u} \rightarrow \mathbf{v}]} \sum_{p \in \pi} M_p.$$

It is useful, however, to first replace the maximum by a sum over all path tuples. For $\beta > 0$, define the positive-temperature partition function

$$Z_M^\beta(\mathbf{u}; \mathbf{v}) = \sum_{\pi \in Q[\mathbf{u} \rightarrow \mathbf{v}]} \exp \left\{ \beta \sum_{p \in \pi} M_p \right\}.$$

The parameter β controls how strongly the largest-weight path is favoured. As $\beta \rightarrow \infty$, the largest contribution dominates the sum, and

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z_M^\beta(\mathbf{u}; \mathbf{v}) = L_M(\mathbf{u}; \mathbf{v}).$$

Thus a positive-temperature identity for Z_M^β implies the corresponding last-passage identity after taking this limit.

The advantage of the positive-temperature formulation is that disjoint path partition functions are governed by the Lindström–Gessel–Viennot determinant identity [GV85]. Namely, whenever the endpoints are ordered compatibly,

$$Z_M^\beta(\mathbf{u}; \mathbf{v}) = \det \left[Z_M^\beta(u_i; v_j) \right]_{i,j=1}^k. \quad (36)$$

This determinant identity is the algebraic input behind the isometry.

We now describe the discrete RSK transform at positive temperature. For $i = 1, \dots, m$ and $j \leq i$, let

$$u_i^j = ((1, n)^j, {}^j(i, 1)),$$

where $(1, n)^j$ denotes j copies of the lower-left point $(1, n)$, while ${}^j(i, 1)$ denotes the j -tuple of adjacent upper endpoints ending at $(i, 1)$. Thus $Z_M^\beta(u_i^j)$ is the partition function for j disjoint paths

from the lower-left corner to the j adjacent endpoints at the upper boundary ending at horizontal coordinate i .

The positive-temperature RSK transform $\Phi_\beta(M)$ is the triangular array whose entries are chosen so that these nested partition functions from the lower-left corner are preserved. Equivalently, its entries are defined by the ratios

$$\Phi_\beta(M)_{i,j} = \begin{cases} \frac{Z_M^\beta(u_i^j)}{Z_M^\beta(u_i^{j-1})}, & i = j, \\ \frac{Z_M^\beta(u_i^j) Z_M^\beta(u_{i-1}^{j-1})}{Z_M^\beta(u_i^{j-1}) Z_M^\beta(u_{i-1}^j)}, & j < i. \end{cases}$$

This formula may look a little artificial at first sight, but its purpose is simple: it is engineered so that, when one computes partition functions in the triangular array $\Phi_\beta(M)$, the nested lower-left-corner partition functions agree with those of the original array M . That is,

$$Z_{\Phi_\beta(M)}^\beta(u_i^j) = Z_M^\beta(u_i^j), \quad 1 \leq j \leq i \leq m. \quad (37)$$

This identity is not a separate deep input. It is the basic verification of the definition of $\Phi_\beta(M)$: the ratio formula above is obtained by solving for the entries of the triangular array so that (37) holds. Equivalently, one proves (37) by induction on i and j , using the elementary recurrence for positive-temperature path partition functions in a triangular environment. We suppress this routine verification and use (37) as the defining preservation property of the transform.

Taking $\beta \rightarrow \infty$ gives the corresponding zero-temperature transform $\Psi(M)$. Its entries are obtained by replacing products and quotients in the definition of $\Phi_\beta(M)$ by sums and differences of last-passage values:

$$\Psi(M)_{i,j} = \begin{cases} L_M(u_i^j) - L_M(u_i^{j-1}), & i = j, \\ L_M(u_i^j) + L_M(u_{i-1}^{j-1}) - L_M(u_i^{j-1}) - L_M(u_{i-1}^j), & j < i. \end{cases}$$

This is the discrete counterpart of the melon map. In particular, for the discrete model, proving the RSK isometry amounts to showing that bottom-to-top passage values are unchanged when M is replaced by $\Psi(M)$.

We prove the corresponding positive-temperature statement. For $1 \leq i \leq j \leq m$, set

$$R_M(i, j) = Z_M^\beta((i, n); (j, 1)), \quad R_\Phi(i, j) = Z_{\Phi_\beta(M)}^\beta((i, n); (j, 1)).$$

These are upper-triangular matrices of one-path partition functions from the bottom line to the top line. The goal is to prove

$$R_M(i, j) = R_\Phi(i, j) \quad \text{for all } i \leq j.$$

This is exactly the one-path positive-temperature isometry; the k -path version then follows from the determinant identity (36).

The preservation identity (37) gives more information than just equality of some special partition functions. By applying (36), it says that certain contiguous minors of R_M and R_Φ are equal. Indeed, the j -path partition function $Z_M^\beta(u_i^j)$ is the determinant of the $j \times j$ matrix of one-path partition functions from the first j lower starting points to the j adjacent upper endpoints ending at i . Thus (37) identifies the corresponding contiguous minors of R_M and R_Φ .

These minor identities determine the entries of the upper-triangular matrix one diagonal at a time. The 1×1 minors give the first diagonal entries. Once the entries in the earlier diagonals have been identified, equality of the next family of contiguous minors determines the next diagonal entries. Inducting over the width of the interval $j - i$, we obtain

$$R_M(i, j) = R_\Phi(i, j) \quad \text{for all } i \leq j.$$

Therefore

$$Z_M^\beta((i, n); (j, 1)) = Z_{\Phi_\beta(M)}^\beta((i, n); (j, 1))$$

for every $i \leq j$, and by the Lindström–Gessel–Viennot identity the same holds for all disjoint k -tuples of ordered endpoints.

Finally, taking

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log(\cdot)$$

turns the positive-temperature identity into the corresponding last-passage identity:

$$L_M(\mathbf{x}, n; \mathbf{y}, 1) = L_{\Psi(M)}(\mathbf{x}, n; \mathbf{y}, 1).$$

Passing from step-function approximations to the original cadlag functions gives the desired identity

$$f[(\mathbf{x}, n) \rightarrow (\mathbf{y}, 1)] = Wf[(\mathbf{x}, n) \rightarrow (\mathbf{y}, 1)].$$

□

Hence, by the RSK Isometry, if we are only interested in starting values $x \geq 0$, we only need to apply the map W once at $x = 0$.

Note that, although we have proved tightness of S_n , it is still premature to call a subsequential limit an Airy sheet. At this point we have not shown that different subsequences give the same limiting law. In the next section we will give a definition of the Airy sheet distribution and show that every subsequential limit of S_n has this law.

Before doing so, let us explain why the Airy line ensemble should determine the answer. The $n^{1/3}$ -scale fluctuations of the passage value are produced near the Airy edge of the melon. The endpoint y determines where the optimiser finishes in this edge window. The starting displacement x determines the direction from which the optimiser enters the window.

Here is the point geometrically. Suppose that the optimiser is to finish near the Airy edge at spatial coordinate y . If, in the finite model, its starting point has been shifted by $2xn^{2/3}$, then by the time the optimiser first enters the Airy scaling window it is no longer expected to be on one of the top few curves. Instead, the deterministic parabolic shape predicts that it enters the Airy window from a lower line index, and from a spatial location which depends on x . In the Airy-line-ensemble coordinates, this incoming direction is represented by a point of the form

$$\left(-\sqrt{\frac{k}{2x}}, k \right),$$

with k large. Thus the one-sided Airy-line-ensemble picture asks us to start far down in the ensemble, subtract a deterministic centering, and then let $k \rightarrow \infty$. This is the source of the infinite bulk last passage problem.

More precisely, one is led to the following informal formulation. Let $\{B_i\}_{i \in \mathbb{N}}$ be independent Brownian motions started at zero, and let WB^n be the melonisation of $(B_1, \dots, B_n)$. Suppose that the

edge-scaled melon can be coupled to a parabolically shifted Airy line ensemble $\mathcal{A}$ so that, uniformly on compact subsets,

$$n^{-1/3} \left(WB_i^n(n + 2xn^{2/3}) - 2n - 2xn^{2/3} \right) \longrightarrow \mathcal{A}_i(x).$$

Then there should be a deterministic centering function $a : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ such that, for each fixed $x > 0$ and $y \in \mathbb{R}$, the Airy-line-ensemble quantity

$$\mathcal{A} \left[\left(-\sqrt{\frac{k}{2x}}, k \right) \rightarrow (y, 1) \right] - a \left(\sqrt{\frac{k}{2x}}, k \right)$$

has an almost sure limit as $k \rightarrow \infty$. Denote this limit by $\mathcal{S}(x, y)$. The Brownian last passage fluctuations should then converge in distribution to the same random variable:

$$n^{-1/3} \left(B^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] - 2n - 2(y - x)n^{2/3} \right) \xrightarrow{d} \mathcal{S}(x, y).$$

Thus, in this one-sided picture, the Airy sheet is determined by an infinite last passage problem through the Airy line ensemble. Making this precise requires detailed control of the ensemble deep in the bulk. The construction in the next section avoids putting this bulk problem at the centre of the presentation. Instead, it rotates the left-hand side of the finite Brownian model and rewrites the limiting problem as a finite-index variational problem near the Airy edge. This is not a different limiting object; it is a more convenient route to the same Airy sheet.

6.3. Last passage percolation through Dyson Brownian motion

Applying Theorem 6.3 to independent Brownian motions $(B_1, \dots, B_n)$, we translate the last passage problem in the independent Brownian environment to a last passage problem in Dyson Brownian motion.

COROLLARY 6.4. For $n \in \mathbb{N}$, let $B^n = (B_1, \dots, B_n)$ be independent Brownian motions all started at 0. Let $WB^n = \Gamma(B_1, \dots, B_n)$. Then WB^n is equal in law to n -Dyson Brownian motion, and, for any admissible $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{<}^k$,

$$B^n[(\mathbf{x}, n) \rightarrow (\mathbf{y}, 1)] = WB^n[(\mathbf{x}, n) \rightarrow (\mathbf{y}, 1)].$$

Already from here one can see how the Airy line ensemble enters the problem. The melon WB^n is Dyson Brownian motion, and the Airy line ensemble arises from its upper edge near time n , on the spatial scale $n^{2/3}$ and the fluctuation scale $n^{1/3}$. Thus we look at times of the form

$$t = n + 2xn^{2/3}, \quad x = O(1).$$

The deterministic edge location is obtained by expanding the Dyson edge shape around $t = n$. To first order this gives

$$2n + 2xn^{2/3}.$$

The next term in the expansion is of order $n^{1/3}$, and it is on this scale that the Airy fluctuations and the parabolic correction live. Thus one should think schematically of

$$WB_i^n(n + 2xn^{2/3}) = 2n + 2xn^{2/3} + n^{1/3} \{\text{parabolic Airy-scale fluctuation}\}.$$

This is why the centering below subtracts $2n + 2xn^{2/3}$. The word “parabolic” refers to the residual $n^{1/3}$ -scale shape seen after this linear centering, not to an additional $n^{2/3}$ -order term.

We now study last passage through the melon from a point whose horizontal coordinate is displaced by $2xn^{2/3}$ from the left edge to a point whose horizontal coordinate is displaced by $2yn^{2/3}$ from the right edge. The leading deterministic contribution is

$$2n + 2(y - x)n^{2/3},$$

and the remaining fluctuations are of order $n^{1/3}$:

$$WB^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] = 2n + 2(y - x)n^{2/3} + O(n^{1/3}).$$

Indeed the following lemma is fairly easy to prove.

LEMMA 6.5. Define

$$S_n(x, y) = n^{-1/3} \left(B^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] - 2n - 2(y - x)n^{2/3} \right).$$

Then S_n is a tight family of random variables.

PROOF SKETCH. First note that $S_n(0, 0)$ converges to a Tracy–Widom distribution [TW94]. Furthermore, $S_n(x, \cdot)$ and $S_n(\cdot, y)$ are both rescaled Dyson Brownian motions. This gives one-point control and a modulus of continuity for S_n , which is enough to conclude tightness. $\square$

Although we have proved tightness of S_n , it is still premature to call a subsequential limit an Airy sheet. At this point we have not shown that different subsequences give the same limiting law. In the next section we will give a definition of the Airy sheet distribution and show that every subsequential limit of S_n has this law.

Before doing so, let us explain why the Airy line ensemble should determine the answer. The $n^{1/3}$ -scale fluctuations of the passage value are produced near the Airy edge of the melon. The endpoint y determines where the optimiser finishes in this edge window. The starting displacement x determines the direction from which the optimiser enters it.

The geometry is as follows. In the one-sided Airy-line-ensemble picture, the optimiser is not seen as a short path between two edge points. It enters the Airy window from below: that is, from a line index k which tends to infinity. The horizontal location of this incoming point is fixed by matching the deterministic parabolic shape to the slope imposed by the starting displacement x . This matching gives an incoming Airy-line-ensemble point of the form

$$\left(-\sqrt{\frac{k}{2x}}, k \right).$$

The square root should be read as a parabolic scaling relation: going down to line k in the Airy ensemble corresponds to moving left by order $\sqrt{k/x}$ in the Airy spatial coordinate. Thus the one-sided picture asks us to start far down in the ensemble, subtract a deterministic centering, and then let $k \rightarrow \infty$. This is the source of the infinite bulk last passage problem.

The following conjectural formulation expresses the absolute-limit picture one would like to have. It is close to the geometric intuition behind the DOV construction, but DOV make this picture rigorous through differences rather than through this absolute limit.

CONJECTURE 6.6. Let $\{B_i\}_{i \in \mathbb{N}}$ be independent Brownian motions started at zero, and let WB^n be the melonisation of $(B_1, \dots, B_n)$. Suppose that the edge-scaled melon can be coupled to a

parabolically shifted Airy line ensemble $\mathcal{A}$ so that, uniformly on compact subsets,

$$n^{-1/3} \left(W B_i^n(n + 2xn^{2/3}) - 2n - 2xn^{2/3} \right) \longrightarrow \mathcal{A}_i(x).$$

Then there is a deterministic centering function $a : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ such that, for each fixed $x > 0$ and $y \in \mathbb{R}$, the Airy-line-ensemble quantity

$$\mathcal{A} \left[\left(-\sqrt{\frac{k}{2x}}, k \right) \rightarrow (y, 1) \right] - a \left(\sqrt{\frac{k}{2x}}, k \right)$$

converges almost surely as $k \rightarrow \infty$. Denote this almost sure limit by $\mathcal{S}(x, y)$. Then the Brownian last passage fluctuations converge in distribution to this same random variable:

$$n^{-1/3} \left(B^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] - 2n - 2(y - x)n^{2/3} \right) \xrightarrow{d} \mathcal{S}(x, y).$$

Thus, in this one-sided picture, the Airy sheet is expected to arise from an infinite last passage problem through the Airy line ensemble. The difficulty is that the displayed last passage value contains a large deterministic part depending on how far down the ensemble the path begins. One would like to subtract the correct centering and take an absolute limit, as in the conjecture above, but this is not the route taken in [DOV22].

Instead, DOV avoid proving the existence of this absolute renormalised limit by working with differences of last passage values. For a fixed remote starting point in the Airy line ensemble, they compare the values of travelling to two different top endpoints. The divergent centering from the remote source is then common to both terms and cancels. In this Busemann-like sense, the Airy sheet is identified through difference energies:

$$\mathcal{A} \left[\left(-\sqrt{\frac{k}{2x}}, k \right) \rightarrow (z, 1) \right] - \mathcal{A} \left[\left(-\sqrt{\frac{k}{2x}}, k \right) \rightarrow (y, 1) \right] = \mathcal{S}(x, z) - \mathcal{S}(x, y),$$

for all sufficiently large k , in the precise rational formulation of their definition. This difference formulation captures the same geometric idea while avoiding the need to construct the absolute bulk-to-edge limit directly.

The construction in the next section takes a different route for these notes. Rather than placing the bulk Airy-line-ensemble problem at the centre, we rotate the left-hand side of the finite Brownian model and rewrite the limiting problem as a finite-index variational problem near the Airy edge. This is not a different limiting object; it is a more convenient way to arrive at the same Airy sheet.

6.4. Construction of the Airy sheet

In the one-sided Airy line ensemble picture, the Airy sheet is approached through paths that rise from the bulk of the ensemble before reaching the edge. This is geometrically natural, but it leads to an infinite last passage problem through lower Airy curves, and its rigorous treatment requires substantial bulk estimates. In this section we use a different description. It keeps the relevant optimisation near the Airy edge by replacing the one-sided bulk-to-edge path with a two-sided edge problem.

6.4.1. The Airy diver. Throughout this section, $\mathcal{A}$ denotes the parabolic Airy line ensemble. Fix $x, y \geq 0$. Imagine a diver who starts at spatial location $-x$ on the top curve. The diver moves right, collecting the increments of whichever curves are followed. In a first phase of descent, passage from one curve to the next is from one curve to the adjacent lower one, until spatial location 0 is reached. The diver collects the value of the curve reached at this moment and then begins the ascent. In this latter phase, the diver climbs back up, passing from one curve to the adjacent higher curve, so as to resurface on the top curve and finish the journey at spatial location y . The line on which it crosses spatial location 0 is denoted by k . We will call k the turnaround index.

For a fixed k , the journey thus has three contributions. First there is the weight collected during descent, from $(-x, 1)$ to $(0, k)$. Second there is the height $\mathcal{A}_k(0)$ of the curve at the turnaround point. Third there is the weight collected during the ascent, from $(0, k)$ to $(y, 1)$. The traveller therefore faces a genuine depth problem. A larger value of k gives access to more possible pieces of curve on the two sides of the origin, but the turnaround height $\mathcal{A}_k(0)$ is typically lower. The correct depth is not specified in advance.

To write this cleanly, let us temporarily distinguish the two directions of passage. We write

$$\mathcal{A}^\uparrow[(a, i) \rightarrow (b, j)]$$

for the usual upright last passage value, in which the path moves to the right while its line index decreases. Thus $i \geq j$. We also write

$$\mathcal{A}^\downarrow[(a, i) \rightarrow (b, j)]$$

for the corresponding downward passage value, in which the path moves to the right while its line index increases. Thus $i \leq j$. With this temporary notation, the weight collected by a diver whose turnaround index is k is

$$\mathcal{A}^\downarrow[(-x, 1) \rightarrow (0, k)] + \mathcal{A}_k(0) + \mathcal{A}^\uparrow[(0, k) \rightarrow (y, 1)].$$

The diver's problem is to maximise this quantity over k .

There is one notational inconvenience. The left leg is downward, whereas our standard last passage notation is upright. We therefore reflect the left side. Define

$$\tilde{\mathcal{A}}_j(u) = \mathcal{A}_j(-u).$$

Under this reflection, the downward passage from $(-x, 1)$ to $(0, k)$ in $\mathcal{A}$ becomes the upright passage from $(0, k)$ to $(x, 1)$ in $\tilde{\mathcal{A}}$:

$$\mathcal{A}^\downarrow[(-x, 1) \rightarrow (0, k)] = \tilde{\mathcal{A}}^\uparrow[(0, k) \rightarrow (x, 1)].$$

From now on, passage values without an arrow are understood in the usual upright sense. Thus the same k -dependent collected value is written as

$$\mathcal{A}_k(0) + \tilde{\mathcal{A}}[(0, k) \rightarrow (x, 1)] + \mathcal{A}[(0, k) \rightarrow (y, 1)].$$

6.4.2. The definition and its first checks. In a specification that differs from the standard one offered in [DOV22], but in fact equivalent to it, we now define the Airy sheet by declaring its value to be the optimal collected value of the Airy diver. It is quite unclear why this specification is a good choice, but we will explain how indeed it is.

DEFINITION 6.7. The **Airy Sheet** is a random continuous function $\mathcal{S} : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that

- (1) $\mathcal{S}(\cdot, \cdot)$ has the same distribution as $\mathcal{S}(\cdot + t, \cdot + t)$ for all $t \in \mathbb{R}$.

- (2) $\mathcal{S}$ can be coupled with the parabolic Airy line ensemble $\mathcal{A}$ so that $\mathcal{S}(0, \cdot) = \mathcal{A}_1(\cdot)$, and for all $x, y \geq 0$,

$$\mathcal{S}(x, y) = \max_{k \in \mathbb{N}} \left\{ \mathcal{A}_k(0) + \tilde{\mathcal{A}}[(0, k) \rightarrow (x, 1)] + \mathcal{A}[(0, k) \rightarrow (y, 1)] \right\}. \quad (38)$$

Here $\tilde{\mathcal{A}}_j(u) = \mathcal{A}_j(-u)$. In particular, the maximum index k is almost surely achieved for all $x, y > 0$.

Thus, for $x, y \geq 0$, $\mathcal{S}(x, y)$ is obtained by optimising over the unimodal Airy diver's value for journeys of the form

$$(-x, 1) \longrightarrow (0, k) \longrightarrow (y, 1).$$

For a given turnaround index k , the assigned weight is the sum of the descent contribution, the ascent contribution, and the height $\mathcal{A}_k(0)$ collected at the turnaround point. After the descent is expressed in reflected coordinates, this is precisely the finite-index edge variational problem in (38).

The definition asks for two immediate checks. First, the law specified by the two conditions should be unique. Second, the diver should not be able to profit by going arbitrarily deep; equivalently, the maximum over k should be attained. These are the elementary points recorded next. The deeper question, namely why this diver variational problem is the Brownian LPP scaling limit, is postponed to the convergence proof.

LEMMA 6.8. The Airy Sheet is unique in law.

PROOF. By Definition 6.7, the law of

$$\{\mathcal{S}(x, y) : x, y > 0\}$$

is determined by the law of the Airy line ensemble. Now take any finite collection of points $(x_i, y_i) \in \mathbb{R}^2$. Choosing r sufficiently large, we have $x_i + r > 0$ and $y_i + r > 0$ for every i . By diagonal stationarity, the joint law of

$$(\mathcal{S}(x_i, y_i))_i$$

is the same as that of

$$(\mathcal{S}(x_i + r, y_i + r))_i,$$

and the latter is determined by the Airy line ensemble formula. Thus all finite-dimensional distributions of $\mathcal{S}$ are uniquely determined. Continuity then determines the law of $\mathcal{S}$ as a random continuous function on $\mathbb{R}^2$. $\square$

LEMMA 6.9 (Corollary 5.3 of [DV21]). There exist $c, \beta > 0$ so that, for all $m > \beta \log(i + 1)$,

$$\mathbb{P} \left(\left| \mathcal{A}_i(0) + \kappa i^{2/3} \right| \geq m i^{-1/3} \right) \leq c e^{-m/5},$$

where $\kappa = (3\pi/2)^{2/3}$.

LEMMA 6.10. There exist constants $c, d > 0$ so that, for all $x > 0$, $k \in \mathbb{N}$, and $r \geq 0$,

$$\mathbb{P}(\mathcal{A}[(0, k) \rightarrow (x, 1)] \geq r) \leq c \exp \left\{ -d \frac{r^2}{kx \log(2k)} \right\}.$$

These two estimates explain why the diver cannot profit by going arbitrarily deep. Lemma 6.9 says that

$$\mathcal{A}_k(0) \approx -\kappa k^{2/3}$$

as $k \rightarrow \infty$. On the other hand, Lemma 6.10 says that the last passage contributions on the two sides are much smaller, of order at most $\sqrt{k \log k}$ on the relevant scale. Thus the expression being maximised in (38) tends to $-\infty$ as $k \rightarrow \infty$, and the maximum is attained at some finite turnaround index.

6.4.3. Why the diver appears from Brownian LPP. We now explain why Definition 6.7 is the correct scaling limit of Brownian last passage percolation. For $x, y \in \mathbb{R}$, define the prelimiting Airy sheet by

$$S_n(x, y) = n^{-1/3} \left(B[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] - 2n - 2(y - x)n^{2/3} \right).$$

This is the same $n^{1/3}$ -scale fluctuation field introduced in the preceding subsection, where its tightness was proved.

THEOREM 6.11. Let $\mathcal{S}$ be any subsequential limit of S_n . Then $\mathcal{S}$ is an Airy Sheet.

PROOF. Since S_n is stationary under diagonal translations, any subsequential limit $\mathcal{S}$ is also stationary under diagonal translations. It is therefore enough to identify the limiting distribution of $S_n(x, y)$ for $x, y \geq 0$.

By the RSK isometry,

$$B[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] = WB^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)],$$

where $WB^n = (WB_1^n, \dots, WB_n^n)$ is equal in distribution to n -Dyson Brownian motion. Applying the metric composition law at the intermediate spatial location n , we obtain

$$\begin{aligned} & WB^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] \\ &= \max_{k \in \llbracket 1, n \rrbracket} \left\{ WB^n[(2xn^{2/3}, n) \rightarrow (n, k)] + WB^n[(n, k) \rightarrow (n + 2yn^{2/3}, 1)] \right\}. \end{aligned} \quad (39)$$

The index k is the line on which the optimiser crosses the middle spatial location. This is the finite- n analogue of the diver's turnaround index. The Airy-diver picture suggests that this index should lie in the Airy edge window. In other words, after scaling, the optimising path should turn around at a line index of order one. This is the technical input needed for the construction.

LEMMA 6.12. For $n \in \mathbb{N}$, let k^n be an index at which the maximum is attained in (39). Then k^n is a tight family of random variables.

We leave the proof of Lemma 6.12 for later and continue the proof of Theorem 6.11. Passing to a further subsequence, we may assume that the maximising indices converge almost surely along the subsequence.

We next identify the two summands in (39). The right-hand summand is already an edge last passage problem; it will become the ascent from the turnaround line to the top curve at y . The left-hand summand has the opposite orientation; it will become the descent, but only after reflection. At finite n , this reflected description is obtained by completing the left-hand path to a bottom-to-top bouquet, applying the RSK isometry, rotating the Brownian environment, and applying the RSK

isometry again. Once these two finite- n edge descriptions are in hand, we pass to the Airy limit and stitch the two edge ensembles at the middle point.

The right-hand term is the simpler one. For fixed k , it is already a last passage value near the upper edge of the melon:

$$WB^n[(n, k) \rightarrow (n + 2yn^{2/3}, 1)].$$

After centering, this term will converge to the Airy last passage value

$$\mathcal{A}^{top}[(0, k) \rightarrow (y, 1)].$$

This is the finite- n source of the ascent phase of the Airy diver, from the turnaround line k at spatial location 0 to the top curve at y .

It remains to identify the left-hand term

$$WB^n[(2xn^{2/3}, n) \rightarrow (n, k)].$$

This is the only mildly delicate combinatorial point in the proof. The term is a path in the melon, hence in Dyson Brownian motion, but the useful rotation symmetry belongs to the original Brownian LPP environment. Moreover, this path does not run from the bottom line all the way to the top line when $k > 1$, so the RSK isometry cannot yet be applied to it directly. We therefore enlarge it to a genuine bottom-to-top bouquet by adjoining the forced bouquet carried by the top $k - 1$ melon curves.

For fixed k , the ordered structure of the melon gives

$$\begin{aligned} WB^n[(2xn^{2/3}, n) \rightarrow (n, k)] &= WB^n[(\{0^{k-1}, 2xn^{2/3}\}, n) \rightarrow (n, 1)^k] \\ &\quad - WB^n[(0^{k-1}, n) \rightarrow (n, 1)^{k-1}]. \end{aligned} \quad (40)$$

Here $0^{k-1} = (0, \dots, 0)$, and $(n, 1)^k$ denotes the k -tuple of upper endpoints on the first k lines. The reason for (40) is that the first $k - 1$ melon paths, which follow the top $k - 1$ curves from time 0 to time n , form a forced optimising bouquet. This bouquet is disjoint from the remaining path from $(2xn^{2/3}, n)$ to (n, k) . Their union is therefore a k -disjoint bottom-to-top bouquet, and subtracting the forced $(k - 1)$ -bouquet recovers the original left-hand path.

Now the two terms on the right-hand side of (40) are bottom-to-top bouquet values, so the RSK isometry applies. We pull them back to the original Brownian LPP environment, rotate that Brownian environment by 180° , and then apply the RSK isometry once more. This gives

$$WRB^n[(0, 1)^k \rightarrow (\{n - 2xn^{2/3}, n^{k-1}\}, n)] - WRB^n[(0, n)^{k-1} \rightarrow (n, 1)^{k-1}]. \quad (41)$$

The first term in (41) may be read as a filled k -path object with one complementary curve left out. The filled object occupies the top k lines in the rotated melon. One path stops at the earlier spatial location $n - 2xn^{2/3}$, while the other $k - 1$ paths continue to the final time and occupy the adjacent upper positions. The complementary path, or shadow curve, follows the pieces left unused by this filled family. It starts near $(n - 2xn^{2/3}, 1)$ and ends at (n, k) .

This shadow curve is the finite- n form of the diver's descent. The completely filled k -path contribution records the value of the top k curves up to the middle slice; subtracting the corresponding $(k - 1)$ -path contribution leaves exactly the height of the k^{th} curve at the turnaround point. This is the finite- n precursor of the value $\mathcal{A}_k(0)$ collected by the diver at the end of the descent. The remaining complementary path is the descent contribution itself, written after rotation and reflection as an upright passage value.

Thus the temporary downward notation reappears here in substance:

$$\text{shadow curve} \leftrightarrow \mathcal{A}^\downarrow[(-x, 1) \rightarrow (0, k)] = \tilde{\mathcal{A}}[(0, k) \rightarrow (x, 1)]$$

in the Airy limit. At finite n , this relation is expressed by the identity

$$WRB^n[(0, 1)^k \rightarrow (\{n - 2xn^{2/3}, n^{k-1}\}, n)] = WRB^n[(0, n)^k \rightarrow (n, 1)^k] + \tilde{W}B^n[(0, k) \rightarrow (2xn^{2/3}, 1)], \quad (42)$$

where

$$\tilde{W}B_i^n(u) = WRB_i^n(n - u).$$

The first term on the right of (42) is the completely filled k -path contribution. When we subtract the corresponding $(k - 1)$ -path contribution in (41), what remains is the height of the k^{th} curve at the middle point, together with the shadow curve contribution.

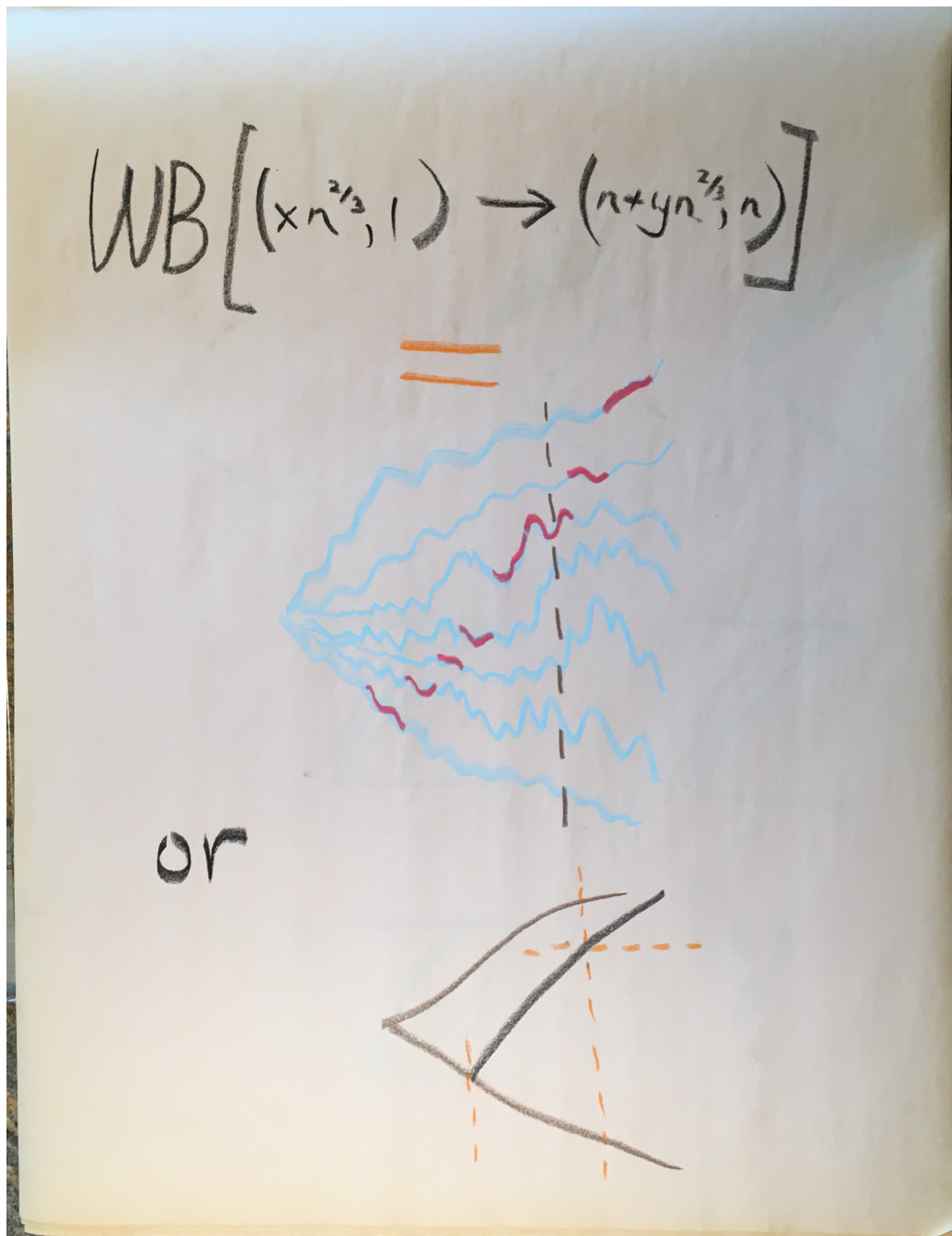


FIGURE 6.1. A red geodesic crosses the Brownian watermelon, starting at $xn^{2/3}$ on the lowest or n -indexed curve and finishing at $n + yn^{2/3}$ on the uppermost curve, with index one. The geodesic crosses n through the dashed black line on the curve with index $n + 1 - k$: on the k^{th} highest curve. The lower picture is of the schematic type appearing on the next two pages.

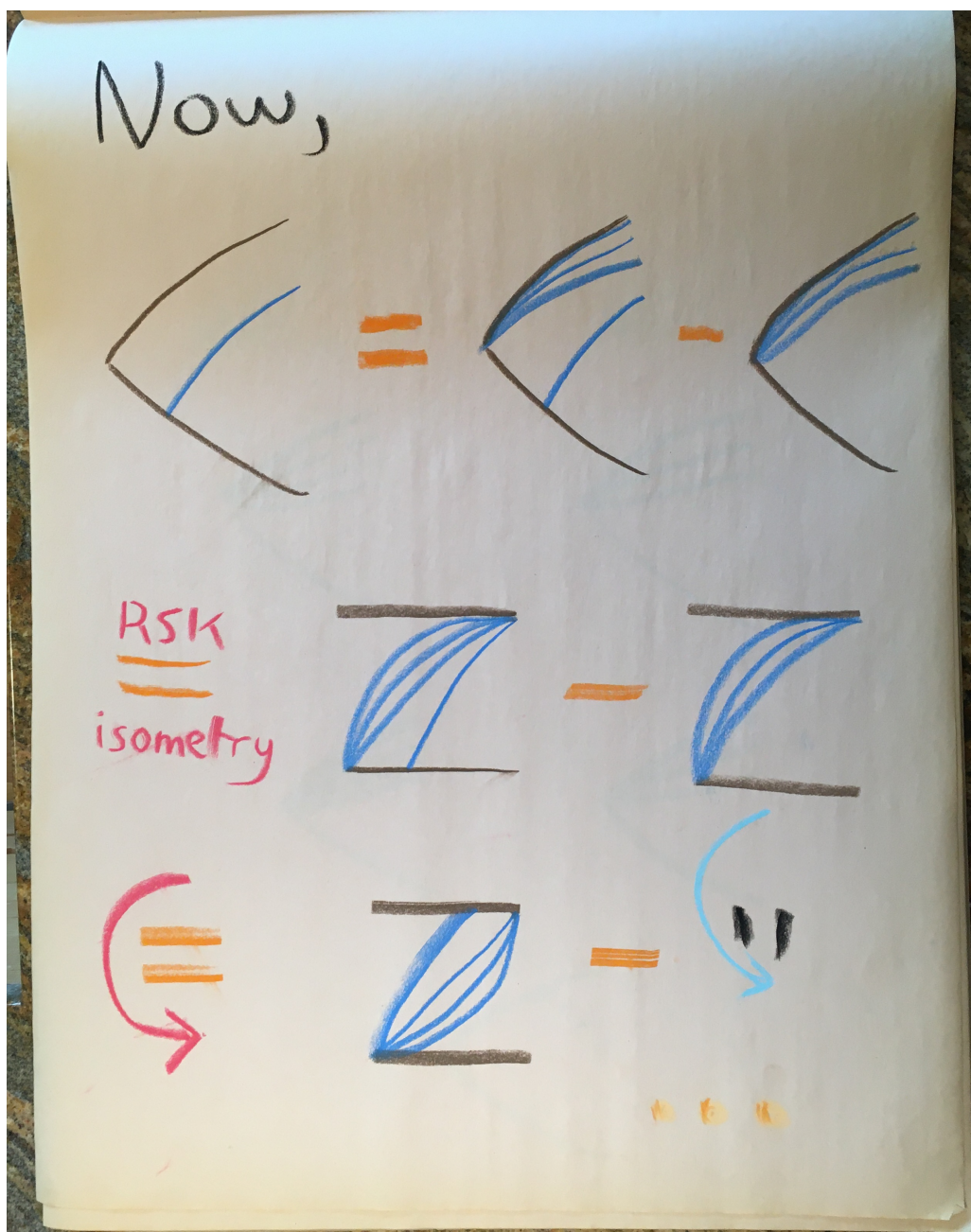


FIGURE 6.2. The left-hand side is the picture from the preceding page truncated at n , so that the blue curve abuts on index $n + 1 - k$. We add, and subtract, k blue geodesics running out from zero. The right-hand terms concern bouquets that run across the entire index space, and this makes the RSK isometry available. Thus we may write the two expressions in terms of the Brownian LPP environment. Carrying the latter term over, a symmetry of Brownian LPP is exploited to rotate the first picture by 180 degrees.

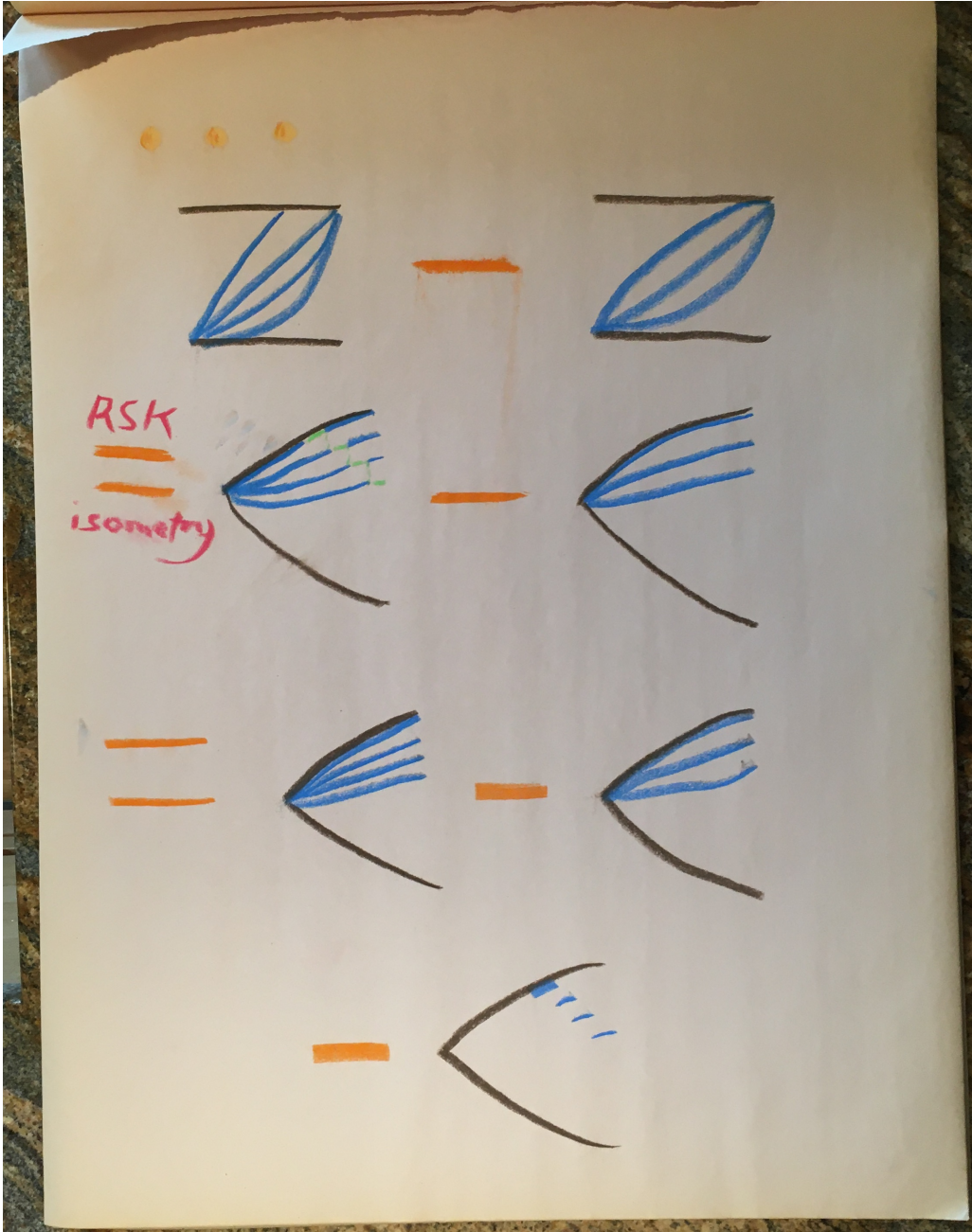


FIGURE 6.3. We carry over the difference term. Applying the RSK isometry to the first term, we obtain k paths, starting in the top k places in the watermelon environment. The uppermost ends early, at $n - xn^{2/3}$, while the others run on to finish in slots in $\llbracket 1, k-1 \rrbracket$ at n . The green path is what is absent in the union of their traces. For the second term, we simply have $k-1$ curves running in uppermost positions between 0 and n in the watermelon. In the final equality, we rearrange. The first difference contributes a curve holding constant with index k in the watermelon from 0 to n . In the latter subtracted term, we see the descending path whose trace is the green region. After scaling, these terms contribute the value $\mathcal{A}_k(0)$ and the increment sum $\mathcal{A}^\downarrow((-x, 1) \rightarrow (0, k))$ in the Airy diver's descent.

We now package these two finite- n edge descriptions into their limiting line ensembles. Define the 180° rotation of B on $[0, n] \times \{1, \dots, n\}$ by

$$RB_i(u) = B_{n-i+1}(n-u), \quad u \in [0, n], \quad i \in \{1, \dots, n\}.$$

We couple the systems so that the following edge-scaled ensembles converge uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{N}$:

$$\begin{aligned} A_{n,i}^{top}(y) &:= n^{-1/3} \left(WB_i^n(n + 2yn^{2/3}) - 2n - 2yn^{2/3} \right) \longrightarrow \mathcal{A}_i^{top}(y), \\ A_{n,i}^{bot}(x) &:= n^{-1/3} \left(WRB_i^n(n + 2xn^{2/3}) - 2n - 2xn^{2/3} \right) \longrightarrow \mathcal{A}_i^{bot}(x). \end{aligned}$$

At the middle point these two limits agree:

$$\mathcal{A}_i^{top}(0) = \mathcal{A}_i^{bot}(0), \quad i \in \mathbb{N}.$$

Thus we may stitch the two limiting ensembles together at zero.

LEMMA 6.13. Under the coupling above, define

$$\mathcal{A}_i(u) = \begin{cases} \mathcal{A}_i^{top}(u), & u \geq 0, \\ \mathcal{A}_i^{bot}(-u), & u < 0. \end{cases}$$

Then $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ is equal in distribution to the parabolic Airy line ensemble.

The stitched object is certainly a line ensemble, but what matters is the conclusion of the lemma: its law is that of the ordinary Airy line ensemble. The finite- n reason is simple. At time n , the top and bottom constructions have the same Dyson point configuration. Given this middle configuration, the Markov property gives independent left and right continuations with the correct transition laws. Replacing the left side by the left side coming from the rotated Brownian environment therefore changes neither the conditional law nor the middle distribution.

PROOF OF LEMMA 6.13. Let B^n , WB^n , and WRB^n be as above. Since RB^n is again a system of Brownian input curves, WRB^n is equal in distribution to Dyson Brownian motion. Moreover, by the Gibbs–Markov property of Dyson Brownian motion, the two sides

$$(WB^n(n-u))_{u \geq 0} \quad \text{and} \quad (WB^n(n+v))_{v \geq 0}$$

are conditionally independent given $WB^n(n)$, and the same statement holds for WRB^n .

Since the two constructions agree at the middle point, $WB^n(n) = WRB^n(n)$, the processes

$$(WB^n(n-u), WB^n(n), WB^n(n+v))_{u,v \geq 0}$$

and

$$(WRB^n(n-u), WB^n(n), WB^n(n+v))_{u,v \geq 0}$$

have the same transition probabilities and the same middle distribution. Thus they have the same law. If we define the stitched Dyson Brownian motion

$$W^n(u) = \begin{cases} WB^n(u), & u \geq n, \\ WRB^n(u), & u < n, \end{cases}$$

then W^n is again equal in distribution to Dyson Brownian motion. Taking the edge scaling limit gives an Airy line ensemble. Under the coupling above, this edge scaling limit is precisely the stitched ensemble $\mathcal{A}$. Hence $\mathcal{A}$ is equal in distribution to the parabolic Airy line ensemble. $\square$

We can now pass the two finite- n edge descriptions to the limit. For fixed k , the right-hand term satisfies

$$n^{-1/3} \left(WB^n[(n, k) \rightarrow (n + 2yn^{2/3}, 1)] - 2yn^{2/3} \right) \longrightarrow \mathcal{A}^{top}[(0, k) \rightarrow (y, 1)].$$

Likewise, the reflected left-hand edge term satisfies

$$n^{-1/3} \left(\tilde{W}B^n[(0, k) \rightarrow (2xn^{2/3}, 1)] + 2xn^{2/3} \right) \longrightarrow \tilde{\mathcal{A}}[(0, k) \rightarrow (x, 1)].$$

Finally,

$$\begin{aligned} n^{-1/3} \left(WRB^n[(0, n)^k \rightarrow (n, 1)^k] - WRB^n[(0, n)^{k-1} \rightarrow (n, 1)^{k-1}] - 2n \right) \\ = n^{-1/3} (WRB_k^n(n) - 2n) \longrightarrow \mathcal{A}_k(0). \end{aligned}$$

Combining these limits, we find that, for every fixed k ,

$$\begin{aligned} n^{-1/3} \left(WB^n[(2xn^{2/3}, n) \rightarrow (n, k)] + WB^n[(n, k) \rightarrow (n + 2yn^{2/3}, 1)] \right. \\ \left. - 2n - 2(y - x)n^{2/3} \right) \\ \longrightarrow \mathcal{A}_k(0) + \tilde{\mathcal{A}}[(0, k) \rightarrow (x, 1)] + \mathcal{A}[(0, k) \rightarrow (y, 1)]. \end{aligned}$$

Finally, Lemma 6.12 allows us to pass from fixed k to the maximising k in the midpoint decomposition. Therefore every subsequential limit of $S_n(x, y)$, for $x, y \geq 0$, has the distribution given by (38). Together with diagonal stationarity, this shows that the subsequential limit is an Airy sheet. $\square$

Now it remains to prove tightness of k_n .

PROOF OF LEMMA 6.12. We record the beginning of the natural argument. Define

$$\begin{aligned} F_n(k) &= n^{-1/3} \left(WB^n[(2xn^{2/3}, n) \rightarrow (n, k)] - WB_k^n(n) + 2xn^{2/3} \right), \\ G_n(k) &= n^{-1/3} \left(WB^n[(n, k) \rightarrow (n + 2yn^{2/3}, 1)] + WB_k^n(n) - 2n - 2yn^{2/3} \right), \\ H_n &= n^{-1/3} \left(WB^n[(2xn^{2/3}, n) \rightarrow (n + 2yn^{2/3}, 1)] - 2n - 2(y - x)n^{2/3} \right). \end{aligned}$$

Then $F_n(k)$ is the normalised cost of reaching line k from the left, after removing the height $WB_k^n(n)$ at the middle point, while $G_n(k)$ is the corresponding normalised cost of travelling from line k to the right, with the same middle height added back. Thus

$$\max_{1 \leq k \leq n} \{F_n(k) + G_n(k)\} = H_n.$$

The random variables H_n are tight by the tightness of the prelimiting Airy sheet variables $S_n(x, y)$ proved in the preceding subsection. The expected conclusion is that $F_n(k) + G_n(k)$ becomes very negative when k is large. This would force the maximising index to remain tight.

Equivalently, one wants to prove that, for every deterministic sequence $k_n \rightarrow \infty$,

$$F_n(k_n) + G_n(k_n) \longrightarrow -\infty$$

in probability. This is a localisation estimate for the midpoint line. It expresses the same edge heuristic used above: an optimiser contributing to the Airy sheet should cross the middle inside the Airy window, not deep in the bulk. Proving this rigorously requires upper bounds on the two terms F_n and G_n , or an appeal to an existing midpoint-localisation result. $\square$

6.5. Constructing the directed landscape

Let us now define the universal limiting object: the directed landscape. Set

$$\mathbb{R}_{<}^4 = \{(x, s; y, t) \in \mathbb{R}^4 : s < t\},$$

and let $\mathcal{C}(\mathbb{R}_{<}^4)$ denote the space of continuous functions $\mathbb{R}_{<}^4 \rightarrow \mathbb{R}$, equipped with the topology of uniform convergence on compact sets.

DEFINITION 6.14. A **directed landscape** is a $\mathcal{C}(\mathbb{R}_{<}^4)$ -valued random variable $\mathcal{L}$ satisfying the following properties:

- (1) (Airy sheet marginals) For each $t \in \mathbb{R}$ and $s > 0$, the random function

$$(x, y) \mapsto \mathcal{L}(x, t; y, t + s^3)$$

is equal in distribution to an Airy sheet of scale s .

- (2) (Independent increments) For any finite collection of disjoint time intervals (t_i, s_i) , $i = 1, \dots, k$, the random functions

$$\mathcal{L}(\cdot, t_i; \cdot, s_i)$$

are independent.

- (3) (Metric composition law) Almost surely, for all $r < s < t$ and $x, y \in \mathbb{R}$,

$$\mathcal{L}(x, r; y, t) = \max_{z \in \mathbb{R}} \{ \mathcal{L}(x, r; z, s) + \mathcal{L}(z, s; y, t) \}.$$

From the definition, the directed landscape inherits the following basic distributional symmetries.

LEMMA 6.15. Let $\mathcal{L}$ be a directed landscape, assuming for now that it exists and is unique in law. Then the following equalities hold in distribution as random functions on $\mathcal{C}(\mathbb{R}_{<}^4)$. Here $r, c \in \mathbb{R}$ and $q > 0$.

- (1) (Time stationarity)

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(x, s + r; y, t + r).$$

- (2) (Spatial stationarity)

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(x + c, s; y + c, t).$$

- (3) (Flip symmetry)

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(-y, -t; -x, -s).$$

- (4) (Skew symmetry)

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(x + cs, s; y + ct, t) + (t - s)^{-1} [(x - y - c(t - s))^2 - (x - y)^2].$$

- (5) (Rescaling)

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} q \mathcal{L}(q^{-2}x, q^{-3}s; q^{-2}y, q^{-3}t).$$

We now construct the directed landscape. The first step is to construct it on dyadic times by stitching together independent Airy sheets. For $k \in \mathbb{Z}$, set

$$S_k = \{(x, t; y, s) \in \mathbb{R}_{<}^4 : 2^k t, 2^k s \in \mathbb{Z}\},$$

and let

$$S = \bigcup_{k \in \mathbb{Z}} S_k.$$

LEMMA 6.16. There exists a directed landscape defined on dyadic times. Its law is unique.

PROOF. The joint law of $\mathcal{L} : S \rightarrow \mathbb{R}$ on any finite collection of dyadic time intervals is uniquely determined by the Airy sheet marginals, independent increments, and the metric composition law. This proves uniqueness on dyadic times.

For existence, we construct the process level by level. Suppose first that we are working on the mesh $2^{-k}\mathbb{Z}$. For each $i \in \mathbb{Z}$, let $\mathcal{S}_i$ be an independent Airy sheet. If

$$t = \frac{i}{2^k}, \quad s = \frac{1}{2^k},$$

define the landscape across the elementary interval $[t, t + s]$ by

$$\mathcal{L}_k(x, t; y, t + s) = s \mathcal{S}_i(x/s^2, y/s^2).$$

For longer intervals on the same mesh, define $\mathcal{L}_k$ by composition: for $i \in \mathbb{Z}$ and $\ell \geq 2$,

$$\mathcal{L}_k\left(x, \frac{i}{2^k}; y, \frac{i + \ell}{2^k}\right) = \max_{x_1, \dots, x_{\ell-1} \in \mathbb{R}} \sum_{j=1}^{\ell} \mathcal{L}_k\left(x_{j-1}, \frac{i + j - 1}{2^k}; x_j, \frac{i + j}{2^k}\right),$$

where $x_0 = x$ and $x_\ell = y$.

By the Airy sheet stitching theorem, this construction is consistent when passing from one dyadic mesh to a finer one: $\mathcal{L}_k$ agrees in law with $\mathcal{L}_{k-1}$ on S_{k-1} . Kolmogorov's extension theorem therefore gives a random function $\mathcal{L} : S \rightarrow \mathbb{R}$, and the construction ensures that the three defining properties hold on dyadic times. $\square$

To extend $\mathcal{L}$ to all times, one needs to prove almost sure continuity on compact subsets of $\mathbb{R}_{<}^4$. This is a technical step rather than a conceptual one: the needed estimates come from the spatial and temporal moduli of continuity of the Airy sheet and Airy process marginals. We record the result without proof.

PROPOSITION 6.17 (Lemma 10.4 and Proposition 10.5 of [DOV22]). Almost surely, $\mathcal{L}$ is continuous on dyadic times and has a unique continuous extension to $\mathbb{R}_{<}^4$.

It remains to check that the metric composition law continues to hold after passing from dyadic times to arbitrary times. The intended argument is to approximate the three times by dyadic times and pass to the limit. The only point requiring care is to rule out maximizers escaping to spatial infinity. This is where the parabolic shape of the directed landscape enters.

LEMMA 6.18 (Corollary 10.7 of [DOV22]). There exists a random constant C satisfying

$$\mathbb{P}(C > m) \leq ce^{-dm^{3/2}}$$

such that, for all $u = (x, t; y, t + s) \in \mathbb{R}_{<}^4$,

$$\left| \mathcal{L}(x, t; y, t + s) + \frac{(y - x)^2}{s} \right| \leq C s^{1/3} \log^{4/3} \left(\frac{2(\|u\| + 2)}{s} \right) \log^{2/3}(\|u\| + 2).$$

We now prove the metric composition law for arbitrary times.

LEMMA 6.19. Almost surely, for all $x, y \in \mathbb{R}$ and $t < r < s$,

$$\mathcal{L}(x, t; y, s) = \max_{z \in \mathbb{R}} \{ \mathcal{L}(x, t; z, r) + \mathcal{L}(z, r; y, s) \}.$$

PROOF. Let t_n, r_n, s_n be dyadic rationals converging respectively to t, r, s , with $t_n < r_n < s_n$. By the dyadic metric composition law, there exists $z_n \in \mathbb{R}$ such that

$$\mathcal{L}(x, t_n; y, s_n) = \mathcal{L}(x, t_n; z_n, r_n) + \mathcal{L}(z_n, r_n; y, s_n).$$

The parabolic bound in Lemma 6.18 prevents $|z_n|$ from tending to infinity: if z_n were far from both x and y , the two terms on the right would carry a large negative parabolic cost, while the left-hand side remains controlled. Hence (z_n) is bounded, and after passing to a subsequence we may assume that $z_n \rightarrow z$.

By continuity,

$$\mathcal{L}(x, t; y, s) = \mathcal{L}(x, t; z, r) + \mathcal{L}(z, r; y, s).$$

For any fixed $z' \in \mathbb{R}$, applying the dyadic composition law with z' as a competitor and then passing to the limit gives

$$\mathcal{L}(x, t; y, s) \geq \mathcal{L}(x, t; z', r) + \mathcal{L}(z', r; y, s).$$

Thus z is a maximizer and the metric composition law holds. $\square$

Finally, we obtain the desired existence and uniqueness statement.

THEOREM 6.20. The directed landscape exists and is unique in law.

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